ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE

ENGINEERING AND TECHNOLOGY

POWER SERIES SUBSPACES OF NUCLEAR FRÉCHET SPACES WITH THE PROPERTIES DN AND Ω

Ph.D. THESIS

Nazlı DOĞAN

Department of Mathematical Engineering

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DN VE Ω ÖZELLİĞİNE SAHİP NÜKLEER FRÉCHET UZAYLARININ KUVVET SERİSİ ALT UZAYLARI

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In memory Sunar Kural Aytuna & To Aydın Aytuna

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POWER SERIES SUBSPACES OF NUCLEAR FRÉCHET SPACES WITH THE PROPERTIES $\underline{DN}$ AND Ω

SUMMARY

Power series spaces constitute an important and well-studied class in the theory of Fréchet spaces. Linear topological invariants $\underline{DN}$ and Ω are enjoyed by many natural Fréchet spaces appearing in analysis. In particular, spaces of analytic functions, solutions of homogeneous elliptic linear partial differential operator with their natural topologies have the properties $\underline{DN}$ and Ω.

It is a well-known fact that the diametral dimension $\Delta(E)$ and the approximate diametral dimension $\delta(E)$ of a nuclear Fréchet space E with the properties $\underline{DN}$ and Ω are between corresponding invariant of power series spaces $\Lambda_1(\varepsilon)$ and $\Lambda_\infty(\varepsilon)$ for some specific exponent sequence ε. This sequence is called associated exponent sequence of E, see Definition 2.4.2.

Concidence of the diametral dimension and/or approximate diametral dimension of E with that of a power series space yields some structural results. For example, in [1], A. Aytuna, J. Krone and T. Terzioğlu proved that a nuclear Fréchet space E with the properties $\underline{DN}$ and Ω contains a complemented copy of $\Lambda_\infty(\varepsilon)$ provided $\Delta(E) = \Delta(\Lambda_\infty(\varepsilon))$ and ε is stable. On the other hand, A. Aytuna, [2], characterized tame nuclear Fréchet spaces E with the properties $\underline{DN}$ and Ω and stable exponent sequence ε, as those that satisfies $\delta(E) = \delta(\Lambda_1(\varepsilon))$.
These results lead us to ask the following two questions: Let E be a nuclear Fréchet space with the properties $\underline{DN}$ and Ω and ε be the associated exponent sequence of E.

1. Is there a complemented subspace of E which is isomorphic to $\Lambda_1(\varepsilon)$ if $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$?
2. If the diametral dimension of E coincides with that of a power series space, then does this imply that the approximate diametral dimension also do the same and vice versa?

The basis of this thesis was motivated by these two questions.
The main purpose of this thesis is to determine the connections between the diametral dimension and the approximate diametral dimension and to investigate power series subspaces of the nuclear Fréchet spaces with the properties $\underline{DN}$ and Ω using these invariants.

In the first chapter, some significant studies in the theory of nuclear Fréchet spaces are mentioned and the aim of this thesis is given. In the second chapter, we introduced preliminary materials and essential theorems.

In the third chapter, we showed that the second question has an affirmative answer when the power series space is of infinite type. Then we searched an answer for the second question in the finite type case and, in this regard, we first proved that the condition $\delta(E) = \delta(\Lambda_1(\varepsilon))$ always implies $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$. For other direction, the existence of a prominent bounded subset in the nuclear Fréchet space E plays a decisive role. Among other things, we proved that $\delta(E) = \delta(\Lambda_1(\varepsilon))$ if and only if E has a prominent bounded subset and $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$.

In the first section of the fourth chapter, we showed that a regular nuclear Köthe space with the properties $\underline{DN}$ and Ω is a power series space if its diametral dimension coincides with that of a power series space of infinite type or its approximate diametral dimension coincides with that of a power series space of finite type.

In the second section of the fourth chapter, we constructed a family $\mathcal{K}$ of nuclear Köthe spaces $K(a_{k,n})$ with the properties $\underline{DN}$ and Ω. First we showed that for an element of the family of $\mathcal{K}$ which is parameterized by a stable sequence α, $\Delta(K(a_{k,n})) = \Delta(\Lambda_1(\alpha))$ and $\delta(K(a_{k,n})) = \delta(\Lambda_1(\alpha))$. Second, we proved that for an element of the family of $\mathcal{K}$ which is parameterized by an unstable sequence α, $\Delta(K(a_{k,n})) = \Delta(\Lambda_1(\varepsilon))$ and $\delta(K(a_{k,n})) \neq \delta(\Lambda_1(\varepsilon))$ for its associated exponent sequence ε. This showed that the second question has a negative answer for power series space of finite type. Furthermore, we proved in Theorem 4.3.1 that the first question has a negative answer, that is, $\Lambda_1(\varepsilon)$ is not isomorphic to any subspace of these Köthe spaces $K(a_{k,n})$, let alone is isomorphic to a complemented subspace, though the condition $\Delta(K(a_{k,n})) = \Delta(\Lambda_1(\varepsilon))$ is satisfied.

In the third section of fourth chapter, motivated by our finding in the third section, we compiled some additional information, for instance, for an element E of the family $\mathcal{K}$ parameterized by an unstable sequence,

- E does not have a prominent bounded set.
- $\Delta(E)$, with respect to the canonical topology, is not barrelled, hence, not ultrabornological.
- Although the equality $\Delta(E) = \Lambda_1(\varepsilon)$ is satisfied and the canonical imbedding from $\Delta(E)$ into $\Lambda_1(\varepsilon)$ has a closed graph, the canonical imbedding from $\Delta(E)$ into $\Lambda_1(\varepsilon)$ is not continuous.

DN VE Ω ÖZELLİĞİNE SAHİP NÜKLEER FRÉCHET UZAYLARININ KUVVET SERİSİ ALT UZAYLARI

ÖZET

Fréchet uzayları yerel konveks uzayların, normlu uzay olmayan en önemli örneklerini içeren bir sınıftır. 1950'li yıllarda A. Grothendieck tarafından nükleerlik tanımını vermesi ile nükleer Fréchet uzayları bir çok çalışmanın ilham kaynağı olmuştur.

Nükleer Fréchet uzaylarının yapı teorisine yön veren iki önemli soru mevcuttur. Bu sorulardan ilki A. Grothendieck tarafından ortaya atılmıştır ve A. Grothendieck her nükleer Fréchet uzayının bazı olup olmadığını sormuştur. Bazı olmayan nükleer uzayların olduğu B. S. Mitiagin ve N. Zobin tarafından ispat edilmiş, böylece A. Grothendieck'in sorusu olumsuz bir şekilde cevaplamıştır.

1960'larda A. S. Dynin and B. S. Mitiagin, bazı olan her nükleer Frćechet uzaylarının bir nükleer Köthe uzayına izomorf olduğunu gösterdiler. Bu yüzden nükleer Köthe uzayları, nükleer Fréchet uzayların yapı teorisinde önemli bir yer kaplamaktadır.

Diğer soru ise A. Pelczynski tarafından sorulmuştur. A. Pelzcynski, nükleer Köthe uzaylarının her tümler uzayının bazı olup olmadığını sordu. 1975'te B. S. Mitiagin ve G. Henkin A. Pelczynski'nin sorusunun sonlu tip kuvvet serisi uzayları için olumlu bir cevabı olduğunu gösterdiler.

Bu soruyu sonsuz tip kuvvet serisi için cevaplamak çok uzun zaman aldı. 1989'da E. Dubinsky ve D. Vogt eğer sonsuz tip kuvvet serisi uysal ise bu sorunun olumlu bir cevabının olduğunu gösterdiler. E. Dubinsky ve D. Vogt bir sonsuz tip kuvvet serisi uzayının uysal olması için bu kuvvet serisini üreten eksponansiyel dizinin kararsız olması gerektiğini gösterdiler. Öte yandan kararlı bir eksponansiyel dizi tarafından üretilen sonsuz tip kuvvet serisi uzayı için cevap ise 1990 yılında A. Aytuna, J. Krone ve T. Terzioğlu tarafından verildi. A. Aytuna, J. Krone ve T. Terzioğlu, kararlı bir eksponansiyel dizi tarafından üretilen sonsuz tip kuvvet serisi uzayının tümler alt uzaylarının yine sonsuz tip kuvvet serisi alt uzayı olduğunu gösterdiler. Bu sorunun sonsuz tip kuvvet serileri için tam cevabı ise 2018 yılında A. K. Dronov ve V. M. Kaplitzkii tarafından verildi. A. K. Dronov ve V. M. Kaplitzkii regüler bazı olan her d_1, nükleer Köthe uzayının her tümler alt uzayının bir bazı olduğunu gösterdiler.

Yukarıda belirttiğimiz gibi nükleer Fréchet uzaylarının yapı teorisi pek çok matematikçi tarafından ele alınan önemli bir alanı oluşturmaktadır.

