ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE
ENGINEERING AND TECHNOLOGY

STABILITY ANALYSIS AND HOPF BIFURCATION IN A DELAY-DYNAMICAL SYSTEM

M.Sc. THESIS

Yasemin ÇALIŞ

Department of Mathematical Engineering

Mathematical Engineering Program

JULY 2020

ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE
ENGINEERING AND TECHNOLOGY

STABILITY ANALYSIS AND HOPF BIFURCATION IN A DELAY-DYNAMICAL SYSTEM

M.Sc. THESIS

Yasemin ÇALIŞ
(509181209)

Department of Mathematical Engineering

Mathematical Engineering Program

Thesis Advisor: Assoc. Prof. Dr. Cihangir ÖZEMİR
Co-advisor: Dr. Ali DEMİRCİ

JULY 2020

İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

GECİKMELİ BİR DİNAMİK SİSTEMİN KARARLILIK ANALİZİ VE HOPF ÇATALLANMASI

YÜKSEK LİSANS TEZİ

Yasemin ÇALIŞ
(509181209)

Matematik Mühendisliği Anabilim Dalı

Matematik Mühendisliği Programı

Tez Danışmanı: Doç. Dr. Cihangir ÖZEMİR
Eş Danışman: Dr. Ali DEMİRCİ

TEMMUZ 2020

Yasemin ÇALIŞ, a M.Sc. student of ITU Graduate School of Science Engineering and Technology 509181209 successfully defended the thesis entitled “STABILITY ANALYSIS AND HOPF BIFURCATION IN A DELAY-DYNAMICAL SYSTEM”, which he/she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

Thesis Advisor :	**Assoc. Prof. Dr. Cihangir ÖZEMİR** Istanbul Technical University	....
Co-advisor :	**Dr. Ali DEMİRCİ** Istanbul Technical University	....
Jury Members :	**Assoc. Prof. Dr. Ahmet KIRIŞ** Istanbul Technical University	....
	Assoc. Prof. Dr. Handan BORLUK Özyeğin University	....
	Assist. Prof. Dr. Ali DİNLER Istanbul Medeniyet University	....

Date of Submission : **15 June 2020**
Date of Defense : **13 July 2020**

FOREWORD

First and foremost, I would like to thank my advisor Assoc. Prof. Cihangir Özemir for the endless supervision and extensive guidance during the master thesis. I would like to express my sincere gratitude for his valuable advice, constructive idea and moral support. I further wish to thank my co-advisor Ali Demirci, Ph.D. for the instructive comments, continuous support and valuable contributions. I deeply appreciate to my advisors their kindnesses, immeasurable patients and understandings.

My special thanks should be given to my friends, Havva SÜLÜK and Tuba OK for their supports and understanding. Finally, I would like to thank my dear family for all their moral supports and unlimited love throughout my life.

July 2020 Yasemin ÇALIŞ

TABLE OF CONTENTS

Page

ABBREVIATIONS

ODE	**:** Ordinary Differential Equation
DDE	**:** Delay Differential Equation
LDDE	**:** Linear Delay Differential Equation

SYMBOLS

x	**:** The Interest Rate
y	**:** The Investment Demand
z	**:** The Price Index
u	**:** Average Profit Margin
K	**:** Feedback Strength
t	**:** Time
τ	**:** Amount of Time-Delay
a	**:**The Saving Amount
b	**:**The Cost Per Investment
c	**:**The Elasticity of Demand

LIST OF TABLES

LIST OF FIGURES

STABILITY ANALYSIS AND HOPF BIFURCATION IN A DELAY-DYNAMICAL SYSTEM

SUMMARY

Nonlinear dynamical systems have had an important place in the financial science for the last decades. These developments have helped the community understand the internal complexity of financial and economical models especially through stability, bifurcation and chaos theory. In literature, there is a great deal of studies and dynamical systems on this field.

In this thesis work, the following dynamical system is considered

$$\dot{x} = z(t) + [y(t) - a]x(t) + u(t), \tag{1a}$$
$$\dot{y} = 1 - by(t) - x^2(t) + K[y(t) - y(t-\tau)], \tag{1b}$$
$$\dot{z} = -x(t) - cz(t), \tag{1c}$$
$$\dot{u} = -dx(t)y(t) - ku(t) \tag{1d}$$

where a,b,c,d,k are nonnegative parameters of the system. Here K is the feedback strength and τ is time delay term, $K,\tau \in \mathbb{R}$ and $K,\tau \geq 0$. State variables of the systems represent the interest rate x, the investment demand y, the price index z and average profit margin u.

The main purpose of this study is to investigate the dynamic response of the system with average profit margin variable and time delay. The topics covered in the thesis study are as follows:

In Section 1, we introduce the model we are considering and we present information on the properties of this system. We give a brief overview on the other financial dynamical systems available in the literature.

In Section 2, we review some basic information about nonlinear stability analysis of dynamical systems, in non-delay and delay case.

Section 3 includes the main work that was carried out in this thesis study. A financial model with the delayed feedback term is considered and the fixed points of this system are obtained. The distributions of the roots of the transcendental type characteristic equation is analyzed at the fixed points. After stability analysis, we determine a critical value for the time delay τ, which we name as τ_0. We show that the system undergoes a Hopf bifurcation at τ_0 theoretically, switching its dynamics from stability to instability under some conditions on the parameters. Furthermore, the information obtained theoretically is represented by numerical simulations. We exhibit the stability condition of the system at the different τ values by graphs.

In Section 4, we summarize our results and we conclude by some future recommendations.

GECİKMELİ BİR DİNAMİK SİSTEMİN
KARARLILIK ANALİZİ VE HOPF ÇATALLANMASI

ÖZET

Dinamik sistemler hayatımızın bir parçasıdır ve zamana göre değişimi modelleyen sistemlerdir. Bu sistemler diferansiyel denklemler ile ifade edilirler ve lineer veya nonlineer olabilirler. Matematiksel olarak bir dinamik sistem,

$$\frac{dx}{dt} = \dot{x} = f(x), \quad x, f \in R^n \tag{2}$$

şeklinde ifade edilir.

Bu tez çalışmasının amacı zaman gecikmeli doğrusal olmayan finansal bir dinamik sistemin nitel davranışlarını araştırmaktır. Faiz oranı, yatırım talebi, fiyat endeksi ve ortalama kar marjı içeren bu sistemin dinamik yapısı incelenip, denge noktalarında stabilite analizi yapılarak Hopf çatallanması incelenmiştir. Ayrıca bu stabilite analizleri sayısal simüslayonlarla desteklenmiştir.

Sistemdeki kaotik bir davranış dış faktörlere bağlı kalmayıp sistemin doğal iç yapısındaki belirsizliklerden meydana gelmektedir. Bu durum ise kaos teorisinin ortaya çıkmasına ve bilim dünyasının dikkatini çekmesine sebep olmuştur. Kaos teorisi ise hava durumu, borsa, türbülans gibi kontrol ve tahmin edilmesi zor olgularda uygulama imkanı bulmaktadır.

Kaos teorisi fen bilimleri ve mühendislik bilimleri yanında ekonomi alanında da önemli bir yere sahiptir. 2007 yılında ABD'de görülen mortgage krizinde olduğu gibi ekonomi dünyasında herhangi bir kriz çıkması durumunda kaos meydana gelmektedir. Dinamik sistem teorisi ve ekonomi-finans bilimleri arasındaki etkileşim hem matematikçiler hem de ekonomi uzmanları için geçmişten günümüze önemli bir araştırma alanıdır.

Literatürde dinamik sistemlere bakıldığında, finans teorisi ile igili birçok matematiksel modeller vardır. Örneğin,

$$\dot{x} = z + (y - a)x, \tag{3a}$$
$$\dot{y} = 1 - by - x^2, \tag{3b}$$
$$\dot{z} = -x - cz \tag{3c}$$

şeklindeki üç bağımlı değişkenli finansal dinamik sistem; üretim, para, sermaye ve iş gücü olmak üzere dört alt değişkenden yola çıkılarak türetilmiş olup, sistemdeki x değişkeni faiz oranını, y değişkeni yatırım talebini ve z değişkeni fiyat endeksini ifade etmektedir. Sabit değerlerden bahsetmek gerekirse, $a \geq 0$ sabit değeri tasarruf miktarını, $b \geq 0$ sabit değeri yatırım başına düşen maliyeti ve $c \geq 0$ sabit değeri ise ticari piyasalarda talebin fiyat esnekliğidir. Yatırım piyasasındaki, yatırım ve tasarruf arasındaki fazlalık ve fiyatlardaki değişiklik faiz oranlarında önemli değişikliklere

sebep olmaktadır. Bu durumu (3a) denklemi ifade etmektedir. (3b) denklemi ise y değişkenindeki herhangi bir değişim oranının yatırım maliyeti ile faiz oranı ile ilgili olduğunu söyler. Son olarak, fiyat endeksinin enflasyon oranlarından etkilenmesinden hareketle (3c) denklemi formülize edilmiştir. Bu çalışmanın esas amacı ise (3) sistemini esas alarak yeni bir sistem oluşturup, yeni sistemin stabilite analizini ve Hopf çatallanmasını araştırmaktır.

Bu hedef doğrultusunda tez çalışmasında işlenen konular aşağıda belirtildiği gibidir.

Bölüm 1'de öncelikle bu tez çalışmasında araştırma yapılacak sistemin nasıl oluşturulduğundan bahsedildi. Sistemin oluşturulmasında literatürdeki iki model ele alınmıştır. Sistemlerden biri, kaotik davranış gösteren (3) denkleminin yatırım talebini ifade eden y değişkeninin denklemine zaman gecikme geribildirimi eklenmesi ile

$$\dot{x} = z(t) + [y(t) - a]x(t), \tag{4a}$$
$$\dot{y} = 1 - by(t) - x^2(t) + K[y(t) - y(t-\tau)], \tag{4b}$$
$$\dot{z} = -x(t) - cz(t) \tag{4c}$$

şeklinde ifade edilen diferansiyel denklem sistemidir. Sistemde $\tau \geq 0$ zaman gecikmesini, K ise geri bildirim gücünü temsil etmektedir. Bu sistem parametrelere bağlı olarak bir ya da üç denge noktasına sahiptir. Sistemde bir denge noktasında stabilite analizi uygulanmış ve Hopf dallanması saptanmıştır, bu kritik değerde y değişkeni periyodik davranış göstermektedir. Sistemdeki a, b, c sabit değerlerine uygun değerler verildiğinde ve τ_0 kritik zaman gecikme değeri olarak alındığında, sistem $\tau \in [0, \tau_0)$ değerleri için stabil davranırken $\tau = \tau_0$ değerinde Hopf dallanması meydana gelmektedir.

Modelimizi inşa ederken esas aldığımız diğer denklem sistemi ise

$$\dot{x} = z + (y-a)x + u, \tag{5a}$$
$$\dot{y} = 1 - by - x^2, \tag{5b}$$
$$\dot{z} = -x - cz, \tag{5c}$$
$$\dot{u} = -dxy - ku \tag{5d}$$

şeklindedir. Bu sistemde faiz oranı sadece yatırım talebi ve fiyat endeksine bağlı olmayıp ortalama kar marjına da bağlıdır. Ayrıca ortalama kar marjı ile faiz oranı doğru orantılıdır. Bu sistem ise (3) sistemine ortalama kar marjını ifade eden u yeni durum değişkeni eklenmesi ile elde edilmiştir. Parametrelerin bazı değer aralıkları için, (3) sistemi bir pozitif Lyapunov üsteline sahipken, (5) sistemi iki pozitif Lyapunov üsteline sahiptir. Dolayısıyla (3) sistemi kaotik bir yapıya sahipken (5) sistemi hiperkaotik bir davranış sergilemektedir.

Bu sistemler ve çalışmamızda esas aldığımız dinamik finansal sistemimiz hakkında bilgiler verilmiştir. Daha sonrasında bu sistemlere paralel olarak literatürdeki diğer sistemler incelenmiştir.

Bölüm 2'de ise lineer ve nonlineer dinamik sistemler, stabilite analizi, linearizasyon ve Hopf dallanması koşulları hakkında bilgiler verilmiştir. Lineer olmayan zaman gecikmeli diferansiyel denklemler hakkında da bilgi verilip, örneklerle anlatım yapılmıştır.

Bölüm 3'te ise sistem (4) ve sistem (5)'in birleştirilmesi ile oluşturulan yeni dinamik finans sistemimiz;

$$\dot{x} = z(t) + [y(t) - a]x(t) + u(t), \tag{6a}$$
$$\dot{y} = 1 - by(t) - x^2(t) + K[y(t) - y(t-\tau)], \tag{6b}$$
$$\dot{z} = -x(t) - cz(t), \tag{6c}$$
$$\dot{u} = -dx(t)y(t) - ku(t) \tag{6d}$$

şeklinde ifade edilmiştir. K geri bildirim gücünü ifade etmekte ve $K, \tau \geq 0$ olup, a, b, c, d, k yine sistemin negatif olmayan parametreleridir.

Tez çalışmasının konusunu, yukarıda anlatılanlar ışığında, şu soruların cevaplanması oluşturmaktadır:

- Sistem (6) denge noktaları civarında nasıl bir davranış gösterir?
- Bu sistemin stabilite analizi yapıldığında Hopf çatallanması meydana gelir mi?
- Sistemde çatallanmaya sebep olan kritik τ_0 değerini analitik olarak hesaplayabilir miyiz?
- (5) sistemine zaman gecikme teriminin eklenmesinin denklemin stabilitesi üzerindeki etkisi nedir?

Sonuç olarak, bu tez çalışmasında ek değişkenler ve zaman gecikme terimlerinin (3) sistemine etkilerini hesaba katarak, (4) ve (5) sistemlerinin birleştirilmesiyle dinamik finans sistemi (6) incelenmiştir. Yeni kurulan bu dinamik sistemin stabilite analizi yapıldıktan sonra sistemde Hopf çatallanmasının meydana geldiği hem analitik olarak hem de sayısal simülasyonlarla gösterilmiştir.

1. INTRODUCTION

The aim of this thesis work is to investigate the qualitative behaviour of a financial dynamical system which contains a time delay. We investigate the dynamic response of this system of which variables are interest rate, investment demand, price index and average profit margin. We perform a stability analysis at the fixed points and show that the system undergoes a Hopf bifurcation. The bifurcation analyses are supported by numerical simulations.

Jun-Hai and Yu-Shu [1] states that chaotic behaviour is the inherent randomness in a given system. Internal properties of the system cause that uncertainty, not the external disturbances. This makes chaos theory "attractive", as *the complicated things can be interpreted as the internal behaviours in themselves with a certain structure and aims, but not as the external and accidental behaviour* [1]. Also, it is the more harder to predict the behavior of the system when the inherent randomness is irregular.

Chaos theory has had an important place in economics besides nature and engineering fields. In economic field, chaos occurs during economic crisis; for instance, as in the USA mortgage crisis. When this crisis happened in the USA in 2007, the chaos had started at the financial world. The interplay between the dynamical systems theory and economic and financial science has been a major subject of research both for the mathematicians and experts of economic fields in the past decades and to date.