Biz bu tez çalışmasında $\underline{DN}$ ve Ω özelliklerine sahip nükleer Fréchet uzaylarının yapı teorisi ile ilgilendik. Fréchet uzaylarının en doğal örnekleri ise $\underline{DN}$ ve Ω özelliklerine

sahiptir. Özellikle, analitik fonksiyon uzayları ve homojen eliptik lineer kısmi diferansiyel operatörlerin çözüm uzayları $\underline{DN}$ ve Ω özelliklerine sahiptir.

$\underline{DN}$ ve Ω özelliğine sahip bir E nükleer Fréchet uzayının çapsal boyutu $\Delta(E)$ ve yaklaşık çapsal boyutu $\delta(E)$ özel bir ε ekponansiyel dizisi tarafından üretilen sonlu tip kuvvet serisi uzayı $\Lambda_1(\varepsilon)$ ve sonsuz tip kuvvet serisi uzayı $\Lambda_\infty(\varepsilon)$ ile ilişkilidir. Bu eksponansiyel diziye nükleer Fréchet uzayının ilişkili eksponansiyel dizisi denir. $\underline{DN}$ ve Ω özelliğine sahip bir E nükleer Fréchet uzayının çapsal boyutunun bu kuvvet serisi uzaylarından birinin çapsal boyutuna eşit olması ise iki ekstrem durum oluşturur.

İlk olarak; $\underline{DN}$ ve Ω özelliğine sahip bir E nükleer Fréchet uzayının çapsal boyutunun bir sonsuz tip kuvvet serisi uzayının çapsal boyutuna eşit olması durumunu düşünürsek, bu durum E uzayının yapı teorisi hakkında önemli bilgiler verir. Örneğin, A. Aytuna, J. Krone ve T. Terzioğlu, [1], $\underline{DN}$ ve Ω özelliğine sahip bir E nükleer Fréchet uzayının $\Delta(E) = \Delta(\Lambda_\infty(\varepsilon))$ ve ε ekponansiyel dizisi kararlı koşulları altında, $\Lambda_\infty(\varepsilon)$ uzayına izomorf bir tümler alt uzayının var olduğunu gösterdiler. Aslında buradaki çapsal boyut koşulu tümler bir altuzay oluşturmak üzere kullanılmıştır. Yine, A. Aytuna, J. Krone ve T. Terzioğlu, [1], $\underline{DN}$ ve Ω özelliğine sahip bir E nükleer Fréchet uzayının ilişkili eksponansiyel dizisi kararlı ise E uzayının daima ilişkili eksponansiyel dizisi tarafından üretilen sonsuz tip kuvvet serisi uzayına izomorf biraltuzayı olduğunu ispat ettiler. Ayrıca, A. Aytuna, J. Krone ve T. Terzioğlu, [1], Kompleks düzlemde alınan herhangi bir bölge üzerinde tanımlanan analitik fonksiyonlar uzayının çapsal boyut ile karakterize edilebildiğini gösterdiler: $D \subseteq \mathbb{C}$ bir bölge ve D bölgesi üzerinde tanımlı analitik fonksiyonlar uzayı $O(D)$ olmak üzere $O(D)$ uzayının çapsal boyutunun bir sonsuz tip kuvvet serisi uzayının çapsal boyutuna eşit olması ile $O(D)$ uzayının bu sonsuz tip kuvvet serisi uzayına izomorf olmasının denk olduğunu gösterdiler.

İkinci ekstrem durum olarak tanımlayacağımız, $\underline{DN}$ ve Ω özelliğine sahip bir E nükleer Fréchet uzayının çapsal boyutunun bir sonsuz tip kuvvet serisi uzayının çapsal boyutuna eşit olması, durumunu için elde edilmiş bir sonuç mevcut değildir. Bu ise bizi aşağıdaki soruyu sormaya yönledirdi: E uzayı $\underline{DN}$ ve Ω özelliğine sahip bir nükleer Fréchet uzayı ve ε, E uzayının ilişkili ekponansiyel dizisi olsun.

1. E uzayı $\underline{DN}$ ve Ω özelliğine sahip bir nükleer Fréchet uzayı ve ε, E uzayının ilişkili ekponansiyel dizisi olsun. Eğer $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$ ise, E uzayının $\Lambda_1(\varepsilon)$ uzayına izomorf bir tümler alt uzayı var mıdır?

Öte yandan, A. Aytuna, [2], her $\underline{DN}$ ve Ω özelliğine sahip uysal E nükleer Fréchet uzayının, $\delta(E) = \delta(\Lambda_1(\varepsilon))$ ve ε eksponansiyel dizisi kararlı olması durumunda $\Lambda_1(\varepsilon)$ uzayına izomorf olduğunu gösterdi. Bu ise bizi aşağıdaki soruyu sormaya yönledirdi:

2. E uzayı $\underline{DN}$ ve Ω özelliğine sahip bir nükleer Fréchet uzayı ve ε, E uzayının ilişkili ekponansiyel dizisi olsun. Eğer E uzayının çapsal boyutu bir kuvvet serisinin çapsal boyutuna eşit ise E uzayının yaklaşık çapsal boyutu da aynı kuvvet serisinin yaklaşık çapsal boyutuna eşit midir ve tersi de doğru mudur?

Bu tezin temeli bu iki soru üzerine kurulmuştur ve bu tezin amacı, $\underline{DN}$ ve Ω özelliğine sahip nükleer Fréchet uzaylarının topolojik değişmezleri arasındaki bağlantıları araştırmak ve bu bağlantıları kullanarak, bazı çapsal koşullar altında, bu uzayların kuvvet serisi alt uzaylarının var olup olmadığını araştırmaktır.

Birinci bölümde nükleer Fréchet uzaylarının bazı önemli sonuçlarına değinilmiş ve bu tez çalışmasının motivasyonu verilmiştir. İkinci bölümde bazı giriş yapıları ve önemli teoremler verilmiştir.

Üçüncü bölümde ikinci sorunun sonsuz tip kuvvet serisi uzayları için olumlu bir cevabının olduğu gösterilmiştir. Sonlu tip kuvvet serileri için ise ilk olarak $\delta(E) = \delta(\Lambda_1(\varepsilon))$ koşulu altında $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$ sağlandığı ispat edildi. Diğer yönün E uzayının çapsal boyutunun topolojisi ile bağlantılı olduğu gösterildi: eğer $\Delta(E)$ uzayının kanonik topolojisinin varilleri dolu ise $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$ iken $\delta(E) = \delta(\Lambda_1(\varepsilon))$ olduğunu ispat ettik. Ayrıca E uzayının üstüne çeşitli çapsal koşullar koyduğumuzda da bu sonucu elde edebileceğimizi gösterdik. Bu bölümde bu soruyu tam bir şekilde karakterize eden sonuçta ise belirgin sınırlı kümelerin varlığının önemli bir rolü vardır: İkinci soru için, $\delta(E) = \delta(\Lambda_1(\varepsilon))$ koşulunun sağlanması için gerek ve yeter koşulun E uzayının belirgin sınırlı bir alt kümesinin olması ve $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$ sağlanması olduğu gösterildi.

Dördüncü bölümün ilk kısımında $K(a_{k,n})$, $\underline{DN}$ ve Ω özelliğine sahip regüler nükleer bir Köthe uzayı olmak üzere $\Delta(K(a_{k,n})) = \Delta(\Lambda_\infty(\varepsilon))$ koşulunun sağlanması için gerek ve yeter koşul $K(a_{k,n})$ uzayının $\Lambda_\infty(\varepsilon)$ uzayına izomorf olması ve benzer şekilde $\delta(K(a_{k,n})) = \delta(\Lambda_1(\varepsilon))$ koşulunun sağlanması için gerek ve yeter koşul $K(a_{k,n})$ uzayının $\Lambda_1(\varepsilon)$ uzayına izomorf olması ifadelerini ispat edildi. Dördüncü bölümün ikinci kısımında $\underline{DN}$ ve Ω özelliğine sahip $K(a_{k,n})$ nükleer Köthe uzayılarından oluşan bir aile kuruldu. Bu aileden herhangi eleman için Kolmogorov çapları uzun bir kombinatoriyel yol ile hesap edildi. Ardından, Kolmogorov çapları için hem alt hem de üst bir kestirimin olduğu ispat edildi. Bu kestirim kullanılarak, bu ailenin kararlı ε ilişkili eksponansiyel dizisi olan bir elemanı $K(a_{k,n})$ için $\Delta(K(a_{k,n})) = \Delta(\Lambda_1(\alpha))$ ve $\delta(K(a_{k,n})) = \delta(\Lambda_1(\alpha))$ olduğunu gösterdik. Dolayısıyla, bu ailenin kararlı ε ilişkili eksponansiyel dizisi olan bir elemanı $K(a_{k,n})$ için ikinci sorunun olumlu bir cevabı vardır. Bu ailenin kararsız ε ilişkili eksponansiyel dizisi olan herhangi bir elemanı $K(a_{k,n})$ için $\Delta(K(a_{k,n})) = \Delta(\Lambda_1(\varepsilon))$ iken $\delta(K(a_{k,n})) \neq \delta(\Lambda_1(\varepsilon))$ olduğunu ispat ettik. Bu ise ikinci sorunun sonlu tip kuvvet serileri için olumsuz bir cevabının olduğunu gösterir. Bu bilgiyi kullanarak, Teorem 4.3.1'te bu Köthe uzaylarının $\Lambda_1(\varepsilon)$ uzayına izomorf bir alt uzayı olmadığını ispat ettik. Dolayısıyla, ilk sorunun da olumsuz bir cevabı vardır. Dördüncü bölümün üçüncü kısımında ikinci bölümde elde ettiğimiz sonuçlardan hareketle bazı ek bilgiler verdik. Mesela bu ailenin kararsız ε ilişkili eksponansiyel dizisi olan herhangi bir elemanı E için