In dynamical systems literature, there are lots of mathematical models related to finance theory. We would like to mention first the financial dynamical system

$$\dot{x} = z + (y - a)x, \tag{1.1a}$$

$$\dot{y} = 1 - by - x^2, \tag{1.1b}$$

$$\dot{z} = -x - cz. \tag{1.1c}$$

Refs. [1], [2], [3], [4], [5], [6] said that this financial dynamic model is formed of four sub-blocks: production, money, stock and labor force, and can be written as three first order differential equations. Three state variables of the system denote interest rate

x, the investment demand y, and the price index z. To mention the constants, $a \geq 0$ is the saving amount, $b \geq 0$ is the cost per investment, and $c \geq 0$ is the elasticity of demand of commercial markets. Two factors cause the major changes in the interest rate x: one of them is contradiction from the investment market, which is the surplus between investment and savings, and the other one is structural adjustment from the prices. This is expressed in (1.1a). The rate of change of y is related with the cost of investment and the interest rate as given in (1.1b). Change in z is affected by inflation rates, therefore, at the same time, it can be expressed by the nominal interest rate and real interest rate, which is formulated in (1.1c) [1].

The model we will be interested in is based on two existing models. When we focus on Ref. [6], we see that besides exhibiting the chaotic behaviour of the model (1.1) for some ranges of the parameters, by calculating the Lyapunov exponents; they also consider the case where there is a time delay feedback in the investment demand;

$$\dot{x} = z(t) + [y(t) - a]x(t), \tag{1.2a}$$

$$\dot{y} = 1 - by(t) - x^2(t) + K[y(t) - y(t-\tau)], \tag{1.2b}$$

$$\dot{z} = -x(t) - cz(t). \tag{1.2c}$$

Here $\tau \geq 0$ is the time delay and K stands for the strength of the feedback. Depending on the parameters, the system may have one or three equilibrium points. They perform the stability analysis of the system in the single equilibrium case and occurrence of a Hopf bifurcation in which the variable y experiences periodic behavior is exhibited. If the constants a, b, c of the system satisfy certain conditions, the authors show that the system is stable for $\tau \in [0, \tau_0)$, where τ_0 is a critical value of the time delay, and the system undergoes a Hopf bifurcation when $\tau = \tau_0$.

The other model that we build our main problem upon is the hyperchaotic system of Ref. [7] which they formulate as

$$\dot{x} = z + (y - a)x + u, \tag{1.3a}$$

$$\dot{y} = 1 - by - x^2, \tag{1.3b}$$

$$\dot{z} = -x - cz, \tag{1.3c}$$

$$\dot{u} = -dxy - ku. \tag{1.3d}$$

Basically, the authors of [7] state that the factors related to interest rate are relevant not only to investment demand and price index but also to the average profit margin: average profit margin and interest rate are proportional. By adding average profit margin as a new state variable u to the system (1.1), they obtain the system (1.3). This newly constructed system has an interesting property: While the system (1.1) has one positive Lyapunov exponent for some range of the parameters, a sign for intrinsic chaotic behaviour, (1.3) is shown in [7] to possess two positive Lyapunov exponents for some region of the parameter space, which is defined in literature as a signal to hyperchaotic behaviour.

Motivated by the two works above, we consider the following system,

$$\dot{x} = z(t) + [y(t) - a]x(t) + u(t), \tag{1.4a}$$

$$\dot{y} = 1 - by(t) - x^2(t) + K[y(t) - y(t-\tau)], \tag{1.4b}$$

$$\dot{z} = -x(t) - cz(t), \tag{1.4c}$$

$$\dot{u} = -dx(t)y(t) - ku(t), \tag{1.4d}$$

which is a combination of (1.2) and (1.3), taking into account a time-delayed feedback in the investment demand variable y and the effect of average profit margin simultaneously in (1.1). Here K is the feedback strength, $K, \tau \geq 0$, and also a, b, c, d, k are the nonnegative parameters of the system (1.4).

To our knowledge, the existing literature does not consider the system (1.4) and in our analysis, we would like to answer the following questions:

(Q1) How is the qualitative behaviour of the system (1.4) around its fixed points?

(Q2) When we follow the route in [6] and do the stability analysis of (1.4), does the system undergo a Hopf bifurcation?

(Q3) Can we analytically determine the critical value of τ_0 that gives the bifurcation?

(Q4) How is the effect of addition of the delay term on the stability of the system (1.3)?

Therefore, the main purpose of this thesis is to search the dynamics of the financial model (1.4) by taking into account the effects of the additional variable and delay-feedback terms in (1.1). After performing stability analysis of the constructed finance system, we theoretically demonstrate that the system undergoes a Hopf-bifurcation and this phenomenon is supported by numerical simulations. We wish that our results on controlled and delayed feedback analysis can be useful for constructing fiscal policy.

The thesis is organized as follows. In the following subsection we present a literature survey. In Section 2 we provide some basic knowledge about nonlinear stability analysis of dynamical systems, in non-delay and delayed case. Section 3 contains the main work, presenting the stability analysis and the investigation of a Hopf bifurcation for the constructed finance system at the fixed points. Bifurcation analyses are demonstrated by numerical simulations. Section 4 is devoted to concluding remarks and future discussions.

1.1 Literature Review

The motivation of the this thesis is based on the system (1.1), and, as we explained above, our model is a combination of (1.2) and (1.3). In addition to the References [1], [2], [3], [4], [5], [6] and [7], in this subsection we will present a brief literature survey on these type of systems.

Refs., [1], [2] and [8] consider the topological structure, Hopf bifurcation and the chaotic situation with different parameter combinations and the effect of any change of the parameters on the economy of the equation (1.1). Another study [5] tackles with this equation (1.1) in view of fractional nonlinear models and its aim is to consider the chaotic behavior in fractional financial systems. Also, Ref. [9] considers synchronization strategies of a three-dimensional chaotic finance system.

By doing the shift $y \to y - \frac{1}{b}$ in the equation (1.1), and adding a delay term to the first equation of the system,

$$\dot{x} = (\frac{1}{b} - a)x + z + xy + k(x(t-\tau) - x(t)), \tag{1.5a}$$

$$\dot{y} = -by - x^2, \tag{1.5b}$$

$$\dot{z} = -x - cz \tag{1.5c}$$

is obtained, which is analyzed in [10]. Another version of this system appears in [11] as

$$\dot{x} = -a(x+y) + K(x(t) - x(t-\tau)), \tag{1.6a}$$

$$\dot{y} = -y - axz, \tag{1.6b}$$

$$\dot{z} = b + axy. \tag{1.6c}$$

In Refs. [6] and [12], time delay is added to the second equation of (1.1), the system becoming

$$\dot{x} = z + [y - a]x, \tag{1.7a}$$

$$\dot{y} = 1 - by - x^2 + K[y(t) - y(t-\tau)], \tag{1.7b}$$

$$\dot{z} = -x - cz \tag{1.7c}$$

to investigate the influence of the time delay on investment demand y. Chen's system [13] is expressed as the following

$$\dot{x} = a(y - x), \tag{1.8a}$$

$$\dot{y} = (c - a)x - xz + cy, \tag{1.8b}$$

$$\dot{z} = xy - bz. \tag{1.8c}$$

By adding a time-delayed term to the second equation of Chen system in [14], they obtain the system

$$\dot{x} = a(y - x), \tag{1.9a}$$

$$\dot{y} = (c - a)x - xz + cy + K(y(t) - y(t-\tau)), \tag{1.9b}$$

$$\dot{z} = xy - bz. \tag{1.9c}$$

In [14], they study both the effect of the delayed feedback on Chen's system and the existence of a Hopf bifurcation.

Another delayed financial model is handled as follows in [15],

$$\dot{x} = (y - a)x + z(t-\tau), \tag{1.10a}$$

$$\dot{y} = 1 - by - x^2, \tag{1.10b}$$

$$\dot{z} = -x - cz \tag{1.10c}$$

where τ represents price change delay.

In the studies [4] and [16], the authors construct the delayed financial system as follows

$$\dot{x} = z + (y-a)x + k_1\{x - x(t-\tau_1)\}, \tag{1.11a}$$

$$\dot{y} = 1 - by - x^2 + k_2\{y - y(t-\tau_2)\}, \tag{1.11b}$$

$$\dot{z} = -x - cz + k_3\{z - z(t-\tau_3)\} \tag{1.11c}$$

where τ_1, τ_2, and τ_3 are time delays and k_1, k_2, and k_3 demonstrate the strengths of the feedbacks. The aim is to investigate the effect of delayed feedbacks on the financial system with time delay terms on the interest rate, the investment demand and the price index of the financial system.

Another system in [17] is constructed by adding the fourth variable ω to an autonomous chaotic system which is proposed by Qi [18]

$$\dot{x} = a(y-x) + eyz - k\omega, \tag{1.12a}$$

$$\dot{y} = cx - dy - xz, \tag{1.12b}$$

$$\dot{z} = xy - bz, \tag{1.12c}$$

$$\dot{\omega} = rx + fyz \tag{1.12d}$$

and the new system has chaotic or hyperchaotic behavior with wide frequency bandwith with suitable parameters.

2. NONLINEAR DYNAMICAL SYSTEMS AND STABILITY

Considering what qualitative information can be obtained about the solutions of the differential equations without solving directly them is so important because many differential equations cannot be solved easily by analytical methods. In this section, we present the essential basics for the stability analysis of nonlinear systems. We start by summarizing the necessary concepts for the linear systems which are related to the notion of stability of solutions.

2.1 Linear Systems

n-dimensional systems of the first order linear homogenous equations has the form

$$\dot{x} = Ax \tag{2.1}$$

where $x \in R^n$ and A is a $n \times n$ matrix. The solution of the linear system (2.1) is proposed in the form $x = \xi e^{rt}$ and when substituted into the equation (2.1), one gets

$$(A - rI)\xi = 0 \tag{2.2}$$

revealing that ξ is an eigenvector of the coefficient matrix A and r is an eigenvalue. The eigenvalues are obtained by solving the roots of polynomial equation

$$\det(A - rI) = 0, \tag{2.3}$$

and the eigenvectors are acquired by substituting the eigenvalues to the equation (2.2). If we develop geometrical interpretations of the solutions to (2.1), we can build up a connection between the geometrical considerations and general solutions for linear systems. Also, this information helps to understand the more complicated nonlinear systems. A **critical point** is the equilibrium solution which is obtained by solving $Ax = 0$ of the equation (2.1). A vector function $x = \phi(t) \in R^n$ which is the solution of equation (2.1) can be visualized as a parametric representation for a curve in the $x_1 \dots x_n$-space. We can interpret this curve as a path or **trajectory** which is traversed by a moving particle whose velocity dx/dt is specified by the differential equation.

For example, when $n = 2$, the x_1x_2-space is called the **phase plane**, the trajectory is a curve in R^2. All solution curves of the system in the phase space in R^n make up the **phase portrait** of the system. For instance, consider the simple uncoupled system

$$\dot{x} = Ax, \qquad A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \quad x \in R^2 \tag{2.4}$$

for which we simply get

$$x_1(t) = c_1 e^{-t}, \quad x_2(t) = c_2 e^{2t}, \tag{2.5}$$

or, in a different notation,

$$x(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} c \tag{2.6}$$

with $c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = x(0)$. Under the evolution of this dynamics, every point $c = x(0) \in R^2$ travels to the point $x(t) \in R^2$. Therefore, the **dynamical system** which is determined by (2.4) is just the mapping $\phi : R \times R^2 \to R^2$

$$\phi(t,c) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} c. \tag{2.7}$$

The behavior of the linear system can be determined according to the eigenvalues of the coefficient matrix A. Also, differential equations can be classified depending on their phase potraits and trajectories. For $\det A \neq 0$, the following theorem is an easy way to determine if the linear system has whether saddle, node, focus or center at the origin.

Theorem 1 *Let $\delta = \det A$ and $\tau = \operatorname{trace} A$. Consider the linear system*

$$\dot{x} = Ax, \quad x \in R^n. \tag{2.8}$$

(a) The equation (2.8) *has a saddle at the origin for $\delta < 0$.*

(b) The equation (2.8) *has a node at the origin when $\delta > 0$ and $\tau^2 - 4\delta \geq 0$; which is stable for $\tau < 0$ and unstable for $\tau > 0$.*

(c) Assume that $\delta > 0$, $\tau^2 - 4\sigma < 0$, and $\tau \neq 0$ then (2.8) *has a focus at the origin; which is stable for $\tau < 0$ and unstable for $\tau > 0$.*

(d) The equation (2.8) *has a center at the origin when $\delta > 0$ and $\tau = 0$.*

Note that in case (b), $\tau^2 \geq 4|\delta| > 0$; i.e., $\tau \neq 0$.

A stable node or focus of (2.8) is called a **sink** of the linear system and an unstable node or focus of (2.8) is called a **source** of the linear system. Additionally, for illustrating the case in R^2, when the linear system is taken as the following

$$\dot{x} = Ax, \quad x \in R^2 \tag{2.9}$$

with $\det(A - rI) = 0$ and $\det A \neq 0$, we can classify the stability properties of linear systems in R^2 in the following table.

Table 2.1 : Stability Properties of Linear Systems

Eigenvalues	Type of Critical Point	Stability
$r_1 > r_2 > 0$	Node	Unstable
$r_1 < r_2 < 0$	Node	Asimptotically stable
$r_2 < 0 < r_1$	Saddle Point	Unstable
$r_1 = r_2 > 0$	Proper or improper node	Unstable
$r_1 = r_2 < 0$	Proper or improper node	Asymptotically stable
$r_1, r_2 = \lambda \mp i\mu$	Spiral point	
$\lambda > 0$		Unstable
$\lambda < 0$		Asymptotically stable
$r_1 = i\mu, r_2 = -i\mu$	Center	Stable

2.2 Nonlinear Systems

2.2.1 Stability and Instability, Linearization

In this part, we mention the stability properties of the nonlinear systems at the equilibrium points and mathematical definitions of stability and unstability are given. Let us consider the autonomous sytem

$$\dot{x} = f(x), \quad x, f \in R^n \tag{2.10}$$

in which the function f does not depend on the time variable t explicitly. Let us mention that any non-autonomous system $\dot{x} = f(x,t)$ in R^n can be considered as an autonomous system in R^{n+1}, by setting $t = x_{n+1}$, so having $\dot{x} = f(x, x_{n+1})$ with $\dot{x}_{n+1} = 1$.

In the stability analysis of a nonlinear system of type (2.10) we first find the critical/equilibrium points of the system. These are time-independent solutions of the system, i.e., $\dot{x} = 0$, which are found by solving the (nonlinear) system $f(x) = 0$. After that we determine the behavior of (2.10) in the vicinity of its critical points. This local behavior of the non-autonomous system near a hyperbolic equilibrium point x_0 is determined by the behavior of the linear system

$$\dot{x} = Ax, \tag{2.11}$$

with the matrix $A = Df(x_0)$, x_0 denoting the equilibrium point.

Definition 1 *The solution of* $f(x_0) = 0, \quad x_0 \in R^n$ *is called an equilibrium point or critical point of* (2.10). *The point* x_0 *is a* ***hyperbolic point*** *if and only if* $Re(\lambda) \neq 0$ *for* $\forall\lambda \in Df(x_0)$.

If x_0 is a critical point of (2.10), then $f(x_0) = 0$ and, by Taylor's Theorem,

$$f(x) = Df(x_0)x + \frac{1}{2}D^2 f(x_0)(x,x) + \ldots. \tag{2.12}$$

The linear function $Df(x_0)x$ is the first approximation to the nonlinear function $f(x)$ in the neighborhood of $x = x_0$. The behavior of the nonlinear system (2.10) in some neighbourhood of the equilibrium point x_0 will be determined by the behavior of its linearization at $x = x_0$ approximately.
For $\forall t \in R,$ assume that ϕ_t is the flow of the equation (2.10). If for all $\varepsilon > 0$ there exists a $\sigma > 0$ such that for all $x \in N_\sigma(x_0)$ and $t \geq 0$ we have

$$\phi_t(x) \in N_\varepsilon(x_0), \tag{2.13}$$

an equilibrium point x_0 is called **stable**. If the critical point is not stable, it is called **unstable**. If x_0 is stable and if there exists a $\sigma > 0$ such that $\forall x \in N_\sigma(x_0)$ we have $\lim_{t\to\infty} \phi_t(x) = x_0$, it is called **asymptotically stable**. If these mathematical definitions are thought geometrically, it means that all solutions starting sufficiently close to x_0 must stay close to x_0 in case of a stable equilibrium point. Additionally, if the paths of the solutions that start close enough to x_0 converge to x_0 in some sense while $t \to \infty$, then the equilibrium point is asymptotically stable.

Theorem 2 *If x_0 is a sink of the nonlinear system* (2.10) *and $Re(\lambda_j) < -a < 0$ for all of the eigenvalues $Df(x_0)$, then given $\varepsilon > 0$ there exists a $\sigma > 0$ such that for all $x \in N_\sigma(x_0)$, the flow $\phi_t(x)$ of* (2.10) *satisfies*

$$|\phi_t(x) - x_0| \leq \varepsilon e^{-at} \tag{2.14}$$

for all $t \leq 0$.

The hyperbolic equilibrium points are stable, asymptotically stable or unstable. If one of the eigenvalues is equal to 0 or pure imaginary like $\lambda = \pm ib$, x_0 can be stable but not asymptotically stable.

Theorem 3 *If x_0 is a stable equilibrium point of* (2.10)*, no eigenvalue of $Df(x_0)$ has positive real part.*

2.2.2 Stable, Unstable and Center Manifolds for Nonlinear Systems

Let us remember that, given the nonlinear equation

$$\dot{x} = f(x), \quad x \in R^n, \tag{2.15}$$

in order to obtain information about the nature of solutions around an equilibrium point $x = x_0$, we analyze the linear system

$$\dot{y} = Ay, \quad y \in R^n, \tag{2.16}$$

where $A = Df(x_0)$. At this point, the stable, unstable and center manifold theorem will help us to get information about the solutions of the nonlinear system. For that theory, we need to transform (2.15) another form. First we shift the fixed point $x = x_0$ of (2.15) to the origin by the transformation $y = x - x_0$. Then (2.15) takes the form

$$\dot{y} = f(x_0 + y), \quad y \in R^n. \tag{2.17}$$

After expanding $f(x_0 + y)$ to Taylor series at $x = x_0 \leftrightarrow y = 0$, we get

$$\dot{y} = Df(x_0)y + R(y), \quad y \in R^n, \tag{2.18}$$

where $R(y) = O(|y|^2)$, using the fact that $f(x_0) = 0$. According to elementary linear algebra theory [Hirsch and Smale 1974], there exists a linear transformation T that transforms the linear equation (2.16) into block diagonal form

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} A_s & 0 & 0 \\ 0 & A_u & 0 \\ 0 & 0 & A_c \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \tag{2.19}$$

where $T^{-1}y \equiv (u,v,w) \in R^s \times R^u \times R^c, \quad s+u+c=n,$ A_s is an $s \times s$ matrix having eigenvalues with negative real parts, A_u is an $u \times u$ matrix having eigenvalues with positive real parts and A_c is an $c \times c$ matrix having eigenvalues zero real parts. It follows that equation (2.19) is a linear vector field which has a s-dimensional invariant stable manifold, a u-dimensional invariant unstable manifold and a c-dimensional invariant center manifold at the origin. If this linear transformation is used in (2.18), one gets

$$\dot{u} = A_s u + R_s(u,v,w), \tag{2.20a}$$

$$\dot{v} = A_u v + R_u(u,v,w), \tag{2.20b}$$

$$\dot{w} = A_c w + R_c(u,v,w). \tag{2.20c}$$

This non-linear vector field will be analyzed at the next theorem.

Theorem 4 *Suppose* (2.20) *is* C^r, $r \geq 2$. *The equation* (2.20) *has a* C^r *s-dimensional invariant, local stable manifold denoted as* $W^s_{loc}(0)$, *a* C^r *u-dimensional invariant, local unstable manifold denoted as* $W^u_{loc}(0)$ *and a* C^r *c-dimensional invariant, local center manifold* $W^c_{loc}(0)$ *at the fixed point* $(u,v,w)=0$, *all of which intersect at* $(u,v,w)=0$. *This situation means geometrically that these manifolds are tangent to the respective invariant subspaces of the linear vector field* (2.19) *at the origin. Then, these manifolds are locally demonstrated as graphs. Especially, we have*

$$\begin{aligned} W^s_{loc}(o) = \{&(u,v,w) \in R^s \times R^u \times R^c | v = h^s_v(u), w = h^s_w(u); \\ &Dh^s_v(0) = 0,\ Dh^s_w(0) = 0; \quad |u| \ sufficiently\ small\} \end{aligned} \tag{2.21a}$$

$$\begin{aligned} W^u_{loc}(0) = \{&(u,v,w) \in R^s \times R^u \times R^c | u = h^u_u(v), w = h^u_w(v); \\ &Dh^u_u(0) = 0,\ Dh^u_w(0) = 0; \quad |v| \ sufficiently\ small\} \end{aligned} \tag{2.21b}$$

$$\begin{aligned} W^c_{loc}(0) = \{&(u,v,w) \in R^s \times R^u \times R^c | u = h^c_u(w), v = h^c_v(w); \\ &Dh^c_u(0) = 0,\ Dh^c_v(0) = 0; \quad |w| \ sufficiently\ small\} \end{aligned} \tag{2.21c}$$

where $h_v^s(u)$, $h_w^s(u)$, $h_u^u(v)$, $h_w^u(v)$, $h_u^v(c)$, and $h_v^c(w)$ are C^r functions. In addition, trajectories in $W_{loc}^s(0)$ and $W_{loc}^u(0)$ have the same properties asymptotically with trajectories in E^s and E^u, respectively. That is, trajectories of (2.20) *with initial conditions in $W_{loc}^s(0)$ (resp., $W_{loc}^u(0)$) approach the origin at an exponential rate asymptotically as $\lim_{t\to+\infty}$ (resp., $t \to -\infty$).*

We need to make some explanations about the theorem. First of all, the "local" term used many times tells that manifolds are defined within some neighbourhood of the fixed point and that is why they have a boundary. Also, $Dh_v^s(0) = 0$, $Dh_w^s(0) = 0$, etc. means that nonlinear manifolds are tangent to the related linear manifolds at the origin.

2.2.3 Hopf Bifurcation

What do we understand exactly by the term of bifurcation? It, generally, includes the concept of "topological equivalence". When the parameters are varied, the phase portrait changes its topological structure and we say that bifurcation has occured. Hopf bifurcation is a critical point where system's stability switches and a periodic solution arises [19].

Assume that a two dimensional system has a stable fixed point. When does it lose its stability as a parameter μ varies? The eigenvalues of the Jacobian matrix are the key. If the equilibrium point is stable, λ_1 and λ_2 both lie in the left half-plane $Re(\lambda) < 0$. There are two possibilities: either the eigenvalues are both real and negative or complex conjugates with negative real parts. To make the fixed point unstable, it is necessary that one or both of the eigenvalues passes to the right half- plane as μ varies.

Hopf bifurcation is defined according to the properties of the eigenvalues as follows.

Consider a system

$$\dot{x} = f_\mu(x), \quad x \in R^n, \quad \mu \in R \tag{2.22}$$

where μ is a parameter. Suppose the system has an equilibrium (x_0, μ_0), and $f \in C^\infty$ [20], [21].

Assume that

- The Jacobian matrix $D_x f_{\mu_0}(x_0)$ has a simple pair of purely imaginary eigenvalues and other eigenvalues have negative real parts.

Then there is a smooth curve of equilibria $(x(\mu),\mu)$ with $x(\mu_0) = x_0$. The eigenvalues $\lambda(\mu),\bar{\lambda}(\mu)$ of $J(\mu) = D_x f_\mu(x(\mu))$ which are purely imaginary at $\mu = \mu_0$ vary smoothly with μ. Moreover, if

- $\frac{d(Re(\lambda(\mu)))}{d\mu}|_{\mu=\mu_0} \neq 0$

then there is a Hopf bifurcation. Whether it is supercritical, subcritical, or degenerate depends on the higher order terms in system (2.22).

2.3 Nonlinear Delay Differential Equations, Linear Stability Analysis and Hopf Bifurcation

A general delay differential equation (DDE) with a single constant delay term can be shown as the following

$$\dot{X} = F(t,X(t),X(t-\tau)) \tag{2.23}$$

with $X,F \in \mathbb{R}^n$. In specific cases, depending on the value of delay, nonlinearity, the number of dynamical variables etc., analytical solutions of such a system can be determined. Of course, in many cases, numerical investigations is inevitable to search for solutions of this system.

The information about the solution of a DDE can be obtained via local stability analysis at the critical point. The stability of an equilibrium point of a DDE is decided by examining whether the near trajectories are getting close to or going far away from the equilibrium point.

In analogy with the non-delay systems, a fixed/critical/equilibrium point X^* of (2.23) is a solution that does not change in time, therefore for which we have $X(t) = X(t-\tau) = X^*$ for every t. Then, X^* is found by solving

$$f(t,X(t) = X^*,X(t-\tau) = X^*) = 0, \qquad X^* = (x_1^*,x_2^*,...,x_n^*)^T. \tag{2.24}$$

To investigate the stability of the equilibrium point, we perturb it in usual way by infinitesimally varying the solution in the vicinity of the critical point X^* by a time dependent function $\delta X(t)$, existing over an interval of at least the values of the longest delay, τ_{max}, in the case of multiple delays. Restricting to the autonomous case and denoting $X = X(t)$ and $X_\tau = X(t-\tau)$, we have

$$X = X^* + \delta X, \quad X_\tau = X^* + \delta X_\tau. \tag{2.25}$$

Then

$$\dot{X} = \delta\dot{X} = f(X^* + \delta X, X^* + \delta X_\tau), \tag{2.26}$$

where δX's are the infinitesimal displacements from the equilibrium point over the interval $(t_0 - \tau, t_0)$. Eq. (2.26) can be linearized at the equilibrium point as

$$\delta\dot{X} = J_0\,\delta X + J_\tau\,\delta X_\tau, \tag{2.27}$$

$$(J_0)_{i,j} = \left(\frac{\partial F_i}{\partial x_j}\right)|_{x_j = x_j^*} \quad for \quad i,j = 1,2,...,n, \tag{2.28}$$

$$(J_\tau)_{i,j} = \left(\frac{\partial F_i}{\partial x_{\tau j}}\right)|_{x_{\tau j} = x_j^*} \quad for \quad i,j = 1,2,...,n, \tag{2.29}$$

using the Taylor series expansion, J_0 being the Jacobian matrix with respect to X whereas J_τ represents the Jacobian matrix with respect to X_τ, which are both evaluated at $X = X_\tau = X^*$. Assume that $\delta X(t)$ is solved by exponential functions of time along with the exponents given by the eigenvalue of the corresponding Jacobian matrix,

$$\delta X(t) = Ae^{\lambda t} \tag{2.30}$$

where A is a constant column matrix. Substituting the equation (2.30) into the equation (2.27) and collecting the coefficients of $e^{\lambda t}$, one obtains the matrix equation

$$\lambda A = (J_0 + e^{-\lambda\tau} J_\tau)A. \tag{2.31}$$

This equation obviously can be satisfied with a nonzero vector A if

$$|J_0 + e^{-\lambda\tau} J_\tau - \lambda I| = 0, \tag{2.32}$$

where I is the identity matrix.

The characteristic equation (2.32) of a delay-differential equation system consists of two parts: The same terms in a characteristic equation of a non-delay system and the transcendental part originating from the delay terms containing τ. As we know if all eigenvalues of the characteristic equation are negative, it is a stable fixed point. When one of the eigenvalues is with a positive real part, the equilibrium point is unstable. If one of the eigenvalues is zero, the stability is undetermined, so to decide the stability of the equilibrium point we need to take into account the higher order terms in Taylor expansion (2.26). In case of degree n polynomial characteristic equations, there are exactly n roots (counting the repeated ones, too). Characteristic equations that include both transcendental terms and polynomials may have infinite number of roots in the complex plane. Therefore, to find these roots is a difficult task. However, to determine the stability of the equilibrium points, in the case of transcendental characteristic equations, some methodologies have been developed.

2.3.1 Example 1

Let us the illustrate the above topic with a basic example in a DDE of the form

$$\dot{x} = -bx(t) + af(x(t-\tau)), \tag{2.33}$$

where a and b are positive parameters and f is a nonlinear function. For the simple functional form

$$f(x(t-\tau)) = -x(t-\tau), \tag{2.34}$$

the general DDE (2.33) becomes the linear DDE (LDDE),

$$\dot{x} = -bx(t) - ax(t-\tau). \tag{2.35}$$

The corresponding characteristic equation of the Eq. (2.35) is as follows;

$$\lambda + b + ae^{-\lambda\tau} = 0. \tag{2.36}$$

Let $\lambda = \alpha + i\beta$ be the eigenvalue associated with the equilibrium point and the critical stability curve is the ones on which $\alpha = 0$ as one can expect that there is a change in stability when the value of α crosses the imaginary axis at $\lambda = i\beta$. Substituting $\lambda = i\beta$ to the equation (2.36) and after simple algebra, it is obtained that;

$$\beta = \pm\sqrt{a^2 - b^2} \tag{2.37}$$

and

$$\beta\tau = \arccos(-\frac{b}{a}) + 2n\pi \tag{2.38}$$

where n is any integer $(0, \pm 1, \pm 2, \ldots)$. As a result, we find that

$$\tau(n) = \frac{2n\pi + \arccos(-\frac{b}{a})}{\sqrt{a^2 - b^2}}. \tag{2.39}$$

To determine those curves for $\tau > 0$ which encompass the stable regions, the critical curves should be the ones on which $\frac{d\lambda}{d\tau} > 0$.