- E uzayı herhangi bir belirgin sınırlı alt kümeye sahip değildir.
- E uzayının çapsal boyutu $\Delta(E)$ doğal topolojisine göre varilleri dolu değildir ve ultrabornolojik değildir.
- $\Delta(E) = \Lambda_1(\varepsilon)$ ve $\Delta(E)$ uzayından $\Lambda_1(\varepsilon)$ uzayına giden kapsama tasvirinin grafiği kapalı olmasına rağmen, bu kapsama tasviri sürekli değildir.

1. INTRODUCTION

Fréchet spaces are one of the leading class of locally convex spaces and include most of the important examples of non-normable locally convex spaces. In the fifties, by pioneering works of A. Grothendieck's, nuclear Fréchet spaces are introduced and it became one of the important sources of inspiration for research.

A. Grothendieck posed an important question about the existence of a basis in a nuclear Fréchet space. This question answered negatively by B. S. Mitiagin and N. M. Zobin, [3], that is, there exists a nuclear Fréchet space with no Schauder basis.

In the sixties, Dynin and Mitiagin gave a theorem which states that if a nuclear Fréchet space has a Schauder basis, then it is canonically isomorphic to a nuclear Köthe space. Then, it is crucial to investigate the structure of nuclear Fŕechet spaces in terms of Köthe spaces and power series spaces which constitute an important and well-studied class in the theory of Köthe spaces.

Another important problem is posed by A. Pelczynski [4]: Does every complemented subspace of a nuclear Köthe space have a basis? In 1975, B. S. Mitiagin and G. Henkin, [5], solved Pelczynski's problem positively for power series spaces of finite type. On the other hand, it took a long time to solve Pelczynski's problem for power series space of infinite type.

In 1989, E. Dubinsky and D. Vogt, [6], showed that if $\Lambda_\infty(\alpha)$ is tame, then every complemented subspace of $\Lambda_\infty(\alpha)$ has a basis. Also they proved that $\Lambda_\infty(\alpha)$ is tame if α is unstable. In 1990, A. Aytuna, J. Krone, T. Terzioğlu , [1], showed that a complemented subspace of an infinite type power series space $\Lambda_\infty(\alpha)$ with stable α, is indeed an infinite type power series space, therefore it has a basis. Finally, in 2018, A. K. Dronov and V. M. Kaplitzkii, [7] showed that every complemented subspace of a nuclear Köthe space E with a regular basis of type (d_1) has a basis so, every complemented subspace of $\Lambda_\infty(\alpha)$ has a basis.

As mentioned above, the study of whether the (complemented) subspaces of nuclear Fréchet spaces have a basis has been handled by several mathematicians. Also, various topological invariants were introduced to determine the structure of nuclear Fréchet spaces. For instance, D. Vogt and his school defined DN and Ω-type invariants and characterized entirely subspaces and quotient spaces of stable power series spaces in terms of these invariants and diametral dimension.

In this thesis, we are mainly interested in the class of nuclear Fréchet spaces with the properties $\underline{DN}$ and Ω which comprises many natural nuclear Fréchet spaces such as spaces of analytic functions, solutions of homogeneous elliptic linear partial differential operators.

A. Aytuna, J. Krone, T. Terzioğlu in [8] showed that if E is a nuclear Fréchet space with the properties $\underline{\text{DN}}$ and Ω, then there exists a sequence (unique up to equivalence) ε such that

$$\Delta(\Lambda_1(\varepsilon)) \subseteq \Delta(E) \subseteq \Delta(\Lambda_\infty(\varepsilon)) \tag{1.1}$$

where $\Delta(E)$ denotes the diametral dimension of E. The sequence ε was called associated exponent sequence of E.

Furthermore, A. Aytuna, J. Krone, T. Terzioğlu showed that if E is a nuclear Fréchet space with the properties $\underline{\text{DN}}$ and Ω, possessing stable associated exponent sequence ε and $\Delta(E) = \Delta(\Lambda_\infty(\varepsilon))$, then E has a complemented subspace which is isomorphic to infinite type power series space $\Lambda_\infty(\varepsilon)$.

In this thesis, we deal with the other extreme, namely, the main question of this thesis is:

Question 1.0.1 *Let E be a nuclear Fréchet space with the properties* $\underline{DN}$ *and* Ω *and* ε *be the associated exponent sequence of E. Is there a (complemented) subspace of E which is isomorphic to* $\Lambda_1(\varepsilon)$ *if* $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$*?*

This problem led us to examine the relationship between the diametral dimension and the other invariants. The most appropriate topological invariants for comparison with the diametral dimension is the approximate diametral dimension.

By using the same calculation as in [8], for a nuclear Fréchet space E with properties $\underline{DN}$ and Ω, it is easy to show

$$\delta(\Lambda_\infty(\varepsilon)) \subseteq \delta(E) \subseteq \delta(\Lambda_1(\varepsilon)) \tag{1.2}$$

for approximate diametral dimension $\delta(E)$ of E. Then, we always have

$\Delta(\Lambda_1(\varepsilon)) \subseteq \Delta(E)$ and $\delta(E) \subseteq \delta(\Lambda_1(\varepsilon))$ for a nuclear Fréchet space E with properties $\underline{DN}$ and Ω. If we assumed that Question 1.0.1 have an affirmative answer, that is, $\Lambda_1(\varepsilon)$ is isomorphic to a complemented subspace of E provided $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$, then this would give us $\delta(\Lambda_1(\varepsilon)) \subseteq \delta(E)$ and we would find $\delta(E) = \delta(\Lambda_1(\varepsilon))$. Therefore, the condition $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$ would give us the equality $\delta(E) = \delta(\Lambda_1(\varepsilon))$. This leads to ask the following question:

Question 1.0.2 *Let E be a nuclear Fréchet space with the properties $\underline{DN}$ and Ω. If diametral dimension of E coincides with that of a power series space, then does this imply that the approximate diametral dimension also do the same and vice versa?*

This thesis is mainly concerned with these questions.

In the first chapter, some significant studies in the theory of nuclear Fréchet spaces are mentioned and the aim of this thesis is given. In the second chapter, we give preliminary materials and essential theorems.

In the third chapter, we showed that Question 1.0.2 has an affirmative answer when power series space is of infinite type. Then we searched an answer for the Question 1.0.2 in the finite type case and, in this regard, we first prove that the condition $\delta(E) = \delta(\Lambda_1(\varepsilon))$ always implies $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$. For other direction, the existence of a prominent bounded subset in the nuclear Fréchet space E plays a decisive role for the answer of Question 1.0.2. Among other things, we prove that $\delta(E) = \delta(\Lambda_1(\varepsilon))$ if and only if E has a prominent bounded set and $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$.

In the first section of the fourth chapter, we showed that a regular nuclear Köthe space with the properties $\underline{DN}$ and Ω is a power series space if its diametral dimension coincides with that of a power series space of infinite type or its approximate diametral dimension coincides with that of a power series space of finite type.

In the second section of the fourth chapter, we constructed a family $\mathcal{K}$ of nuclear Köthe spaces $K(a_{k,n})$ with the properties $\underline{DN}$ and Ω. First we showed that for an element of the family of $\mathcal{K}$ which is parameterized by a stable sequence α,

$\Delta(K(a_{k,n})) = \Delta(\Lambda_1(\alpha))$ and $\delta(K(a_{k,n})) = \delta(\Lambda_1(\alpha))$. Second, we proved that for an element of the family of $\mathcal{K}$ which is parameterized by an unstable sequence α, $\Delta(K(a_{k,n})) = \Delta(\Lambda_1(\varepsilon))$ and $\delta(K(a_{k,n})) \neq \delta(\Lambda_1(\varepsilon))$ for its associated exponent sequence ε. This showed that the second question has a negative answer for power series space of finite type. Furthermore, we proved in Theorem 4.3.1 that the first question has a negative answer, that is, $\Lambda_1(\varepsilon)$ is not isomorphic to any subspace of these Köthe spaces $K(a_{k,n})$, let alone is isomorphic to a complemented subspace, though the condition $\Delta(K(a_{k,n})) = \Delta(\Lambda_1(\varepsilon))$ is satisfied. In the third section of fourth chapter, motivated by our finding in the third section, we compiled some additional information, for instance, for an element E of the family $\mathcal{K}$ parameterized by an unstable sequence,

- E does not have a prominent bounded set.
- $\Delta(E)$, with respect to the canonical topology, is neither barrelled nor ultrabornological.
- Although the equality $\Delta(E) = \Lambda_1(\varepsilon)$ is satisfied and the canonical imbedding from $\Delta(E)$ into $\Lambda_1(\varepsilon)$ has a closed graph, the canonical imbedding from $\Delta(E)$ into $\Lambda_1(\varepsilon)$ is not continuous.