The above characteristic equation (2.36) is evaluated by the derivative of the equation with respect to τ

$$\frac{d\lambda}{d\tau} - \lambda a e^{-\lambda\tau} - \lambda a e^{-\lambda\tau}\frac{d\lambda}{d\tau} = 0. \tag{2.40}$$

After arranging (2.40), one gets

$$\frac{d\lambda}{d\tau} = \frac{-\lambda(\lambda+b)}{1+\tau(\lambda+b)}. \tag{2.41}$$

From the above characteristic equation (2.36), evaluate

$$Re(\frac{d\lambda}{d\tau})|_{\alpha=0} = \beta^2 D^{-1}, \tag{2.42}$$

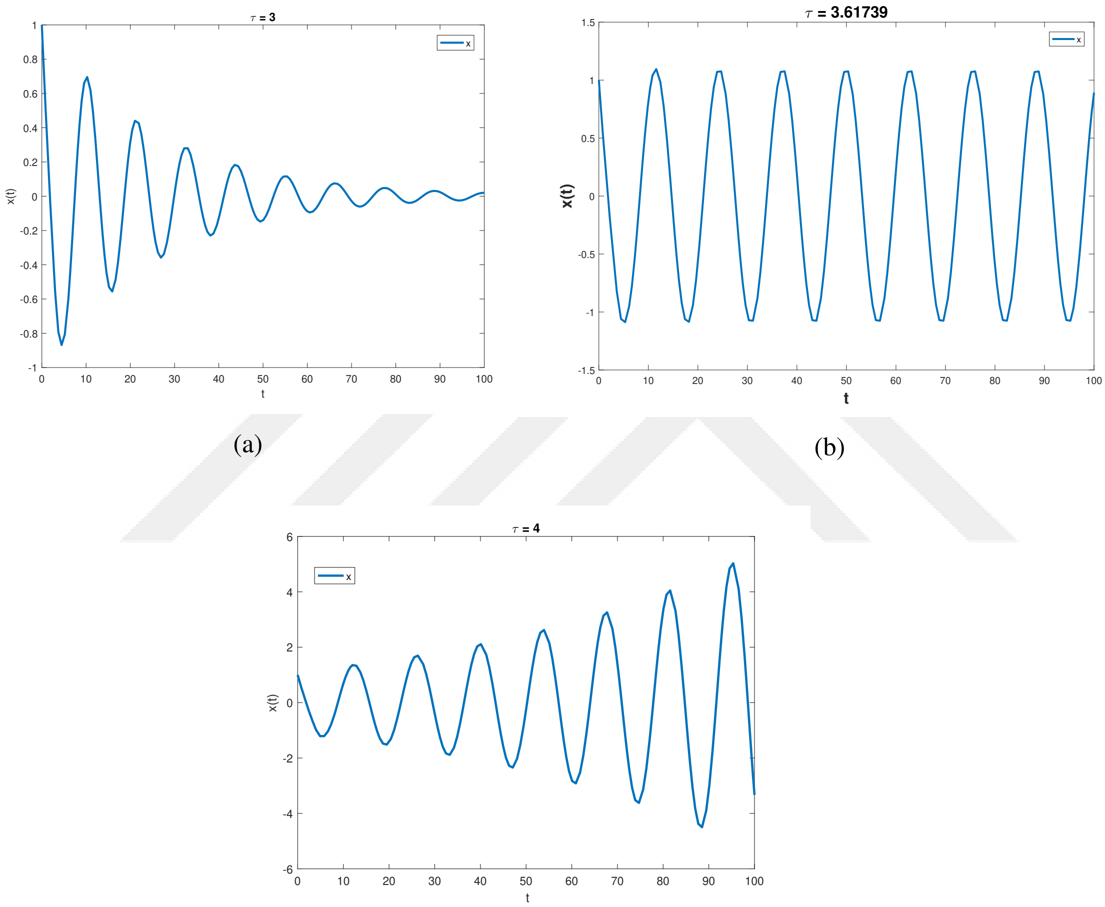
where

$$D = (1+\tau b)^2 + \tau^2\beta^2. \tag{2.43}$$

Therefore

$$\frac{d\alpha}{d\tau} > 0. \tag{2.44}$$

Let us show the existence of a Hopf bifurcation by numerical simulations of the LDDE (2.36). To give an example, the values of the parameters are choosen as $a = 0.5$, $b = 0.1$ and the existence of a bifurcation analysis is found out as a function of the delay time τ. The origin is the equilibrium point for the LDDE (2.36), so there exists a stable equilibrium point up to the value of delay time $\tau < 3.61739$ according to the given values of the parameters. In Figure 2.1a, the solution of the equation exhibits a damped oscillatory decay to the equilibrium point for $\tau = 3.0$. Although, the value of τ reachs to the critical value, $\tau = 3.61739$, the LDDE exhibits only periodic oscillations (Hopf oscillations) as shown in Figure 2.1b and this is called Hopf bifurcation curve. For $\tau = 4.0$, the LDDE exhibits undamped growing oscillations as depicted in Figure 2.1c. Thus the change from stability to instability across the critical τ value confirms the existence of a Hopf bifurcation.



(c)

Figure 2.1 : Numerical simulations of the system (2.33) for different τ values (a) $\tau = 3.0$, (b) $\tau = 3.61739$, (c) $\tau = 4.0$. Here we take the parameter values as $a = 0.5$, $b = 0.1$. Also, the initial condition is taken as $x(0) = 1$.

3. ANALYSIS AND NUMERICAL SIMULATIONS OF THE MODEL

In this section, we will analyze the new financial model which we construct as follows

$$\dot{x} = z(t) + [y(t) - a]x(t) + u(t), \tag{3.1a}$$

$$\dot{y} = 1 - by(t) - x^2(t) + K[y(t) - y(t-\tau)], \tag{3.1b}$$

$$\dot{z} = -x(t) - cz(t), \tag{3.1c}$$

$$\dot{u} = -dx(t)y(t) - ku(t). \tag{3.1d}$$

The model with the time delay term will be investigated for stability conditions at the equilibrium points and the existence of a Hopf bifurcation will be shown. Also, the results will be supported by numerical simulations.

3.1 Stability of Equilibrium Points and Hopf Bifurcation

By solving the equations

$$z^* + [y^* - a]x^* + u^* \;=\; 0, \tag{3.2a}$$

$$1 - by^* - x^{*2} + K[y^*(t) - y^*(t-\tau)] \;=\; 0, \tag{3.2b}$$

$$-x^* - cz^* \;=\; 0, \tag{3.2c}$$

$$-dx^*y^* - ku^* \;=\; 0, \tag{3.2d}$$

and taking into account that $y^* = y^*(t) = y^*(t-\tau)$, we find the equilibrium points of the system. The results are given in the following Lemmas.

Lemma 1 *In the case*

$$\frac{kb + abck + cd - ck}{c(d-k)} \leq 0, \tag{3.3}$$

the system has a unique equilibrium,

$$P_0\left(0, \frac{1}{b}, 0, 0\right). \tag{3.4}$$

Lemma 2 *If the parameters of the system satisfy*

$$\frac{kb+abck+cd-ck}{c(d-k)} > 0, \tag{3.5}$$

the system has three equilibrium points;

$$P_0\left(0,\frac{1}{b},0,0\right) \tag{3.6}$$

and

$$P_{1,2}\left(\mp\sqrt{\frac{kb+abck}{c(d-k)}+1},\frac{(1+ca)k}{c(k-d)},\pm\frac{1}{c}\sqrt{\frac{kb+abck}{c(d-k)}+1},\mp\frac{d(1+ca)}{c(d-k)}\sqrt{\frac{kb+abck}{c(d-k)}+1}\right). \tag{3.7}$$

By the change of variables

$$X=x,\quad Y=y-\frac{1}{b},\quad Z=z,\quad U=u \tag{3.8}$$

the equilibrium point P_0 is shifted to

$$P_0\,(0,0,0,0). \tag{3.9}$$

After this transformation, the new system can be arranged as;

$$\dot{X}=Z(t)+[Y(t)+\frac{1}{b}-a]X(t)+U(t), \tag{3.10a}$$

$$\dot{Y}=-bY(t)-X^2(t)+K[Y(t)-Y(t-\tau)], \tag{3.10b}$$

$$\dot{Z}=-X(t)-cZ(t), \tag{3.10c}$$

$$\dot{U}=-dX(t)[Y(t)+\frac{1}{b}]-kU(t). \tag{3.10d}$$

3.1.1 Stability Analysis and Hopf Bifurcation for P_0

We work with the system (3.10), for which the equlibrium point is the origin. Remember that the characteristic equation for a delayed system is

$$|J_0+e^{-\lambda\tau}J_\tau-\lambda I|=0. \tag{3.11}$$

At the equilibrium point $P_0(0,0,0,0)$, we find J_0 and J_τ for Eq. (3.10) to be

$$J_{0,0}=\begin{pmatrix} \frac{1}{b}-a & 0 & 1 & 1 \\ 0 & -b+K & 0 & 0 \\ -1 & 0 & -c & 0 \\ -\frac{d}{b} & 0 & 0 & -k \end{pmatrix} \tag{3.12}$$

and

$$J_{\tau,0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -K & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.13}$$

Solving (3.11), the characteristic equation is obtained. The characteristic equation which is a fourth degree exponential polynomial equation is as follows

$$\left[\lambda - (-b + K - Ke^{-\lambda\tau})\right](\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3) = 0 \tag{3.14}$$

where

$$p_1 = k + a + c - \tfrac{1}{b}, \tag{3.15a}$$

$$p_2 = ck + ak + ac - \tfrac{k+c-d}{b} + 1, \tag{3.15b}$$

$$p_3 = (1 + ac - \tfrac{c}{b})k + \tfrac{cd}{b}. \tag{3.15c}$$

Now we can state the following result (one can look at Refs. [6], [22]).

Lemma 3 *According to the Routh-Hurwitz criterion, when the conditions*

$$p_1 > 0, \quad p_3 > 0, \quad p_1p_2 > p_3 \tag{3.16}$$

hold, the three roots of the characteristic equation (3.14) *originating from the algebraic term in the second paranthesis, have negative real parts; i.e., the roots are on the left half plane, for all* $\tau \geq 0$.

Remark 1 *When* $\tau = 0$*, the transcendental part of* (3.14) *reduces to*

$$\lambda + b = 0, \tag{3.17}$$

b represents the cost per investment. When $b > 0$*, the root of the equation* (3.17) *is negative. Therefore, in case* $\tau = 0$*, when the conditions of Lemma 3 are met, the system stable at the equilibrium* P_0.

Let us underline that although the system (3.1) *is analyzed in [7], they do not provide any stability criterion as we state here.*

See that Lemma 3 guarantees, for all values of $\tau \geq 0$, the eigenvalues originating from the algebraic part of (3.14) have negative real parts when the conditions in (3.16) are satisfied. Therefore stability is at our disposal for this part of the characteristic equation. Then we just need to analyze the root of the transcendental part of (3.14), which is

$$\lambda + b - K + Ke^{-\lambda\tau} = 0. \tag{3.18}$$

The machinery in the rest of this subsection shares the same calculations and lines with the computations in [6]. We assume that the root of the transcendental equation is of the form

$$\lambda(\tau) = \alpha(\tau) + \beta(\tau)i. \tag{3.19}$$

Suppose that for some critical value $\tau = \tau^*$, we have $\alpha(\tau^*) = 0$ and $\beta(\tau^*) \neq 0$. Then the system (3.1) undergoes a Hopf bifurcation at the origin provided the transversality condition is satisfied. It is clear that, if $\lambda = i\omega$ $(\omega > 0)$ is a root of (3.18), it must satisfy

$$i\omega + b - K + K(\cos\omega\tau - i\sin\omega\tau) = 0. \tag{3.20}$$

If the imaginary and real parts are separated, we obtain;

$$b - K + K\cos\omega\tau \quad = 0, \tag{3.21a}$$

$$\omega - K\sin\omega\tau \qquad = 0. \tag{3.21b}$$

Eliminating the trigonometric terms, one obtains

$$\omega^2 = 2Kb - b^2 \tag{3.22}$$

and hence

$$\omega_+ = \sqrt{2Kb - b^2}. \tag{3.23}$$

It is clear that
if $K > b/2$, $\quad \omega > 0$ is determined uniquely,
if $K \leq b/2$, $\quad$ there is no real ω.
Using (3.23) in (3.21), we get

$$\tau_j = \frac{1}{\omega_+}\arccos\frac{K-b}{K} + \frac{2j\pi}{\omega_+}, \quad j = 0, 1, 2, \ldots \tag{3.24}$$

We prove that $\tau = \tau_j$, $\lambda(\tau_j) = i\omega_+$ is a pure imaginary root of the transcendental equation (3.18).

In order to show the existence of the Hopf bifurcation, we need to check the transversality condition. This is done in the following Lemma.

Lemma 4 *$\lambda(\tau_j)$ satisfies the transversality condition; that is,* $\dfrac{dRe(\lambda(\tau_j))}{d\tau} > 0$ *for* $j = 0, 1, 2, \ldots$.

Proof: Consider $\lambda = \lambda(\tau)$ in (3.18),

$$\lambda + b - K + Ke^{-\lambda\tau} = 0, \tag{3.25}$$

evaluate the derivative of the equation with respect to τ

$$\frac{d\lambda}{d\tau} + Ke^{-\lambda\tau}(-\frac{d\lambda}{d\tau}\tau - \lambda) = 0. \tag{3.26}$$

After arranging (3.26) one gets

$$\frac{d\lambda}{d\tau} = \frac{K\lambda e^{-\lambda\tau}}{1 - K\tau e^{-\lambda\tau}}. \tag{3.27}$$

If we substitute $\lambda = i\omega_+$, and $\tau = \tau_j$, we get

$$\frac{d\lambda}{d\tau} = \frac{K\omega_+ i}{e^{i\omega_+\tau_j} - K\tau_j}. \tag{3.28}$$

Since

$$e^{i\omega\tau} = \cos\omega\tau + i\sin\omega\tau, \tag{3.29}$$

we obtain,

$$\text{Re}\left\{\frac{d\lambda}{d\tau}\right\}_{\tau=\tau_j} = \frac{K\omega_+\sin\omega_+\tau_j}{(\cos\omega_+\tau_j - K\tau_j)^2 + (\sin\omega_+\tau_j)^2}. \tag{3.30}$$

Using

$$\omega = K\sin\omega\tau \tag{3.31}$$

from (3.21), one gets

$$\text{Re}\left\{\frac{d\lambda}{d\tau}\right\}_{\tau=\tau_j} = \frac{\omega_+^2}{(\cos\omega_+\tau_j - K\tau_j)^2 + (\sin\omega_+\tau_j)^2} > 0 \tag{3.32}$$

which proves the Lemma.

To analyze the roots of the exponential polynomial equation (3.18), the result of the following lemma which is proved by Ruan & Wei (2003) [23] is needed.

Lemma 5 *Consider the exponential polynomial equation ;*

$$\begin{aligned}
P(\lambda\tau_1, \ldots, e^{-\lambda\tau_m}) &= \lambda^n + p_1^{(0)}\lambda^{n-1} + \ldots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\
&+ [p_1^{(1)}\lambda^{n-1} + \ldots + p_{n-1}^{(1)}\lambda + p_n^{(1)}]e^{-\lambda\tau_1} + \ldots \\
&+ [p_1^{(m)}\lambda^{n-1} + \ldots + p_{n-1}^{(m)}\lambda + p_n^{(m)}]e^{-\lambda\tau_m} = 0,
\end{aligned} \tag{3.33}$$

where $\tau_i \geq 0$ $(i = 1,2,...,m)$ *and* $p_j^{(i)}$ $(i = 0,1,2,...,m; j = 1,2,...,n)$ *are constants. As* $(\tau_1, \tau_2, ..., \tau_m)$ *vary, the sum of the order of the zeros of* $P(\lambda, e^{-\lambda\tau_1}, ..., e^{-\lambda\tau_m})$ *on the open right half-plane can change only if a zero appears on or crosses the imaginary axis.*

Remark 2 *Let us summarize the steps we have gone through so far and comment on how to arrive at Theorem 5 which follows on the next page.*

- When $\tau = 0$: If condition (3.16) of Lemma 3 holds and if $b > 0$, then all the eigenvalues of the linearization of the system (3.10) have negative real parts, hence (3.10) is stable at P_0. (3.14) is an algebraic equation of degree four. Since all the eigenvalues have negative real parts, the sum of the multiplicities of zeros of (3.14) (let us call this *LS*) on the left half plane is equal to 4, write $LS = 4$. Since there is no eigenvalue with a positive real part, the sum of multiplicities of the eigenvalues on the right half plane (let us call this *RS*) is 0, write $RS = 0$.

- Lemma 5 is an only if statement, and it says, if *RS* changes, then there arises a pure imaginary root of (3.14). This is equivalent to the following statement: If (3.14) does not have any imaginary root, then *RS* does not change.

- Suppose $K > b/2$. Then, we have a pure imaginary root of the characteristic equation.

  - The smallest value of τ such that (3.14) has a pure imaginary root is $\tau = \tau_0$, where $\lambda(\tau_0) = i\omega_+$. Therefore, when $\tau \in [0, \tau_0)$, *RS* does not change and remains the same as $RS = 0$. Hence, when $\tau \in [0, \tau_0)$, there is no eigenvalue with positive real part. $\lambda = 0$ is already not an eigenvalue since we put the conditions $b > 0$ and $p_3 > 0$. Hence when $\tau \in [0, \tau_0)$, all the eigenvalues have negative real parts, and the system is stable at the equilibrium point P_0.
  - We have shown that $\mathrm{Re}\big(\lambda(\tau_0)\big) = \mathrm{Re}(i\omega_+) = 0$ and $\dfrac{d\mathrm{Re}(\lambda(\tau))}{d\tau}|_{\tau=\tau_0} > 0$. This means, at $\tau = \tau_0$, $\mathrm{Re}(\lambda(\tau))$ is an increasing function of τ. $\mathrm{Re}\big(\lambda(\tau)\big)$ must pass from negative to positive values at $\tau = \tau_0$. Hence, on the interval (τ_0, τ_1), $\mathrm{Re}(\lambda)$ takes on positive values. For some value of τ in (τ_0, τ_1), we have $RS \geq 1$. Since on the interval (τ_0, τ_1) there does not occur a pure imaginary

root, RS does not change, hence $RS \geq 1$ for every $\tau \in (\tau_0, \tau_1)$. Therefore, for $\tau \in (\tau_0, \tau_1)$ the system has at least one eigenvalue with positive real part and it is unstable at the equilibrium point P_1.

- Suppose $K \leq b/2$. Then, we do not have a pure imaginary root of the characteristic equation.
  - If condition (3.16) of Lemma 3 hold and if $b > 0$, since no imaginary root occurs for any value of $\tau \geq 0$, there does not appear any eigenvalue on the right half plane and the system remains stable for all values of $\tau \geq 0$.

Based on the arguments that we tried to explain above, considering the Lemmas [1-5], the following Theorem can be obtained.

Theorem 5 *We assume that the conditions of* (3.16) *hold and* $b > 0$.

If $K > b/2$,

(i) *The equilibrium point* P_0 *of the system* (3.1) *is stable for* $\tau \in [0, \tau_0)$ *while it behaves unstable for* $\tau \in (\tau_0, \tau_1)$.

(ii) *The system* (3.1) *undergoes a Hopf bifurcation at the equilibrium point* P_0 *when* $\tau = \tau_0$.

If $K \leq b/2$,

(iii) *The equilibrium point* P_0 *of the system* (3.1) *is stable for* $\tau \geq 0$.

3.1.2 Stability Analysis and Hopf Bifurcation for P_1

In the previous subsection, the stability condition of the system (3.1) is considered at the equilibrium point $P_0 = (0, \frac{1}{b}, 0, 0)$, or equaivalently, of the system (3.10) at the point $P_0(0,0,0,0)$. There remains to consider the stability of the other two equilibrium points. We shall deal with P_1 only, the analysis follows similar lines for the equilibrium point P_2.

Remember that the equilibrium point P_1 was

$$P_1\left(+\sqrt{\frac{kb+abck}{c(d-k)}+1},\frac{(1+ca)k}{c(k-d)},-\frac{1}{c}\sqrt{\frac{kb+abck}{c(d-k)}+1},+\frac{d(1+ca)}{c(d-k)}\sqrt{\frac{kb+abck}{c(d-k)}+1}\right).$$

The related Jacobians J_0 and J_τ at the equilibrium P_1 are found to be

$$J_{0,1}=\begin{pmatrix} \frac{k+acd}{ck-cd} & \sqrt{\frac{kb+abck}{c(d-k)}+1} & 1 & 1 \\ -2\sqrt{\frac{kb+abck}{c(d-k)}+1} & -b+K & 0 & 0 \\ -1 & 0 & -c & 0 \\ -\frac{dk+acdk}{ck-cd} & -d\sqrt{\frac{kb+abck}{c(d-k)}+1} & 0 & -k \end{pmatrix} \tag{3.34}$$

and

$$J_{\tau,1}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -K & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.35}$$

The characteristic equation

$$|J_0+e^{-\lambda\tau}J_\tau-\lambda I|=0 \tag{3.36}$$

yields

$$\lambda^4+a_1\lambda^3+b_1\lambda^2+c_1\lambda+d_1+(a_2\lambda^3+b_2\lambda^2+c_2\lambda)e^{-\lambda\tau}=0 \tag{3.37}$$

where

$$\begin{aligned}
\theta &= \sqrt{\frac{kb+abck}{c(d-k)}+1},\\
a_1 &= \frac{(acd+k+c(d-k)(b+c+k-K))}{(c(d-k))},\\
b_1 &= \frac{1}{c(d-k)}(c^2(ad+(d-k)(b+k-K)))\\
&\quad+\frac{1}{c(d-k)}(k(b-d+k-K)+c(k(-bk+kK-2\theta^2)+d(1+(a+k)(b-K)+2\theta^2))),\\
c_1 &= \frac{cd+k(-d+k)+c^2(ad+(d-k)k)(b-K)}{c(d-k)}+2(c-d+k)\theta^2, \qquad (3.38)\\
d_1 &= 2c(-d+k)\theta^2,\\
a_2 &= K,\\
b_2 &= \frac{((acd+k+c(d-k)(c+k))K)}{(c(d-k))},\\
c_2 &= \frac{((cd+k(-d+k)+c^2(ad+(d-k)k))K)}{(c(d-k))}.
\end{aligned}$$

We see that the characteristic equation is different than that of P_0, and we will make use of works of Ruan and Wei (2003) [22] and Li and Wei (2005) [24] in order to discover the distribution of zeros of the system (3.37), which is a fourth degree transcendental polynomial equation.

If $i\omega$, $(\omega > 0)$ is a root of Eq. (3.37), it must satisfy

$$\omega^4 - a_1\omega^3 i - b_1\omega^2 + c_1\omega i + d_1 + (-a_2\omega^3 i - b_2\omega^2 + c_2\omega i)e^{-i\omega\tau} = 0. \tag{3.39}$$

After separating the real and imaginary parts of the equation, we get

$$\omega^4 - b_1\omega^2 + d_1 = (a_2\omega^3 - c_2\omega)\sin(\omega\tau) + b_2\omega^2\cos(\omega\tau), \tag{3.40a}$$

$$a_1\omega^3 - c_1\omega = (c_2\omega - a_2\omega^3)\cos(\omega\tau) + b_2\omega^2\sin(\omega\tau). \tag{3.40b}$$

Taking the squares of both equations and adding up obtain

$$\begin{aligned}\omega^8 &+ (a_1^2 - 2b_1 - a_2^2)\omega^6 + (b_1^2 + 2d_1 - 2a_1c_1 - b_2^2 + 2a_2c_2)\omega^4 \\ &+ (c_1^2 - 2b_1d_1 - c_2^2)\omega^2 + d_1^2 = 0.\end{aligned} \tag{3.41}$$

Let $z = \omega^2$ and denote $p = a_1^2 - 2b_1 - a_2^2$, $q = b_1^2 + 2d_1 - 2a_1c_1 - b_2^2 + 2a_2c_2$, $u = c_1^2 - 2b_1d_1 - c_2^2$ and $v = d_1^2$. Then Eq. (3.41) becomes

$$z^4 + pz^3 + qz^2 + uz + v = 0. \tag{3.42}$$

We need to give a remark here. The transcendental-polynomial characteristic equation (3.37) is different and more complicated than the characteristic equation considered in [24] . Their characteristic equation is in the form

$$\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d + re^{-\lambda\tau} = 0, \tag{3.43}$$

which appears as Eq. (2.1) in [24]. Although our characteristic equation (3.37) was different than theirs, we followed their lines in the search of a root $\lambda = i\omega$, $\omega > 0$ and obtained exactly the same fourth-degree equation (3.42) up to a difference in all of the constants, of course. Therefore, apart from this line, we will adapt the development from Eq. (2.5) of [24]. Let us call

$$h(z) = z^4 + pz^3 + qz^2 + uz + v. \tag{3.44}$$

After differentiating $h(z)$, we have $h'(z) = 4z^3 + 3pz^2 + 2qz + u$. Set

$$4z^3 + 3pz^2 + 2qz + u = 0. \tag{3.45}$$

Let $y = z + \frac{p}{4}$. Then equation (3.45) becomes

$$y^3 + p_1 y + q_1 = 0, \tag{3.46}$$

where $p_1 = \dfrac{q}{2} - \dfrac{3}{16}p^2,\ \ q_1 = \dfrac{p^3}{32} - \dfrac{pq}{8} + \dfrac{u}{4}.$

The roots of the equation (3.46) are

$$y_1 = \sqrt[3]{-\frac{q_1}{2} + \sqrt{D}} + \sqrt[3]{-\frac{q_1}{2} - \sqrt{D}}, \tag{3.47}$$

$$y_2 = \sqrt[3]{-\frac{q_1}{2} + \sqrt{D}\sigma} + \sqrt[3]{-\frac{q_1}{2} - \sqrt{D}\sigma^2}, \tag{3.48}$$

$$y_3 = \sqrt[3]{-\frac{q_1}{2} + \sqrt{D}\sigma^2} + \sqrt[3]{-\frac{q_1}{2} - \sqrt{D}\sigma}, \tag{3.49}$$

$$with \tag{3.50}$$

$$D = (\frac{q_1}{2})^2 + (\frac{p_1}{3})^3, \quad \sigma = \frac{-1+\sqrt{3}i}{2}, \tag{3.51}$$

$$and \quad therefore \tag{3.52}$$

$$z_i = y_i - \frac{p_i}{4},\ i = 1,2,3. \tag{3.53}$$

Lemma 6 *[24] Observe that $v = d_1^2 \geq 0$.*

(i) When $D \geq 0$, the equation (3.42) has positive roots iff $z_1 > 0$ and $h(z_1) < 0$.

(ii) When $D < 0$, the equation (3.42) has a positive root iff there exists at least one $z^* \in z_1, z_2, z_3$, such that $z^* > 0$ and $h(z^*) \leq 0$.

Proof: (i) For $D \geq 0$, (3.46) has the unique real root y_1, hence the equation (3.45) has the unique real root z_1 . Also, z_1 is the unique stationary point of $h(z)$ and the minimum point of $h(z)$ because $h(z)$ is a differentiable function and

$$\lim_{z\to\infty} h(z) = \infty. \tag{3.54}$$

The sufficiency is clear, so we just prove the necessity. Now, we analyze the situations of either $z_1 \leq 0$ or $z_1 > 0$ and $h(z_1) > 0$. If we assume that $z_1 \leq 0$, since $h(0) = v \geq 0$ is the minimum of $h(z)$ for $z \geq 0$, it follows that $h(z)$ has no positive real zeros. If we assume that $z_1 > 0$ and $h(z_1) > 0$, as $min_{z>0}\{h(z)\} = h(z_1) > 0$, it follows that $h(z)$ has no positive real zeros.

(ii) For $D < 0$, the equation (3.46) has the roots y_1, y_2 and y_3, it follows that the equation (3.45) has three roots z_1, z_2 and z_3 where at least one of them is real. Without loss of generality, if we suppose that z_1, z_2 and z_3 are all real that means $h(z)$ has at most three stationary points at z_1, z_2 and z_3. Since the remainder of the proof is parallel to that of (i), we can skip it. [24]

Assume that Eq. (3.42) has positive roots. We can suppose that it has four positive roots which we will denote as z_i^*, $i = 1,2,3,4$. It follows that the equation (3.41) has four positive roots, denoted by $\omega_i = \sqrt{z_i^*}$, $i = 1,2,3,4$. From (3.40) we solve

$$\begin{aligned}\tau_k^{(j)} = \frac{1}{\omega_k}\Big[\arccos\Big(\frac{-\omega^4 a_1 a_2 + \omega^4 b_2 - \omega^2 b_1 b_2 + \omega^2 a_2 c_1 + \omega^2 a_1 c_2 - c_1 c_2 + b_2 d_1}{\omega^4 a_2^2 + \omega^2 b_2^2 - 2\omega^2 a_2 c_2 + c_2^2}\Big) \\ + 2(j-1)\pi\Big], \qquad k = 1,2,3,4; \quad j = 1,2,....\end{aligned} \tag{3.55}$$

Then $\mp i\omega_k$ is a pair of purely imaginary roots of equation (3.37) when $\tau = \tau_k^{(j)}$, $k = 1,2,3,4$; $j = 1,2,....$ Obviously,

$$\lim_{j\to\infty} \tau_k^{(j)} = \infty, \qquad k = 1,2,3,4. \tag{3.56}$$

Then, it can be defined as

$$\tau_0 = \tau_{k_0}^{(j_0)} = \min_{1\le k\le 4, 1\le j}\{\tau_k^{(j)}\}, \quad \omega_0 = \omega_{k_0}, \quad z_0 = z_{k_0}^*. \tag{3.57}$$

Lemma 7 *Assume that* $a_1 + a_2 > 0$, $(a_1 + a_2)(b_1 + b_2) - (c_1 + c_2) > 0$, $d_1 > 0$ *and* $(c_1 + c_2)[(a_1 + a_2)(b_1 + b_2) - (c_1 + c_2)] - (a_1 + a_2)^2 d_1 > 0$.

(i) For $\tau \in [0, \tau_0)$, if one of following conditions holds: *(a)* $v \ge 0$, $D \ge 0$, $z_1 > 0$ and $h(z_1) \le 0$; *(b)* $v \ge 0$, $D < 0$ and there exists a $z^* \in \{z_1, z_2, z_3\}$ such that $\exists z^* > 0$ and $h(z^*) \le 0$, then all roots of equation (3.37) have negative real parts.

(ii) For all $\tau \ge 0$, if the conditions (a) *and* (b) of (i) are not satisfied, then all roots of equation (3.37) have negative real parts.

Proof: When $\tau = 0$, equation (3.37) becomes

$$\lambda^4 + a_1\lambda^3 + b_1\lambda^2 + c_1\lambda + d_1 + (a_2\lambda^3 + b_2\lambda^2 + c_2\lambda) = 0. \tag{3.58}$$

According to the Routh-Hurwitz criterion, all roots of equation (3.58) have negative real parts if and only if

$$a_1 + a_2 > 0, \qquad (a_1 + a_2)(b_1 + b_2) - (c_1 + c_2) > 0, \quad d_1 > 0, \tag{3.59}$$

$$(c_1 + c_2)[(a_1 + a_2)(b_1 + b_2) - (c_1 + c_2)] - (a_1 + a_2)^2 d_1 > 0. \tag{3.60}$$

Now, under this condition, at $\tau = 0$, (3.37) has no eigenvalue with positive real part, $RS = 0$. Lemma 6 proposes if and only if conditions for (3.37) to have a pure imaginary root. If $(a) - (b)$ are not satisfied, Lemma 6 states that (3.37) has no pure imaginary root for all $\tau \geq 0$. Therefore, according to Lemma 5, RS does not change for (3.37) for all $\tau \geq 0$, which means it does not have any eigenvalue with positive real part for $\tau \geq 0$. $\lambda = 0$ is not an eigenvalue since we assumed $d_1 > 0$. Therefore, all the eigenvalues are with negative real parts for $\tau \geq 0$. This proves *(ii)*.