2. PRELIMINARIES

In this section, after establishing terminology and notation, we collect some basic facts and definitions that are needed them in the sequel.

We will use the standard terminology and notation of [9] and [10]. A complete Hausdorff locally convex space E whose topology defined by countable fundamental system of seminorms $(\|.\|_k)_{k\in\mathbb{N}}$ is called a *Fréchet space*. Without loss of generality, we will assume the sequence $(\|.\|_k)_{k\in\mathbb{N}}$ is increasing. For each $k \in \mathbb{N}$, $E_k := \widehat{(E/\ker\|.\|_k, \|.\|_k)}$ is called the local Banach space with respect to the seminorm $\|.\|_k$ and we denote the closed unit ball of E_k by U_k. Since $\ker\|.\|_{k+1} \subseteq \ker\|.\|_k$ for all $k \in \mathbb{N}$, there exists a natural continuous map $i^k_{k+1} : E_{k+1} \to E_k$ which is referred to as *linking map*. Then, a Fréchet space E can be considered as a projective limit of the projective system $\left(E_k, i^k_{k+1}\right)_{k\in\mathbb{N}}$.

Nuclear locally convex spaces were defined by A. Grothendieck in [11]. It is generally accepted by many mathematicians that his definition is not practical to check whether a given locally convex space is nuclear or not. In consequence, several mathematicians reformulated the definition of nuclearity in terms of nuclear maps, Hilbert-Schmidt maps, diametral dimension, etc. In this thesis, we call a *nuclear Fréchet space E* as a Fréchet space that admits a representation as the projective limit of a sequence of separable Hilbert spaces E_k with Hilbert-Schmidt linking maps i^k_{k+1}.

2.1 Diametral Dimension and Approximate Diametral Dimension

For a Fréchet space E, we will denote the class of all neighborhoods of zero in E and the class of all bounded sets in E by $\mathscr{U}(E)$ and $\mathscr{B}(E)$, respectively. If U and V are absolutely convex sets of E and U absorbs V, that is, $V \subseteq CU$ for some $C > 0$, and L is a subspace of E, then we set;

$$\delta(V,U,L) = \inf\{t > 0 : V \subseteq tU + L\}. \tag{2.1}$$

The n^{th} *Kolmogorov diameter* of V with respect to U is defined as;

$$d_n(V,U) = \inf\{\delta(V,U,L) : \dim L \leq n\} \qquad n = 0,1,2,... \tag{2.2}$$

Here is a list of some useful properties of Kolmogorov diameters: for details see [12, Pg. 208, Proposition 1 and Pg. 209, Corollary 5]

Proposition 2.1.1 *Let E be a Fréchet space and U and V be two absolutely convex sets such that U absorbs V. Then, for every $n = 0,1,2,...$*

- $d_{n+1}(V,U) \leq d_n(V,U)$.
- *If $V_1 \subseteq V$ and $U \subset U_1$, then $d_n(V_1,U_1) \leq d_n(V,U)$.*
- $d_n(\lambda V, \beta U) = \dfrac{\lambda}{\beta} d_n(V,U)$ *for all $\lambda, \beta > 0$.*
- *V is precompact with respect to U if and only if $\lim_{n\to\infty} d_n(V,U) = 0$.*

Definition 2.1.2 *The diametral dimension of E is defined as*

$$\begin{aligned}\Delta(E) &= \left\{(t_n)_{n\in\mathbb{N}} : \forall\, U \in \mathscr{U}(E)\ \exists\, V \in \mathscr{U}(E)\ \lim_{n\to\infty} t_n d_n(V,U) = 0\right\} \\ &= \bigcap_{U\in\mathscr{U}(E)} \ \bigcup_{V\in\mathscr{U}(E)} \Delta(V,U)\end{aligned} \tag{2.3}$$

where $\Delta(V,U) = \left\{(t_n)_{n\in\mathbb{N}} : \lim_{n\to\infty} t_n d_n(V,U) = 0\right\}$.

Let $U_1 \supset U_2 \supset \cdots \supset U_p \supset \cdots$ be a base of neighborhoods of zero of Fréchet space E. The diametral dimension of E can be represented as

$$\Delta(E) = \left\{(t_n)_{n\in\mathbb{N}} : \forall p \in \mathbb{N}\ \ \exists\, q > p\ \ \lim_{n\to\infty} t_n d_n\left(U_q, U_p\right) = 0\right\}. \tag{2.4}$$

Demeulenaere et al. [13] showed that the diametral dimension of a nuclear Fréchet space can also be represented as

$$\Delta(E) = \left\{(t_n)_{n\in\mathbb{N}} : \forall\, p \in \mathbb{N}\ \ \exists\, q > p\ \ \sup_{n\in\mathbb{N}} |t_n|\, d_n\left(U_q, U_p\right) < +\infty\right\}. \tag{2.5}$$

Definition 2.1.3 *The approximate diametral dimension of a Fréchet space E is defined as*

$$\delta(E) = \left\{ (t_n)_{n\in\mathbb{N}} : \exists\, U \in \mathscr{U}(E)\ \exists\, B \in \mathscr{B}(E)\ \lim_{n\to\infty} \frac{t_n}{d_n(B,U)} = 0 \right\} = \bigcup_{U\in\mathscr{U}(E)} \bigcup_{B\in\mathscr{B}(E)} \delta(B,U) \tag{2.6}$$

where $\delta(B,U) = \left\{ (t_n)_{n\in\mathbb{N}} : \lim_{n\to\infty} \frac{t_n}{d_n(B,U)} = 0 \right\}$.

It follows from Proposition 6.6.5 of [14] that for a Fréchet space E with the base of neighborhoods $U_1 \supset U_2 \supset \cdots \supset U_p \supset \cdots$, the approximate diametral dimension can be represented as;

$$\delta(E) = \left\{ (t_n)_{n\in\mathbb{N}} : \exists p \in \mathbb{N}\ \forall\, q > p\ \lim_{n\to\infty} \frac{t_n}{d_n\left(U_q, U_p\right)} = 0 \right\}. \tag{2.7}$$

Let E and G be two Fréchet spaces and U and V be absolutely convex subsets of E such that $V \subseteq rU$ for some $r > 0$. If there is a linear map $T : E \to G$, then for all $n \in \mathbb{N}$

$$d_n(T(V), T(U)) \leq d_n(V,U) \tag{2.8}$$

holds and so we have the following proposition:

Proposition 2.1.4 *Let E be a Fréchet space and F be a subspace or a qoutient of E. Then,*

- $\Delta(E) \subseteq \Delta(F)$.
- $\delta(F) \subseteq \delta(E)$.

Hence the diametral dimension and the approximate diametral dimension are linear topological invariants.

Proof. [14, Proposition 6.6.7 and Proposition 6.6.25] □

The concept of the approximative dimension of a linear metric space which is based on ε-capacity of compact sets in the space was introduced by Kolmogorov and Pelcyznski, see also [15], [16] and [14]. The relation between invariants introduced above and

ε-capacity of compact sets in the space was discovered by Mityagin, see [17] and [18]. Among other things, Mityagin conducted a detailed study of these invariants and used them characterize nuclear locally convex space. The concept of approximate diametral dimension as stated above was given and studied by Bessaga, Pelczynski and Rolewicz, [19].

For the proof of these and for additional properties of the diametral dimension/approximate diametral dimension, we refer the reader to [18], [19], [20, Chapter 9], [14, Chapter 6.6], and [21] .

The properties of the canonical topology on diametral dimension of a nuclear Fréchet space:

Let E be a nuclear Fréchet space with the increasing systems of seminorms $(\|.\|_k)_{k\in N}$. Then the diametral dimension

$$\begin{aligned}\Delta(E) &= \left\{(t_n)_{n\in\mathbb{N}} : \forall\, p\in\mathbb{N}\ \exists q>p\ \lim_{n\to\infty} t_n d_n\left(U_q,U_p\right)=0\right\}\\ &= \bigcap_{p\in\mathbb{N}}\ \bigcup_{q>p}\Delta\left(U_q,U_p\right)\end{aligned} \tag{2.9}$$

is the projective limit of inductive limits of Banach spaces $\Delta\left(U_q,U_p\right)$ with the norm $\|(t_n)_n\| = \sup_{n\in\mathbb{N}} |t_n|\, d_n(U_q,U_p)$. Hence $\Delta(E)$ is a topological vector space with respect to that topology which will be called *the canonical topology*.