Now let us prove *(i)*. Since one of *(a)-(b)* holds, then by Lemma 6, there is a pure imaginary root of (3.37) at $\tau = \tau_0$. Since there is no pure imaginary root of (3.37) for all values of $\tau \in [0, \tau_0)$, by Lemma 5, RS does not change on $0 \leq \tau < \tau_0$, hence all the eigenvalues of (3.37) have negative real parts when $\tau \in [0, \tau_0)$. This proves *(i)*.

Let

$$\lambda(\tau) = \alpha(\tau) + i\omega(\tau) \tag{3.61}$$

be the root of equation (3.37) satisfying $\alpha(\tau_0) = 0,\ \omega(\tau_0) = \omega_0$.

Lemma 8 *Assume that $h'(z_0) \neq 0$. Then, $\mp i\omega_0$ is a simple (i.e., not multiple) pure imaginary root of the equation* (3.37) *when $\tau = \tau_0$. Additionally, if the conditions of Lemma 7-(i) hold, the following transversality condition holds:*

$$\frac{d(Re\lambda(\tau))}{d\tau}\Big|_{\tau=\tau_0} \neq 0 \tag{3.62}$$

and the sign of $d(Re\lambda(\tau))/d\tau|_{\tau=\tau_0}$ is consistent with that of $h'(z_0)$.

Proof: Denote

$$R(\lambda) = \lambda^4 + a_1\lambda^3 + b_1\lambda^2 + c_1\lambda + d_1, \tag{3.63}$$

$$Q(\lambda) = a_2\lambda^3 + b_2\lambda^2 + c_2\lambda. \tag{3.64}$$

Then (3.37) can be represented as

$$R(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0, \tag{3.65}$$

and (3.41) can be written as following:

$$R(i\omega)\bar{R}(i\omega) - Q(i\omega)\bar{Q}(i\omega) = 0. \tag{3.66}$$

Then, together with (3.42) and (3.44), we get

$$h(\omega^2) = R(i\omega)\bar{R}(i\omega) - Q(i\omega)\bar{Q}(i\omega). \tag{3.67}$$

Differentiating both sides of (3.67) with respect to the ω, we obtain

$$2\omega h'(\omega^2) = -i\{R\bar{R}' - R'\bar{R} + Q'\bar{Q} - Q\bar{Q}'\}. \tag{3.68}$$

If $i\omega_0$ is not a simple root, then it must satisfy

$$\frac{d}{d\lambda}[R(\lambda) + Q(\lambda)e^{-\lambda\tau}]|_{\lambda=i\omega_0} = 0, \tag{3.69}$$

that is,

$$R'(i\omega_0) + Q'(i\omega_0)e^{-i\omega_0\tau_0} - \tau_0 Q(i\omega_0)e^{-i\omega_0\tau_0} = 0. \tag{3.70}$$

With (3.65), we have

$$\tau_0 = \frac{-R'(i\omega_0)}{R(i\omega_0)} + \frac{Q'(i\omega_0)}{Q(i\omega_0)}. \tag{3.71}$$

Thus, by (3.66) and (3.67), we get

$$\begin{aligned}
Im(\tau_0) &= Im\left(\frac{Q'(i\omega_0)}{Q(i\omega_0)} - \frac{R'(i\omega_0)}{R(i\omega_0)}\right) = Im\left(\frac{Q'(i\omega_0\bar{Q}(i\omega_0)}{Q(i\omega_0)\bar{Q}(i\omega_0)} - \frac{R'(i\omega_0)\bar{R}(i\omega_0)}{R(i\omega_0)\bar{R}(i\omega_0)}\right) \\
&= Im\left(\frac{Q'(i\omega_0)\bar{Q}(i\omega_0) - R'(i\omega_0)\bar{R}(i\omega_0)}{R(i\omega_0)\bar{R}(i\omega_0)}\right) \\
&= \frac{-i\left[Q'(i\omega_0)\bar{Q}(i\omega_0) - R'(i\omega_0)\bar{R}(i\omega_0) - \bar{Q}'(i\omega_0)Q(i\omega_0) + \bar{R}'(i\omega_0)R(i\omega_0)\right]}{2R(i\omega_0)\bar{R}(i\omega_0)} = \frac{\omega_0 h'(\omega_0^2)}{|R(i\omega_0)|^2}.
\end{aligned} \tag{3.72}$$

It is obtained that $h'(\omega_0^2) = 0$, because τ_0 is real and $\mathrm{Im}(\tau_0) = 0$. We have a contradiction to the assumption $h'(\omega_0^2) \neq 0$. This is the proof of the first conclusion.

Now, we need to prove that $\left.\frac{d(Re(\lambda\tau))}{d\tau}\right|_{\tau=\tau_0,\lambda=i\omega_0} \neq 0.$

Differentiating both sides of (3.65) with respect to τ,

$$\frac{d}{d\tau}[R(\lambda) + Q(\lambda)e^{-\lambda\tau}] = 0, \tag{3.73}$$

we obtain;

$$\frac{d\lambda}{d\tau}[R'(\lambda) + Q'(\lambda)e^{-\lambda\tau} - \tau Q(\lambda)e^{-\lambda\tau}] - \lambda Q(\lambda)e^{-\lambda\tau} = 0 \tag{3.74}$$

which implies

$$
\begin{aligned}
\frac{d\lambda}{d\tau} &= \frac{\lambda Q(\lambda)e^{-\lambda\tau}}{[R'(\lambda)+Q'(\lambda)e^{-\lambda\tau}-\tau Q(\lambda)e^{-\lambda\tau}]} \\
&= \frac{\lambda Q(\lambda)}{R'(\lambda)e^{\lambda\tau}+Q'(\tau)-\tau Q(\lambda)} = \frac{\lambda Q(\lambda)[\bar{R}'(\lambda)e^{\bar{\lambda}\tau}+\bar{Q}'(\lambda)-\tau\bar{Q}(\lambda)]}{|R'(\lambda)e^{\lambda\tau}+Q'(\lambda)-\tau Q(\lambda)|^2} \\
&= \frac{\lambda[-R(\lambda)\bar{R}'(\lambda)e^{\lambda\tau}e^{\bar{\lambda}\tau}+Q(\lambda)\bar{Q}'(\lambda)-\tau\bar{Q}(\lambda)Q(\lambda)]}{|R'(\lambda)e^{\lambda\tau}+Q'(\lambda)-\tau Q(\lambda)|^2} \\
&= \frac{\{\lambda[-R(\lambda)\bar{R}'e^{\lambda\tau}e^{\bar{\lambda}\tau}(\lambda)+Q(\lambda)\bar{Q}'(\lambda)-\tau|Q(\lambda)|^2]\}}{|R'(\lambda)e^{\lambda\tau}+Q'(\lambda)-\tau Q(\lambda)|^2}.
\end{aligned}
\tag{3.75}
$$

It follows together with (3.68) that

$$
\begin{aligned}
&\frac{d(Re\lambda(\tau))}{d\tau}\Big|_{\tau=\tau_0,\lambda=i\omega_0} = \frac{Re\{\lambda[-R(\lambda)\bar{R}'(\lambda)e^{\lambda\tau}e^{\bar{\lambda}\tau}+Q(\lambda)\bar{Q}'(\lambda)-\tau|Q(\lambda)|^2]\}_{\tau=\tau_0,\lambda=i\omega_0}}{|R'(\lambda)e^{\lambda\tau}+Q'(\lambda)-\tau Q(\lambda)|^2_{\tau=\tau_0,\lambda=i\omega_0}} \\
&= \frac{i\omega_0}{2}\frac{[-R(i\omega_0)\bar{R}'(i\omega_0)+Q(i\omega_0)\bar{Q}'(i\omega_0)+\bar{R}(i\omega_0)R'(i\omega_0)-\bar{Q}(i\omega_0)Q'(i\omega_0)]}{|R'(i\omega_0)e^{i\omega_0\tau_o}+Q'(i\omega_0)-\tau_0 Q(i\omega_0)|^2} \\
&= \frac{i\omega_0}{2}\frac{2\omega_0 h'(\omega_0^2)}{i|R'(i\omega_0)e^{i\omega_0\tau_o}+Q'(i\omega_0)-\tau_0 Q(i\omega_0)|^2} \\
&= \frac{\omega_0^2 h'(\omega_0^2)}{|R'(i\omega_0)e^{i\omega_0\tau_o}+Q'(i\omega_0)-\tau_0 Q(i\omega_0)|^2} \neq 0.
\end{aligned}
\tag{3.76}
$$

The statement of Lemma 8 appears in [24]. We adapted the proof from [15] doing some corrections. Now we can state the following Theorem.

Theorem 6 *The values of* ω_0, τ_0, z_0 *and* $\lambda(\tau)$ *are given in equations* (3.57) *and* (3.61). *Suppose that*

- $a_1+a_2>0$, $(a_1+a_2)(b_1+b_2)-(c_1+c_2)>0$, $\quad d_1>0$,
- $(c_1+c_2)[(a_1+a_2)(b_1+b_2)-(c_1+c_2)]-(a_1+a_2)^2 d_1>0$.

(i) When the assumptions (a)-(b) of Lemma (7) *do not hold, the roots of equation* (3.37) *have negative real parts for all* $\tau \geq 0$. *The system* (3.1) *is stable for all* $\tau \geq 0$.

(ii) If either (a) or (b) is satisfied, roots of Eq. (3.37) *have negative real parts for* $\tau \in [0,\tau_0)$. *It is obtained that the roots of* (3.37) *is* $\mp\omega_0 i$ *and other roots have negative real parts when* $\tau=\tau_0$ *and* $h'(z_0)\neq 0$. *Additionally,* $\frac{dRe\lambda(\tau_0)}{d\tau}>0$ *and if* τ_1 *is taken the first value of* $\tau>\tau_0$ *such that Eq.* (3.37) *has purely imaginary root, Eq.* (3.37) *has at least one root with positive real part for* (τ_0,τ_1). *Therefore, at the equilibrium point* P_1, *the system* (3.1),

– *is stable for* $\tau \in [0, \tau_0)$,

– *undergoes a Hopf bifurcation when* $\tau = \tau_0$,

– *is unstable for* $\tau \in (\tau_0, \tau_1)$.

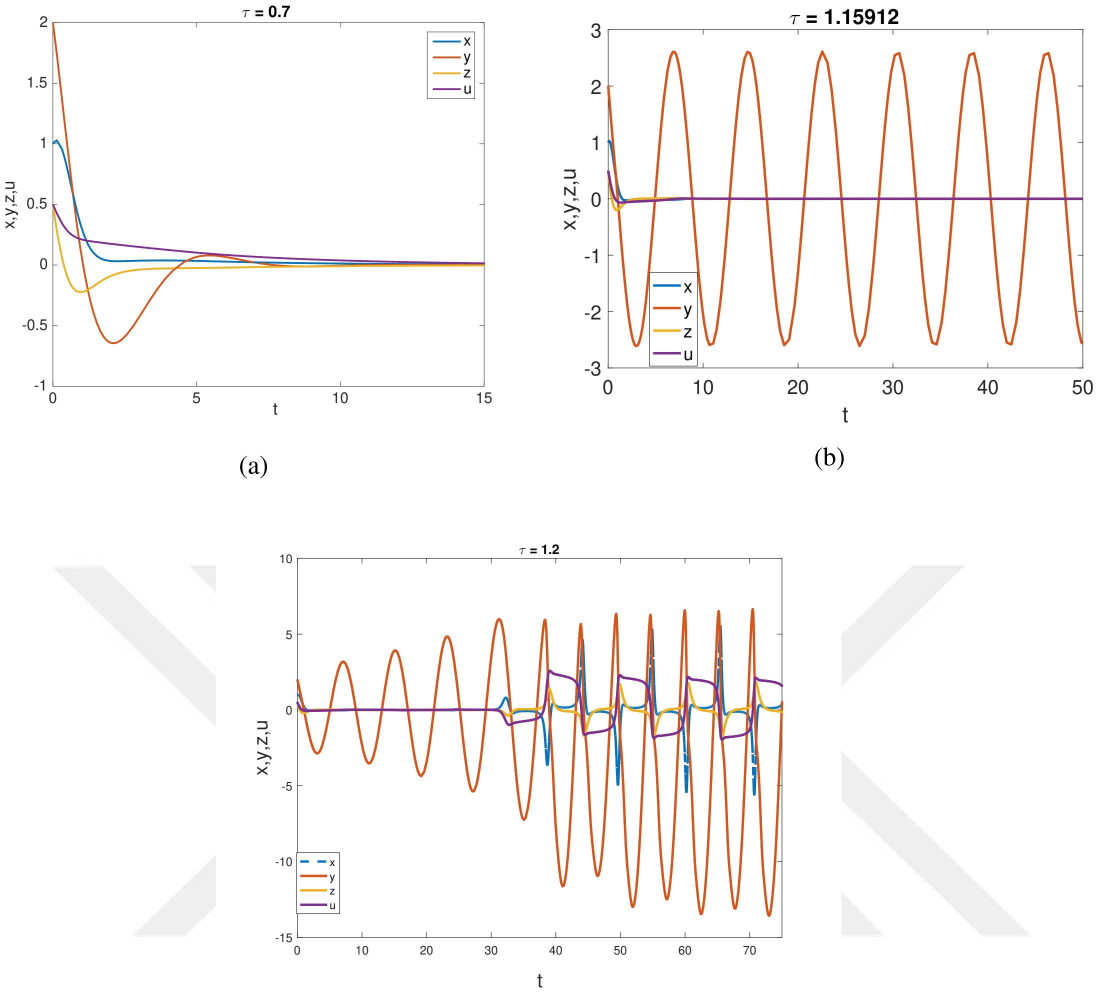
Let us note that arriving at Theorem 6 from Lemmas 5-8 is follows the same lines of reasoning we tried to explain in Remark 2 for Theorem 5. Therefore we do not repeat them here. In Lemma 8, it is shown that $\lambda = \mp i\omega_0$ is a simple (pure imaginary) root of the characteristic equation (3.1). Being a simple root of the eigenvalue equation, not a multiple root, is a necessary condition for the existence of the derivative $\dfrac{d(Re\lambda(\tau))}{d\tau}$, from which we check the transversality condition for a Hopf bifurcation to occur.

3.2 Numerical Simulations of the System

In this subsection we support our theoretical results, with plots of the time series of the dependent variables of the system for different values of the time-delay τ. Our simulations are well in accordance with the theoretical findings of the previous subsection. We give the dependent variable versus time plots for values of the time-delay τ for which

- $\tau < \tau_0$, and the system is stable at the fixed points P_0, P_1,
- $\tau = \tau_0$, and the system undergoes a Hopf bifurcation exhibited by the periodic behaviour in the variable y for P_0 and in x and y for P_1,
- $\tau > \tau_0$, and the system becomes unstable, demonstrated by the unbounded development in the graph of the variable y for P_0 and in x and y for P_1.

After these numerical investigations, we end this subsection by plotting phase portraits of the system corresponding to the cases summarized above.

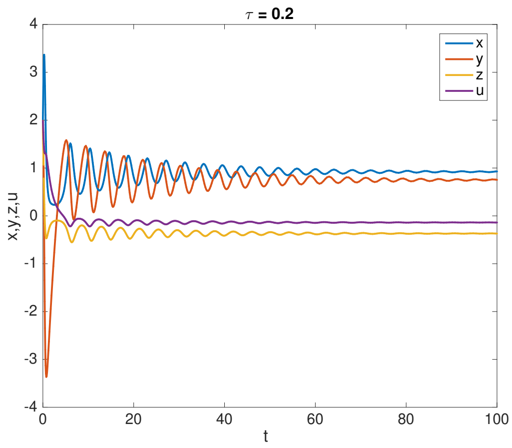


(a) (b)

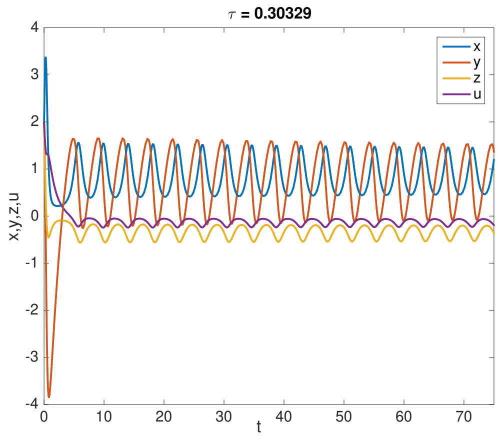
(c)

Figure 3.1 : Numerical simulations of the system (3.1) for different τ values (a) $\tau = 0.7$, (b) $\tau = 1.15912$, (c) $\tau = 1.2$. Here we take the parameter values as $a = 5$, $b = 0.4$, $c = 1.5$, $d = 0.2$, $k = 0.17$ and $K = 1$. Also, the initial conditions are taken as $x(0) = 1$, $y(0) = 2$, $z(0) = u(0) = 0.5$.

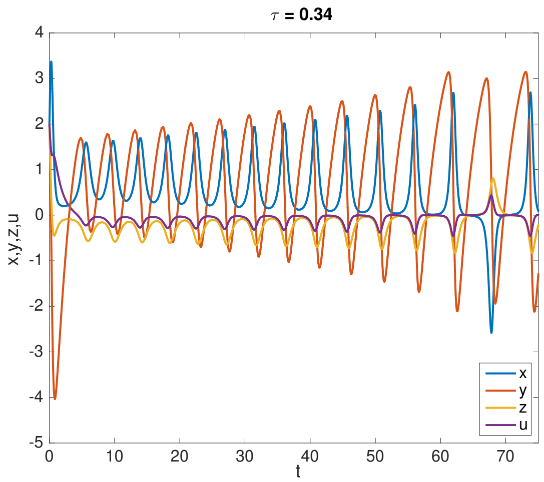
In Figure 3.1, the parameter values and initial conditions are chosen for testing the theoretical results which were obtained for the critical point P_0. Numerical results given in (a), (b) and (c) are examples for stable, Hopf bifurcation and unstable cases, respectively. These results are compatible with predictions of our theoretical results.



(a)



(b)



(c)

Figure 3.2 : Numerical simulations of the system (3.1) for different τ values (a) $\tau = 0.2$, (b) $\tau = 0.030329$, (c) $\tau = 0.034$. Here we take the parameter values as $a = 0.2$, $b = 0.2$, $c = 2.5$, $d = 0.2$, $k = 1$ and $K = 1$. Also, the initial conditions are taken as $x(0) = y(0) = z(0) = u(0) = 2$.

In Figure 3.2, the parameter values and initial conditions are chosen for testing the theoretical results which were obtained for the critical point P_1. Numerical results given in (a), (b) and (c) are examples for stable, Hopf bifurcation and unstable cases, respectively. These results are compatible with predictions of our theoretical results.

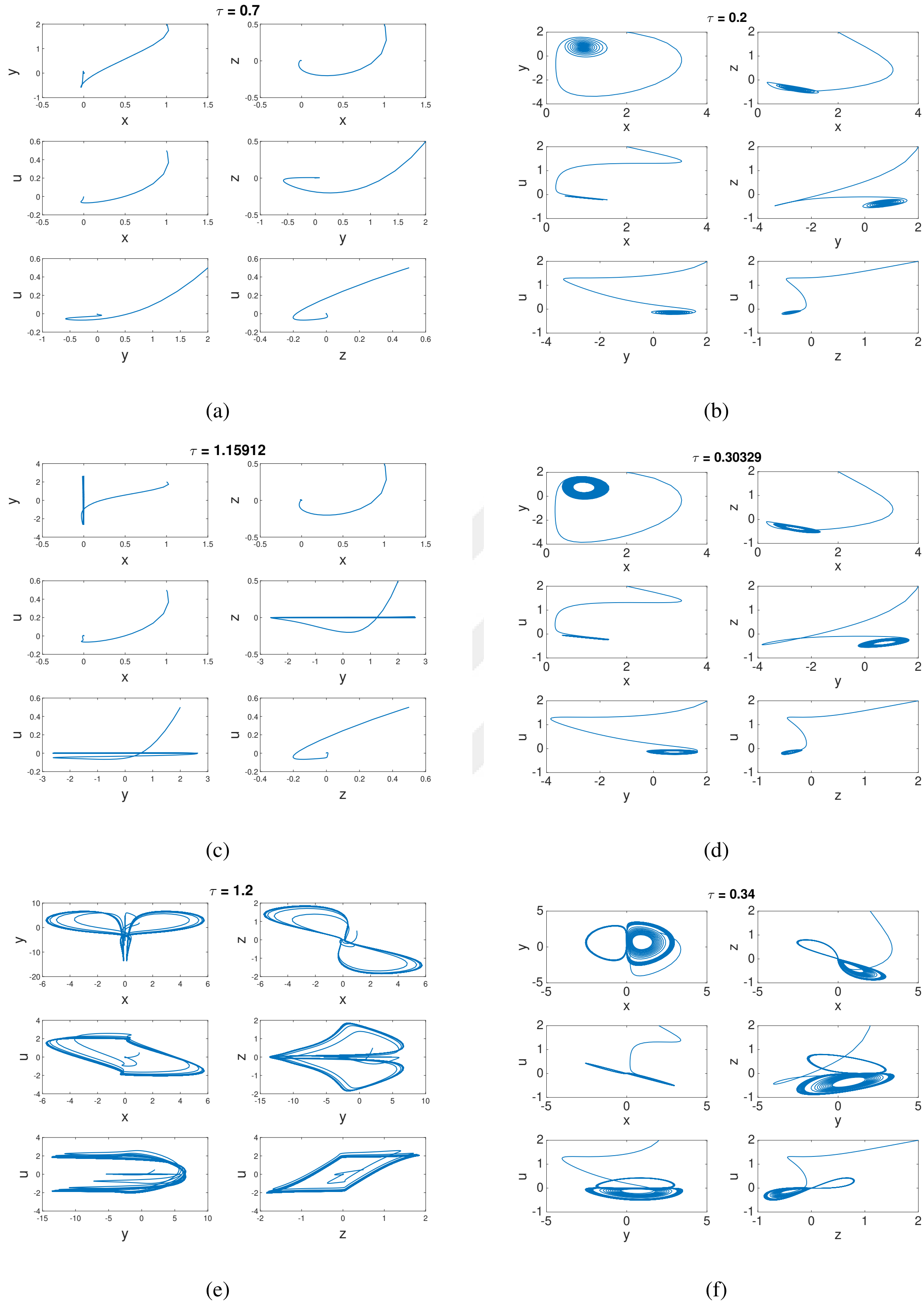


Figure 3.3 : Two dimensional phase portraits obtained from the numerical solutions of the system (3.1) for the chosen parameter values and initial conditions. (a), (c), (e) for P_0, (b), (d), (f) for P_1.

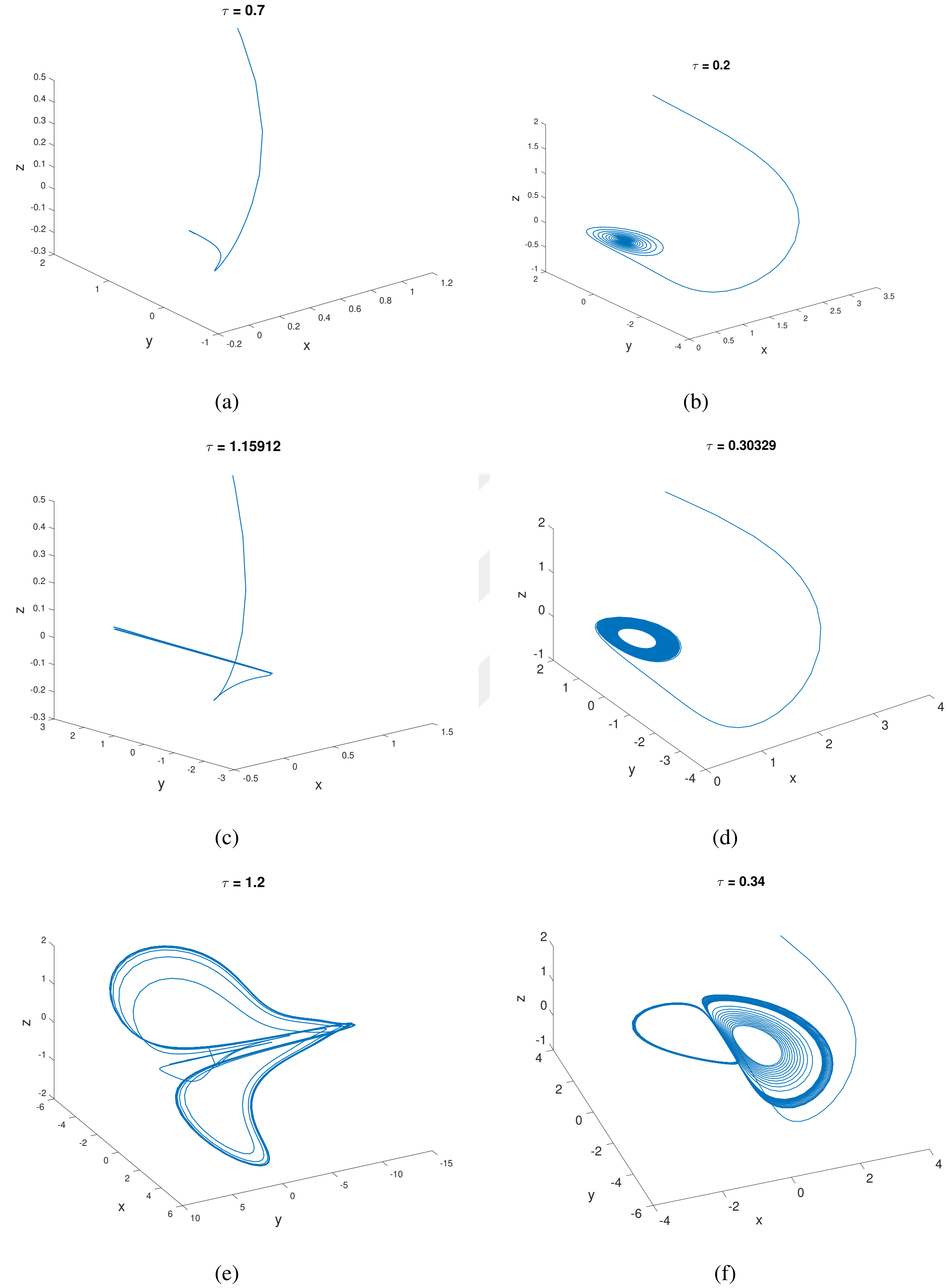


Figure 3.4 : Three dimensional phase portraits of the variables x, y and z obtained from the numerical solutions of the system (3.1) for the chosen parameter values and initial conditions. (a), (c), (e) for P_0, (b), (d), (f) for P_1.

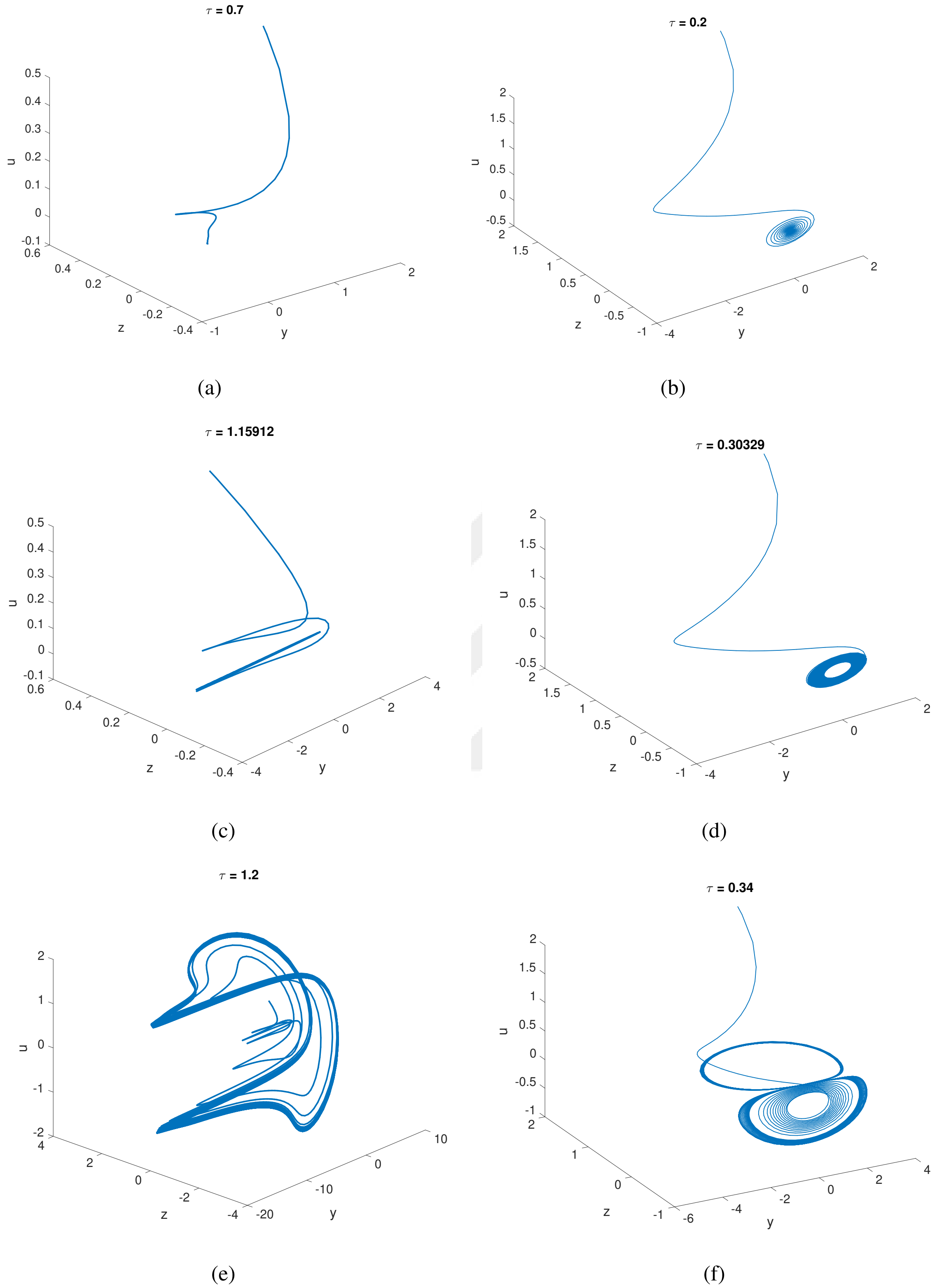


Figure 3.5 : Three dimensional phase portraits of the variables y, z and u obtained from the numerical solutions of the system (3.1) for the chosen parameter values and initial conditions. (a), (c), (e) for P_0, (b), (d), (f) for P_1.