In the fourth chapter, we will give some results provided that the canonical topology of $\Delta(E)$ is barrelled. So, we take a closer look the canonical topology of $\Delta(E)$ and Theorem 2.1.5 gives a condition to the barrelledness of the canonical topology of $\Delta(E)$. Before, we give some definitions that are needed in the sequel.

Let X be a locally compact σ-compact topological space. The space of continuous functions on X will be denoted by $C(X)$. Recall that a function $h: X\to\mathbb{R}$ is said to vanish at infinity on X if for every $\varepsilon>0$ there is a compact set K in X such that $|h(x)|<\varepsilon$ for every $x\in X-K$. For a strictly positive function f, we define

$$C_{(f,0)}(X) = \{g\in C(X) : f\,|g| \text{ vanishes at infinity on } X\}. \tag{2.10}$$

Let $\mathscr{A} = (f_{m,n})_{m,n\in\mathbb{N}}$ be a double indexed sequnce of strictly increasing positive functions of $C(X)$ satisyfing

$$f_{m,n+1} \leq f_{m,n} \leq f_{m+1,n} \tag{2.11}$$

for every $m,n \in \mathbb{N}$. This gives continuous inclusions $C_{(f_{m,n},0)} \subseteq C_{(f_{m,n+1},0)}$ for each $n \in \mathbb{N}$ and $C_{(f_{m+1,n},0)} \subseteq C_{(f_{m,n+1},0)}$ for each $n \in \mathbb{N}$.
Hence, we can define the following weighted inductive limits

$$(\mathscr{A}_m)_0(X) := \operatorname{ind}_{n\in\mathbb{N}} C_{(f_{m,n},0)}(X) \tag{2.12}$$

for each $m \in \mathbb{N}$. Since the inclusion $(\mathscr{A}_{m+1})_0(X) \subseteq (\mathscr{A}_m)_0(X)$ is continuous for every $m \in \mathbb{N}$, we can define the projective spectra of (LB)-spaces $(\mathscr{A}_m)(X)_0$ with inclusions as linking maps

$$(\mathscr{A}\mathscr{C})_0(X) := \operatorname{proj}_{m\in\mathbb{N}} (\mathscr{A}_m)_0(X). \tag{2.13}$$

The space $(\mathscr{A}\mathscr{C})_0(X)$ is called a *weighted (PLB)-space of continuous functions*.

Now we show that $\Delta(E)$ is a weighted PLB-space of continuous functions. Indeed, for fixed $p,q \in \mathbb{N}$, $\left\{d_n\left(U_q,U_p\right)\right\}_{n\in\mathbb{N}}$ can be identified with the continuous function $f_{p,q} : \mathbb{N} \to \mathbb{R}$ defined by $f_{p,q}(n) = d_n\left(U_q,U_p\right)$ where $\mathbb{N}$ is equipped with discrete topology which is locally compact and σ-compact topological space. Then, for each p,q,

$$\begin{aligned}\Delta\left(U_q,U_p\right) &= \left\{(t_n) : \lim_{n\to\infty} t_n d_n\left(U_q,U_p\right) = 0\right\} \\ &= \left\{(t_n) \in C(\mathbb{N}) : t_n f_{p,q}(n) \ \text{ vanishes at infinity on } \mathbb{N}\right\} \\ &= C_{(f_{p,q},0)}(\mathbb{N}).\end{aligned} \tag{2.14}$$

Since $d_n\left(U_{q+1},U_p\right) \leq d_n\left(U_q,U_p\right) \leq d_n\left(U_q,U_{p+1}\right)$ for all $n \in \mathbb{N}$, the matrix $\left(f_{p,q}\right)_{p,q\in\mathbb{N}} = \left(d_{(\cdot)}\left(U_q,U_p\right)\right)_{p,q\in\mathbb{N}}$ of a double sequence of weights increases with respect to the first indices and decreases with respect to the second indices. Therefore, $\Delta(E)$ is of the form weighted PLB-space of continuous functions $\operatorname{proj}_p \operatorname{ind}_{q>p} C_{(f_{p,q},0)}(\mathbb{N})$, as desired.
The topological properties of weighted PLB-spaces of continuous functions were studied in [22]. In particular, the following theorem gives an information about the canonical topology of diametral dimension $\Delta(E)$ which is a direct consequence of [22, Theorem 3.7].

Theorem 2.1.5 *Let E be a Fréchet space. The following conditions are equivalent:*

1. *$\Delta(E)$ is ultrabornological with respect to the canonical topology.*

2. *$\Delta(E)$ is barrelled with respect to the canonical topology.*

3. *$\Delta(E)$ satisfies (wQ)-condition*

   (wQ): $\forall N \ \exists \ M,n \ \forall K,m, \ \exists \ k,S>0:$

$$\min\left(d_i\left(U_n,U_N\right),d_i\left(U_k,U_K\right)\right) \leq S\, d_i\left(U_m,U_M\right) \qquad \forall i \in \mathbb{N}. \quad (2.15)$$

2.2 Köthe Spaces and Power Series Spaces

A matrix $\left(a_{k,n}\right)_{k,n\in\mathbb{N}}$ of non-negative numbers is called a *Köthe matrix* if it is satisfies that for each $k\in\mathbb{N}$ there exists an $n\in\mathbb{N}$ with $a_{k,n}>0$ and $a_{k,n}\leq a_{k,n+1}$ for all $k,n\in\mathbb{N}$. For a Köthe matrix $\left(a_{k,n}\right)_{k,n\in\mathbb{N}}$,

$$K\left(a_{k,n}\right) = \left\{ x=(x_n): \|x\|_k := \sum_{n=1}^{\infty} |x_n|\, a_{k,n} < +\infty \text{ for all } k\in\mathbb{N} \right\} \qquad (2.16)$$

is called a *Köthe space*. Every Köthe space is a Fréchet space given by the semi-norms in its definition.

Dynin-Mitiagin Theorem [9, Theorem 28.12] states that if a nuclear Fréchet space E with the sequence of seminorms $(\|.\|_k)_{k\in\mathbb{N}}$ has a Schauder basis $(e_n)_{n\in\mathbb{N}}$, then it is canonically isomorphic to a nuclear Köthe space defined by the matrix $(\|e_n\|_k)_{k,n\in\mathbb{N}}$. Therefore, it is important to understand the structure of nuclear Köthe spaces in the theory of nuclear Fréchet spaces.

Nuclearity of a Köthe space was characterized as follows:

Theorem 2.2.1 (Grothendieck-Pietsch Criterion) *$K\left(a_{kn}\right)$ is nuclear Köthe space if and only if for every $k\in\mathbb{N}$, there exists a $l>k$ so that $\sum_{n=1}^{\infty}\frac{a_{k,n}}{a_{l,n}}<\infty$.*

Proof. [9, Theorem 28.15]. □

T. Terzioğlu gave an estimation for n^{th}-Kolmogorov diameters of a Köthe space $K(a_{k,n})$ by using the matrix $(a_{k,n})_{k,n\in\mathbb{N}}$.

Proposition 2.2.2 *Let $K(a_{k,n})$ be a Köthe space and fixed $n \in \mathbb{N}$. Assume $J \subset \mathbb{N}$ with $|J| = n+1$ and $I \subset \mathbb{N}$ with $|I| \leq n$. Then for every p and $q > p$,*

$$\inf\left\{\frac{a_{p,i}}{a_{q,i}} : i \in J\right\} \leq d_n(U_q, U_p) \leq \sup\left\{\frac{a_{p,i}}{a_{q,i}} : i \notin I\right\}. \tag{2.17}$$

Proof. [23, Proposition 1]. □

Definition 2.2.3 *A Köthe space $K(a_{k,n})$ is called regular if the sequence $\left(\frac{a_{p,n}}{a_{q,n}}\right)_{n\in\mathbb{N}}$ is non-increasing for every p and $q > p$.*

Remark 2.2.4 *In the light of the above proposition, we conclude that for any regular Köthe space $K(a_{p,n})$, the n^{th}-Kolmogorov diameter is $d_n\left(U_q, U_p\right) = \frac{a_{p,n+1}}{a_{q,n+1}}$. If, on the other hand, $K(a_{p,n})$ is not regular, then, one can find Kolmogorov diameters by rewriting the sequence $\left(\frac{a_{p,n}}{a_{q,n}}\right)_{n\in\mathbb{N}}$ with terms in a descending order so that the n^{th}-Kolmogorov diameter of $K(a_{p,n})$ is nothing but the $n+1-th$ term of this descending sequence.*

Power series spaces form an important family of Fréchet spaces and they play a significant role in this thesis, for a comprehensive survey see [24]. Let $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ be a non-negative increasing sequence with $\lim_{n\to\infty} \alpha_n = +\infty$. A power series space of finite type is defined by

$$\Lambda_1(\alpha) := \left\{ x = (x_n)_{n\in\mathbb{N}} : \|x\|_k := \sum_{n=1}^{\infty} |x_n| e^{-\frac{1}{k}\alpha_n} < +\infty \text{ for all } k \in \mathbb{N} \right\} \tag{2.18}$$

and a power series space of infinite type is defined by

$$\Lambda_\infty(\alpha) := \left\{ x = (x_n)_{n\in\mathbb{N}} : \|x\|_k := \sum_{n=1}^{\infty} |x_n| e^{k\alpha_n} < +\infty \text{ for all } k \in \mathbb{N} \right\}. \tag{2.19}$$

Power series spaces are actually Fréchet spaces equipped with the seminorms in its definitions. The nuclearity of a power series space of finite type $\Lambda_1(\alpha)$ and of infinite type $\Lambda_\infty(\alpha)$ are equivalent to the conditions $\lim_{n\to\infty} \frac{\ln(n)}{\alpha_n} = 0$ and $\sup_{n\in\mathbb{N}} \frac{\ln(n)}{\alpha_n} < +\infty$, respectively.

Definition 2.2.5 *An exponent sequence α is called* ***finitely nuclear*** *if $\Lambda_1(\alpha)$ is nuclear.*

Diametral dimension and approximate diametral dimension of power series spaces are

$$\Delta(\Lambda_1(\alpha)) = \Lambda_1(\alpha), \qquad \Delta(\Lambda_\infty(\alpha)) = \Lambda_\infty(\alpha)' \tag{2.20}$$

and

$$\delta(\Lambda_1(\alpha)) = \Lambda_1(\alpha)', \qquad \delta(\Lambda_\infty(\alpha)) = \Lambda_\infty(\alpha), \tag{2.21}$$

for details see [19] and [18].

Definition 2.2.6 *An exponent sequence α is called*

$$\begin{array}{lll} \textit{stable} & \textit{if} & \sup_{n\in\mathbb{N}} \frac{\alpha_{2n}}{\alpha_n} < +\infty, \\ \textit{weakly-stable} & \textit{if} & \sup_{n\in\mathbb{N}} \frac{\alpha_{n+1}}{\alpha_n} < +\infty, \\ \textit{unstable} & \textit{if} & \lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = +\infty. \end{array}$$

It follows that α is stable, respectively weakly-stable, if and only if $E \cong E \times E$, respectively, $E \cong E \times \mathbb{K}$ where $E = \Lambda_r(\alpha)$ for $r = 1$ or $r = \infty$, for proofs see [25].

2.3 Dragilev's Invariants, Bessaga's Invariants and Vogt's DN and Ω-Type Invariants

The following topological invariants were defined by M. M. Dragilev, [26] and E. Dubinsky, [27] to be used in the structure theory of Köthe spaces.

A Köthe space $K(a_{k,n})$ is called of type $\mathbf{d_i}$,

$$(\mathbf{d_0}): \quad \forall\, k,n \qquad \frac{a_{k+1,n}}{a_{k,n}} \leq \frac{a_{k+1,n+1}}{a_{k,n+1}}$$

$$(\mathbf{d_1}): \quad \exists\, k \ \ \forall\, j \ \ \exists\, l \qquad \sup_{n\in\mathbb{N}} \frac{a_{j,n}^2}{a_{k,n}\, a_{l,n}} < +\infty$$

$$(\mathbf{d_2}): \quad \forall\, k \ \ \exists\, j \ \ \forall\, l \qquad \sup_{n\in\mathbb{N}} \frac{a_{k,n}\, a_{l,n}}{a_{j,n}^2} < +\infty$$

$$(\mathbf{d_3}): \quad \forall\, k,n \qquad \frac{a_{k+1,n}}{a_{k,n}} \leq \frac{a_{k+2,n}}{a_{k+1,n}}$$

$$(\mathbf{d_4}):\ \forall\, k,n \qquad \frac{a_{k+1,n}}{a_{k,n}} \geq \frac{a_{k+2,n}}{a_{k+1,n}}$$

$$(\mathbf{d_5}):\ \exists\, M \geq 1\ \forall\, k,n \qquad \frac{a_{k+1,n}}{a_{k,n}} \leq \left(\frac{a_{k+2,n}}{a_{k+1,n}}\right)^M$$

if the corresponding conditions holds. One can easily observe that d_0 is equivalent to regularity, see Definition 2.2.3.

Motivated by M. M. Dragilev's work, C. Bessaga, [28], introduced the following conditions (D_1) and (D_2) by using Kolmogorov diameters:

$$(\mathbf{D_1}):\ \exists\, U\ \ \forall\, W\ \ \exists\, V \qquad \lim_{n\to+\infty} \frac{d_n(V,W)}{d_n(W,U)} = 0$$

$$(\mathbf{D_2}):\ \forall\, U\ \ \exists\, W\ \ \forall\, V \qquad \lim_{n\to+\infty} \frac{d_n(W,U)}{d_n(V,W)} = 0$$

Note that if E is a regular nuclear Köthe space, then the conditions $(\mathbf{D_1})$ and $(\mathbf{D_2})$ are equivalent the condition $(\mathbf{d_1})$ and $(\mathbf{d_2})$, respectively, since $d_n(U_q,U_p) = \dfrac{a_{p,n+1}}{a_{q,n+1}}$ for every p, $q > p$ and $n \in \mathbb{N}$.

Another basis-free formulations of properties (d_i), $i = 0,\dots,5$ were given by D. Vogt and his school in [29] and [30], as follows: A Fréchet space $(E, \|.\|_k)_{k\in\mathbb{N}}$ is said to have the property:

(DN) $\exists\, k\ \ \forall\, j\ \ \exists\, l,\ C > 0 \qquad \|x\|_j^2 \leq C\, \|x\|_k\, \|x\|_l \qquad \forall\, x \in E$

($\underline{\text{DN}}$) $\exists\, k\ \ \forall\, j\ \ \exists\, l,\ C > 0,\ 0 < \lambda < 1 \qquad \|x\|_j \leq C\, \|x\|_k^{\lambda}\, \|x\|_l^{1-\lambda} \qquad \forall\, x \in E$

(Ω) $\forall\, p\ \ \exists\, q\ \ \forall\, k\ \ \exists\, C > 0,\ 0 < \tau < 1 \qquad \|y\|_q^* \leq C \|y\|_p^{*\,1-\theta} \|y\|_k^{*\,\theta} \qquad \forall\, y \in E'$

($\overline{\Omega}$) $\forall\, p\ \ \exists\, q\ \ \forall\, k\ \ \exists\, C > 0 \qquad \|y\|_q^{*2} \leq C \|y\|_p^* \, \|y\|_k^* \qquad \forall\, y \in E'$

where $\|y\|_k^* := \sup\{|y(x)| : \|x\|_k \leq 1\} \in \mathbb{R} \cup \{+\infty\}$ is the gauge functional of the polar U_k° for $U_k = \{x \in E : \|x\|_k \leq 1\}$.

These are independent of the choice of the seminorm-system, they are topological invariants of the Fréchet space E.

The DN-types invariants are inherited by subspaces and the Ω-type invariants are inherited by quotient spaces, see [9, Chapter 29].

If E is a Köthe space $K(a_{k,n})$, then d_1 and d_3 implies (DN), d_2 implies $\overline{\Omega}$, d_4 implies Ω and d_5 implies $(\underline{DN})$. Moreover, the reverse of these implications hold true for either the matrix $(a_{k,n})_{k,n\in\mathbb{N}}$ or some matrix $(b_{k,n})_{k,n\in\mathbb{N}}$ that is equivalent to $(a_{k,n})_{k,n\in\mathbb{N}}$, see [31]. Further, in [31], D. Vogt characterized Ω for Köthe spaces in terms of Köthe matrix as follows:

Proposition 2.3.1 *A Köthe space $K\left(a_{k,n}\right)$ has the property Ω if and only if the condition*

$$\forall\, p \;\; \exists\, q \;\; \forall\, k \;\; \exists\, j>0,\; C>0 \qquad \left(a_{p,n}\right)^{j} a_{k,n} \leq C\left(a_{q,n}\right)^{j+1} \qquad \forall n\in\mathbb{N} \tag{2.22}$$

is satisfied.

Proof. [31, Proposition 5.3]. □

By using the technique in [31, 5. 1 Proposition], one can easily obtain the following:

Proposition 2.3.2 *A Köthe space $K\left(a_{k,n}\right)$ has the property $\underline{DN}$ if and only if the condition*

$$\exists\, p_0 \;\; \forall\, p \;\; \exists\, q \;\; \exists\, 0<\lambda<1,\; C>0 \qquad a_{p,n} \leq C\left(a_{p_0,n}\right)^{\lambda}\left(a_{q,n}\right)^{1-\lambda} \qquad \forall n\in\mathbb{N} \tag{2.23}$$

is satisfied.