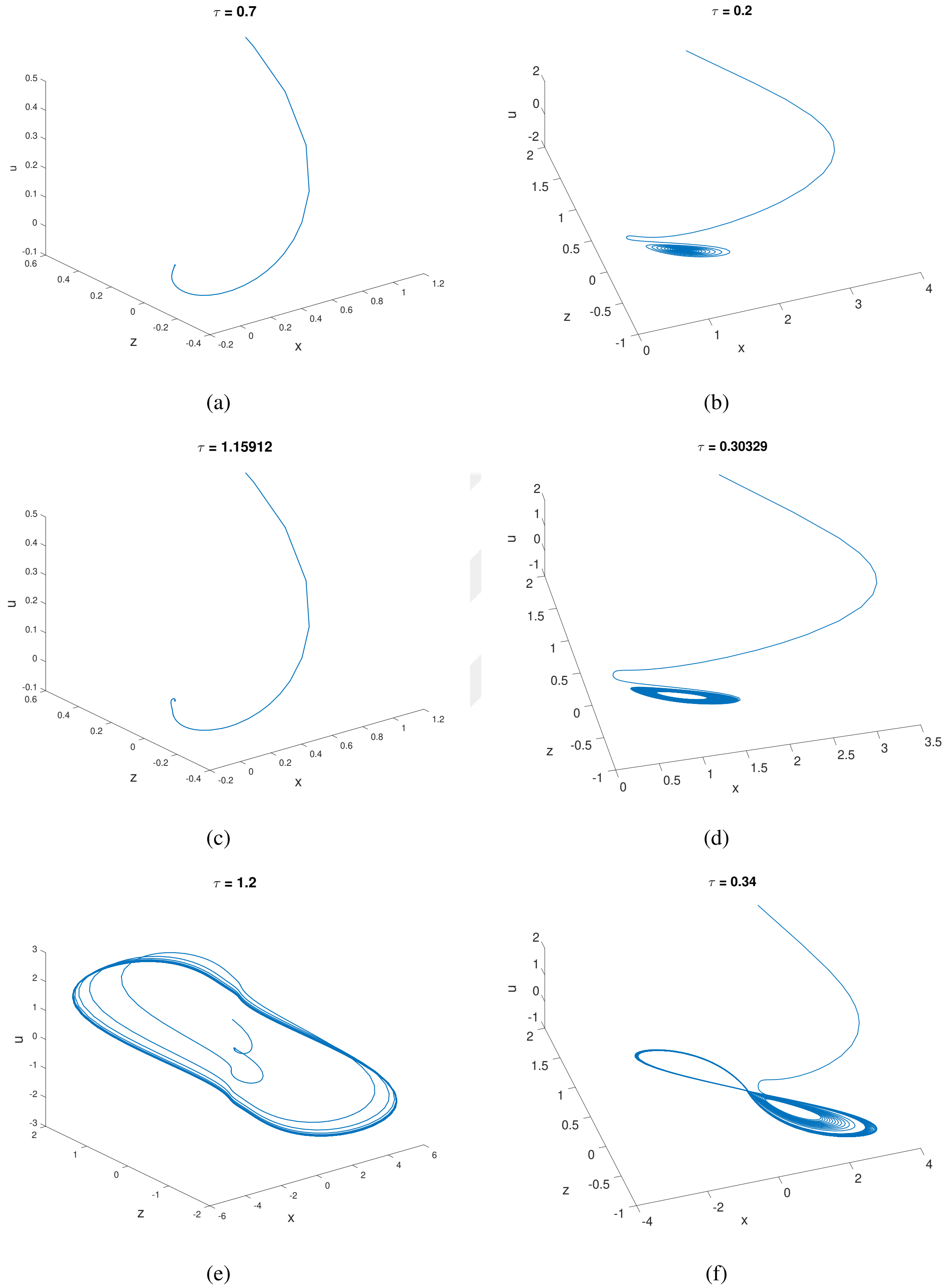


Figure 3.6 : Three dimensional phase portraits of the variables x, z and u obtained from the numerical solutions of the system (3.1) for the chosen parameter values and initial conditions. (a), (c), (e) for P_0, (b), (d), (f) for P_1.

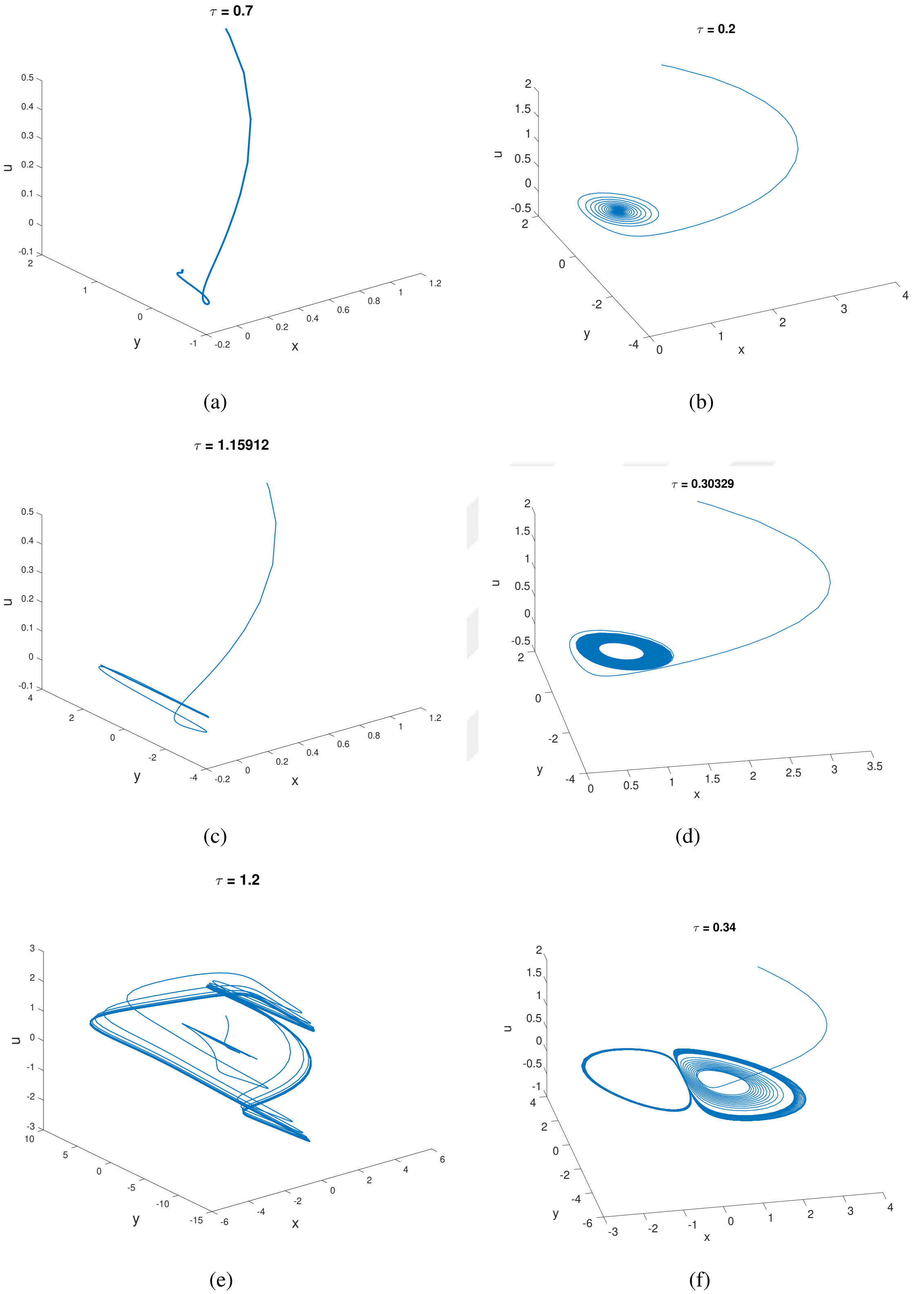


Figure 3.7 : Three dimensional phase portraits of the variables x, y and u obtained from the numerical solutions of the system (3.1) for the chosen parameter values and initial conditions. (a), (c), (e) for P_0, (b), (d), (f) for P_1.

4. CONCLUSIONS AND RECOMMENDATIONS

In this thesis, a new dynamical finance system is established. The new system's basic dynamical behavior, stability and Hopf bifurcation are investigated at the equilibrium points. We analysed the system

$$\dot{x} = z(t) + [y(t) - a]x(t) + u(t), \tag{4.1a}$$

$$\dot{y} = 1 - by(t) - x^2(t) + K[y(t) - y(t-\tau)], \tag{4.1b}$$

$$\dot{z} = -x(t) - cz(t), \tag{4.1c}$$

$$\dot{u} = -dx(t)y(t) - ku(t). \tag{4.1d}$$

We constructed the above model on two existing models in the literature. In the system S_1, which is given in (1.2), there are three state variables, x, y, z and S_1 includes a delay term in the variable y. In the system S_2, which is in (1.3), there are four state variables, x, y, z, u. When $K = 0$ in S_1 and $u = 0$ in S_2, they coincide and become the system (1.1). Our main system S_m, Eq. (4.1) is a composition of S_1 and S_2, with four state variables x, y, z, u, the delay feedback coefficient K, and the parameters a, b, c, d, k. Since S_m is obtained by adding a delay term to S_2, it reflects the delay effect on the system S_2.

Time delay parameter τ is taken as a bifurcation and control parameter in order to search the system's stability behavior. After linearization, the characteristic equations are examined at the equilibrium points and we proved that a Hopf bifurcation exists. If time delay τ passes a critical value, the system experiences a Hopf bifurcation, the stability condition of the system changing from stable to unstable. Through numerical simulations, our main results are confirmed; that the system undergoes a Hopf bifurcation with appropriate parameters and some graphs are shown at different time delay τ values.

The equilibrium points of S_2 and S_m are the same. S_m differs from the system S_2 by the delay feedback term. For the values of the parameters considered in Section 3.2, S_m when $K = 0$, namely the system S_2 is stable. When $K = 1$, S_m is stable for some range of the time-delay term τ; however, it becomes unstable after this critical threshold. Therefore, we can say that, there are cases in which time-delay term has a destabilizing effect on S_2.

We worked out bifurcation analysis of a dynamic finance system and we found that the system has rich dynamic behaviors and responses. Then, this study can be helpful for the relevant fields, especially economy, as a theoretical reference and it deserves to be studied more. As an open problem for further investigation, for instance, we can mention the search for chaotic or hyperchaotic character of the system.

REFERENCES

[1] **Jun-hai, M. and Yu-Shu, C.** (2001). Study for the bifurcation topological structure and the global complicated character of a kind of nonlinear finance system (I), *Applied Mathematics and Mechanics*, *22*(11), 1240–1251.

[2] **Jun-hai, M. and Yu-shu, C.** (2001). Study for the bifurcation topological structure and the global complicated character of a kind of nonlinear finance system (II), *Applied Mathematics and Mechanics*, *22*(12), 1375–1382.

[3] **Ma, J.** (2005). The reconstruction technology of complex nonlinear system, *TianJin: Tianjin University Publishing House*.

[4] **Chen, W.C.** (2008). Dynamics and control of a financial system with time-delayed feedbacks, *Chaos, Solitons & Fractals*, *37*(4), 1198–1207.

[5] **Chen, W.C.** (2008). Nonlinear dynamics and chaos in a fractional-order financial system, *Chaos, Solitons & Fractals*, *36*(5), 1305–1314.

[6] **Gao, Q. and Ma, J.** (2009). Chaos and Hopf bifurcation of a finance system, *Nonlinear Dynamics*, *58*(1-2), 209.

[7] **Yu, H.**, **Cai, G. and Li, Y.** (2012). Dynamic analysis and control of a new hyperchaotic finance system, *Nonlinear Dynamics*, *67*(3), 2171–2182.

[8] **Junhai, M.**, **Yaqiang, C. and Lixia, L.** (2008). Hopf bifurcation and chaos of financial system on condition of specific combination of parameters, *Journal of Systems Science and Complexity*, *21*(2), 250–259.

[9] **Zhao, X.**, **Li, Z. and Li, S.** (2011). Synchronization of a chaotic finance system, *Applied Mathematics and Computation*, *217*(13), 6031–6039.

[10] **Kai, G. and Zhang, W.** (2016). Chaotic Dynamics Analysis for a Class of Delay Nonlinear Finance Systems, *MATEC Web of Conferences*, volume 45, EDP Sciences, p.03005.

[11] **Yang, J.**, **Zhang, E. and Liu, M.** (2016). Bifurcation analysis and chaos control in a modified finance system with delayed feedback, *International Journal of Bifurcation and Chaos*, *26*(06), 1650105.

[12] **Ding, Y.**, **Jiang, W. and Wang, H.** (2012). Hopf-pitchfork bifurcation and periodic phenomena in nonlinear financial system with delay, *Chaos, Solitons & Fractals*, *45*(8), 1048–1057.

[13] **Chen, G. and Ueta, T.** (1999). Yet another attractor, *Int. J. Bifurc. Chaos*, *9*, 1465–1466.

[14] **Song, Y. and Wei, J.** (2004). Bifurcation analysis for Chen's system with delayed feedback and its application to control of chaos, *Chaos, Solitons & Fractals*, *22*(1), 75–91.

[15] **Zhang, X. and Zhu, H.** (2019). Hopf bifurcation and chaos of a delayed finance system, *Complexity*, *2019*.

[16] **Son, W.S. and Park, Y.J.** (2011). Delayed feedback on the dynamical model of a financial system, *Chaos, Solitons & Fractals*, *44*(4-5), 208–217.

[17] **Wu, W. and Chen, Z.** (2010). Hopf bifurcation and intermittent transition to hyperchaos in a novel strong four-dimensional hyperchaotic system, *Nonlinear Dynamics*, *60*(4), 615–630.

[18] **Qi, G.**, **Chen, G.**, **Du, S.**, **Chen, Z. and Yuan, Z.** (2005). Analysis of a new chaotic system, *Physica A: Statistical Mechanics and its Applications*, *352*(2-4), 295–308.

[19] **Strogatz, S.H.** (2018). *Nonlinear dynamics and chaos with student solutions manual: With applications to physics, biology, chemistry, and engineering*, CRC press.

[20] **Guckenheimer, J. and Holmes, P.** (2013). *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, volume 42, Springer Science & Business Media.

[21] **Liu, W.M.** (1994). Criterion of Hopf bifurcations without using eigenvalues, *Journal of Mathematical Analysis and Applications*, *182*(1), 250–256.

[22] **Ruan, S. and Wei, J.** (2001). On the zeros of a third degree exponential polynomial with applications to a delayed model for the control of testosterone secretion, *Mathematical Medicine and Biology*, *18*(1), 41–52.

[23] **Ruan, S. and Wei, J.** (2003). On the zeros of transcendental functions with applications to stability of delay differential equations with two delays, *Dynamics of Continuous Discrete and Impulsive Systems Series A*, *10*, 863–874.

[24] **Li, X. and Wei, J.** (2005). On the zeros of a fourth degree exponential polynomial with applications to a neural network model with delays, *Chaos, Solitons & Fractals*, *26*(2), 519–526.

CURRICULUM VITAE

Name Surname: Yasemin ÇALIŞ

Place and Date of Birth: Şişli-1989

E-Mail: calis.ysmn@gmail.com

EDUCATION:

- **B.Sc.:** 2011, Istanbul University, Faculty of Science, Department of Mathematics