A power series space of finite type $\Lambda_1(\alpha)$ has the properties $\underline{DN}$ and $\overline{\Omega}$, a power series space of infinite type $\Lambda_\infty(\alpha)$ has the properties DN and Ω, see [9, Ch. 29].

Most of the spaces appearing in the theory of nuclear Fréchet spaces have the properties $\underline{DN}$ and Ω. For example, the space $O(M)$ of analytic functions on a Stein manifold M with the topology of uniform convergence on compact subsets of M has the properties $\underline{DN}$ and Ω, see [1], [32] and references therein.

$\underline{\mathrm{DN}}-\Omega$ Compatible Semi-norm System

Let E be a nuclear Fréchet space with the properties $\underline{DN}$ and Ω. We can assume the topology of E defined by an increasing sequence $(\|.\|_k)_{k\in\mathbb{N}}$ of Hilbertian semi-norms satisfying

$$\underline{DN}: \quad \forall\, k \;\; \exists\, C>0,\; 0<\lambda<1 \qquad \|x\|_{k+1} \leq C\, \|x\|_k^{\lambda}\; \|x\|_{k+2}^{1-\lambda} \qquad \forall\, x\in E$$

and

$$\Omega: \quad \forall\, k \;\; \exists\, C>0,\; 0<\tau<1 \qquad \|y\|^*_{k+1} \leq C(\|y\|^*_k)^{1-\theta}(\|y\|^*_{k+2})^{\theta} \qquad \forall\, y\in E'.$$

The nuclear Fréchet space with the increasing sequence $(\|.\|_k)_{k\in\mathbb{N}}$ of Hilbertian semi-norms satisfying the conditions $\underline{DN}$ and Ω as indicated above is called $\underline{DN}-\Omega$ *Compatible Semi-norm System*.

In [33], T. Terzioğlu showed that DN- and Ω-type invariants are related to some conditions on Kolmogorov diameters.

Proposition 2.3.3 *Let E be a nuclear Fréchet space. If E has $\underline{DN}$-property, then the condition*

$$\exists k \quad \forall j \quad \exists l, C>0, 0<\lambda<1 \qquad d_n(U_l,U_k) \leq C\, d_n(U_j,U_k)^{\frac{1}{1-\lambda}} \qquad \forall n\in\mathbb{N} \quad (2.24)$$

is satisfied. If E has Ω-property, then the condition

$$\forall p \quad \exists q \quad \forall k \quad \exists C>0, 0<j<1 \qquad d_n(U_q,U_p) \leq C\, d_n(U_k,U_p)^{j} \qquad \forall n\in\mathbb{N} \quad (2.25)$$

is satisfied.

Proof. [33, Page 4 and 7]. □

As a direct consequence of the above proposition, one can obtain the following:

Corollary 2.3.4 *Let E be a nuclear Fréchet space with $\underline{DN}-\Omega$ compatible semi-norm system $(\|.\|_k)_{k\in\mathbb{N}}$. The following conditions*

$$\forall k \quad \exists C>0, 0<\lambda<1 \qquad d_n(U_{k+2},U_k) \leq C\, d_n(U_{k+1},U_k)^{\frac{1}{1-\lambda}} \qquad \forall n\in\mathbb{N} \quad (2.26)$$

and

$$\forall p,k \quad \exists C>0, 0<j<1 \qquad d_n(U_{p+1},U_p) \leq C\, d_n(U_k,U_p)^{j} \qquad \forall n\in\mathbb{N} \quad (2.27)$$

are satisfied.

2.4 Associated Exponent Sequence and Power Series Subspaces

We end this section by recalling the following results which gives a relation between th diametral dimension/approximate diametral dimension of a nuclear Fréchet spaces with the properties $\underline{DN}$, Ω and that of a power series spaces $\Lambda_1(\varepsilon)$ and $\Lambda_\infty(\varepsilon)$ for some special exponent sequence ε.

Proposition 2.4.1 *Let E be a nuclear Fréchet space with the properties $\underline{DN}$ and Ω. There exists an exponent sequence (unique up to equivalence) $\varepsilon = (\varepsilon_n)$ satisfying:*

$$\Delta(\Lambda_1(\varepsilon)) \subseteq \Delta(E) \subseteq \Delta(\Lambda_\infty(\varepsilon)). \tag{2.28}$$

Furthermore, $\Lambda_1(\alpha) \subseteq \Delta(E)$ implies $\Lambda_1(\alpha) \subseteq \Lambda_1(\varepsilon)$ and $\Delta(E) \subseteq \Lambda'_\infty(\alpha)$ implies $\Lambda'_\infty(\varepsilon) \subseteq \Lambda'_\infty(\alpha)$.

Proof. [8, Proposition 1.1]. □

Definition 2.4.2 *Let E be a nuclear Fréchet space with the properties $\underline{DN}$ and Ω. The sequence ε (unique up to equivalence) in the above proposition is called the* ***associated exponent sequence*** *of E in [8].*

We note that $\Lambda_\infty(\varepsilon)$ is always nuclear provided E is nuclear, but it may happen that $\Lambda_1(\varepsilon)$ is not nuclear. For example, if we take the space of rapidly decreasing sequence $s = \Lambda_\infty(\ln(n))$, the associated exponent sequence of s is $(\ln(n))_{n\in\mathbb{N}}$ and $\Lambda_1(\ln(n))$ is not nuclear.

In the proof of the above proposition, A. Aytuna, J. Krone and T. Terzioğlu showed that there exists an exponent sequence (unique up to equivalence) (ε_n) such that for each $p \in \mathbb{N}$ and $q > p$, there exist $C_1, C_2 > 0$ and $a_1, a_2 > 0$ satisfying

$$C_1\, e^{-a_1\, \varepsilon_n} \leq d_n\left(U_q, U_p\right) \leq C_2\, e^{-a_2\, \varepsilon_n} \tag{2.29}$$

for all $n \in \mathbb{N}$. From this inequality, one can easily obtain

$$\delta(\Lambda_\infty(\varepsilon)) \subseteq \delta(E) \subseteq \delta(\Lambda_1(\varepsilon)). \tag{2.30}$$

In [8], A. Aytuna, J. Krone and T. Terzioğlu showed that for a d-dimensional Stein manifold M, the exponent sequence associated to the space $O(M)$ of analytic functions on M is the sequence $(n^{\frac{1}{d}})_{n\in\mathbb{N}}$. We note that the sequence $(n^{\frac{1}{d}})_{n\in\mathbb{N}}$ is stable and $\Lambda_1(n^{\frac{1}{d}})$ is nuclear. We also note that $\Lambda_\infty(n^{\frac{1}{d}})$ is isomorphic to the space $O(\mathbb{C}^d)$ and $\Lambda_1(n^{\frac{1}{d}})$ is isomorphic to the space $O(\Delta^d)$ where Δ^d denotes the unit polycylinder in $\mathbb{C}^d$, see [34] and references therein.

When it is necessary to explicitly state the associated exponent sequence ε of a nuclear Fréchet space E, we always work with $\underline{DN}$-Ω compatible semi-norm system on E and then, the associated exponent sequence ε of a nuclear Fréchet space E with $\underline{DN}$-Ω compatible semi-norm system will be taken as $\varepsilon = (-\log d_n(U_2,U_1))_{n\in\mathbb{N}}$ where U_i denotes the closed unit ball of the local Banach space E_i, $i = 1,2$, see [8, Pg. 128] .

For a nuclear Fréchet space E with the properties $\underline{DN}$ and Ω and the associated exponent sequence ε, concidence of the diametral dimension of E with that of power series spaces defined by ε form two extreme cases. In [2, Theorem 3.4], A. Aytuna showed that only extreme cases hold for the space $O(M)$ of analytic functions on M with dimension d by proving that either $\Delta(O(M)) = \Delta(\Lambda_\infty(n^{\frac{1}{d}}))$ or $\Delta(O(M)) = \Delta(\Lambda_1(n^{\frac{1}{d}}))$.

The extreme case $\Delta(E) = \Delta(\Lambda_\infty(\varepsilon))$ gives an information about a (complemented) subspace of a nuclear Fréchet space E with the properties $\underline{DN}$ and Ω and stable associated exponent sequence ε. In [8], A. Aytuna, J. Krone and T. Terzioğlu proved that a nuclear Fréchet space E with the properties $\underline{DN}$ and Ω contains a complemented copy of $\Lambda_\infty(\varepsilon)$ provided that $\Delta(E) = \Delta(\Lambda_\infty(\varepsilon))$ and ε is stable.

Theorem 2.4.3 *Let E be a nuclear Fréchet space with the properties $\underline{DN}$ and Ω and stable associated exponent sequence ε. If $\Delta(E) = \Delta(\Lambda_\infty(\varepsilon))$, then E has complemented subspace which is isomorphic to $\Lambda_\infty(\varepsilon)$.*

Proof. [8, Theorem 1.2]. □

There is another observation which contains some information about some subspaces of E without assuming the extreme $\Delta(E) = \Delta(\Lambda_\infty(\varepsilon))$. A. Aytuna, J. Krone and T. Terzioğlu, [8], proved that there is an imbedding from subspaces of $\Lambda_1(\varepsilon)$ with the property DN into E:

Theorem 2.4.4 *Let E be a nuclear Fréchet space with the properties $\underline{DN}$ and Ω and ε the stable associated exponent sequence with $\Lambda_1(\varepsilon)$ nuclear. If Y has property DN and is isomorphic to a subspace $\Lambda_1(\varepsilon)$, then Y is also isomorphic to a subspace of E.*

Proof. [8, Theorem 2.2]. □

As a consequence of the above theorem, we conclude that $\Lambda_\infty(\varepsilon)$ is isomorphic to a subspace of E if $\Lambda_\infty(\varepsilon)$ is isomorphic to a subspace $\Lambda_1(\varepsilon)$. It is well-known that $\Lambda_\infty(\varepsilon)$ is isomorphic to a subspace $\Lambda_1(\varepsilon)$ for stable ε, [35, 4.2 Theorem]. On the other hand, $\Lambda_\infty(\alpha)$ is not isomorphic to a subspace $\Lambda_1(\alpha)$ for an unstable α, [25, 3.3 Corollary]. Although there is no complete characterization of when $\Lambda_\infty(\alpha)$ is isomorphic to a subspace $\Lambda_1(\alpha)$, Z. Nurlu proved that if $\Lambda_\infty(\alpha)$ is isomorphic to a subspace $\Lambda_1(\alpha)$, then α is weakly stable, [36, Proposition 2.6]. But the reverse implication is not true since Z. Nurlu construct a weakly stable α so that $\Lambda_\infty(\alpha)$ is not isomorphic to a subspace $\Lambda_1(\alpha)$ see [36, Example 2.10].

On the other hand, there is no information for the other extreme $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$. This leads us to ask the Question 1.0.1 in Introduction.

3. THE PROPERTIES OF NUCLEAR FRÉCHET SPACES WHOSE DIAMETRAL AND/OR APPROXIMATE DIAMETRAL DIMENSION COINCIDES WITH THAT OF A POWER SERIES SPACE

The main purpose of this chapter is to give an answer to Question 1.0.2 in stated Introduction:

Question 1.0.2 *Let E be a nuclear Fréchet space with the properties* $\underline{DN}$ *and* Ω*. If diametral dimension of E coincides with that of a power series space, then does this imply that the approximate diametral dimension also do the same and vice versa?*

We first relate the Question 1.0.2 to Bessaga's D_1 and D_2-conditions. Then, we give a necessary and sufficient condition confirming Question 1.0.2. However, the answer of Question 1.0.2 in finite type case is negative as seen in the fourth chapter.

Throughout this chapter, we will assume that the sequence $(\|.\|_k)_{k\in\mathbb{N}}$ of semi-norms on a nuclear Fréchet spaces with the properties $\underline{DN}$ and Ω is $\underline{DN}-\Omega$ compatible system.

3.1 Results for the Case of Power Series Space of Infinite Type

The main result in this section is the following theorem which shows that Question 1.0.2 has an affirmative answer when the power series space is of infinite type.

Theorem 3.1.1 *Let E be a nuclear Fréchet space with properties* $\underline{DN}$ *and* Ω *and* ε *be the associated exponent sequence of E. Then* $\Delta(E)=\Delta(\Lambda_\infty(\varepsilon))$ *if and only if* $\delta(E)=\delta(\Lambda_\infty(\varepsilon))$.

For the proof of Theorem 3.1.1 we need the following Lemma characterizing coincidence of $\delta(E)$ with $\delta(\Lambda_\infty(\varepsilon))$, motivated by the following formula given by A. Aytuna in [2]:

Proposition 3.1.2 *Let E be a nuclear Fréchet space E with the properties* $\underline{DN}$, Ω *and associated exponent sequence* ε. *Then*

$$\delta(E) = \delta(\Lambda_1(\varepsilon)) \quad \Leftrightarrow \quad \inf_p \sup_{q \geq p} \limsup_{n \in \mathbb{N}} \frac{\varepsilon_n(p,q)}{\varepsilon_n} = 0 \tag{3.1}$$

where $\varepsilon_n(p,q) = -\log d_n(U_q, U_p)$.

Proof. [2, Corollary 1.10] □

The same characterization can be given for power series spaces of infinite type as follows:

Lemma 3.1.3 *Let E be a nuclear Fréchet space with properties* $\underline{DN}$ *and* Ω *and* $\varepsilon = (\varepsilon_n)_{n\in\mathbb{N}}$ *be the associated exponent sequence of E. Then*

$$\delta(E) = \delta(\Lambda_\infty(\varepsilon)) \quad \Leftrightarrow \quad \inf_{p\in\mathbb{N}} \sup_{q>p} \liminf_{n\in\mathbb{N}} \frac{\varepsilon_n(p,q)}{\varepsilon_n} = +\infty \tag{3.2}$$

where $\varepsilon_n(p,q) = -\log d_n(U_q, U_p)$.

Proof. The approximate diametral dimension $\delta(E)$ can be written as

$$\delta(E) = \bigcup_p \bigcap_{q \geq p} \delta_{pq} \tag{3.3}$$

where $\delta_{pq} = \left\{ (t_n)_{n\in\mathbb{N}} : \sup_{n\in\mathbb{N}} \frac{|t_n|}{d_n(U_q, U_p)} < +\infty \right\}$ is a Banach space with norms $|t_n|_{pq} = \sup_{n\in\mathbb{N}} \frac{|t_n|}{d_n(U_q, U_p)}$. Hence, the approximate diametral dimension can be equipped with the topological inductive limit of Fréchet spaces. Then, the approximate diametral dimension with this topology is barrelled. On the other hand, the inclusion $\delta(E) \subseteq \delta(\Lambda_\infty(\varepsilon)) = \Lambda_\infty(\varepsilon)$ gives us that the identity mapping $i : \delta(E) \to \Lambda_\infty(\varepsilon)$ has a closed graph. Since $\delta(E)$ is barrelled, by using Theorem 5 of [37], we conclude that the identity mapping is continuous. Therefore,

$$\delta(E)=\bigcup_p\bigcap_{q\geq p}\delta_{pq}\hookrightarrow\Lambda_\infty(\varepsilon)\ \text{ is continuous }\ \Leftrightarrow\ \forall p\quad \bigcap_{q\geq p}\delta_{pq}\hookrightarrow\Lambda_\infty(\varepsilon)\ \text{ is continuous}$$

$$\Leftrightarrow\ \forall p\ \ \forall R>1\ \ \exists q\geq p,\ \ C>0\ \ \sup_{n\in\mathbb{N}}|t_n|R^{\varepsilon_n}\leq C\sup_{n\in\mathbb{N}}\frac{|t_n|}{d_n\left(U_q,U_p\right)}\qquad\forall(t_n)\in\delta(E)$$

$$\Leftrightarrow\ \forall p\ \ \forall R>1\ \ \exists q\geq p, C>0\ \ R^{\varepsilon_n}\leq\frac{C}{d_n\left(U_q,U_p\right)}\qquad\forall n\in\mathbb{N}$$

$$\Leftrightarrow\ \forall p\ \ \forall R>1\ \ \ln R\leq\sup_{q\geq p}\liminf_{n\in\mathbb{N}}\frac{\varepsilon_n(p,q)}{\varepsilon_n}$$

$$\Leftrightarrow\ \forall p\ \ \sup_{q\geq p}\liminf_{n\in\mathbb{N}}\frac{\varepsilon_n(p,q)}{\varepsilon_n}=+\infty\quad\Leftrightarrow\quad\inf_{p\in\mathbb{N}}\sup_{q\geq p}\liminf_{n\in\mathbb{N}}\frac{\varepsilon_n(p,q)}{\varepsilon_n}=+\infty. \tag{3.4}$$

Now since $\delta(E)\supseteq\delta(\Lambda_\infty(\varepsilon))$ always holds for the associated exponent sequence ε of E, we have

$$\delta(E)=\delta(\Lambda_\infty(\varepsilon))\qquad\Leftrightarrow\qquad\inf_{p\in\mathbb{N}}\sup_{q\geq p}\liminf_{n\in\mathbb{N}}\frac{\varepsilon_n(p,q)}{\varepsilon_n}=+\infty, \tag{3.5}$$

as desired. □

Proof of Theorem 3.1.1 For the proof of necessity part, we assume that $\delta(E)=\delta(\Lambda_\infty(\varepsilon))$. By Lemma 3.1.3, $\inf_{p\in\mathbb{N}}\sup_{q>p}\liminf_{n\in\mathbb{N}}\frac{\varepsilon_n(p,q)}{\varepsilon_n}=+\infty$. Then we have

$$\forall p\quad\forall M>0\quad\exists q\geq p\qquad\liminf_{n\in\mathbb{N}}\frac{\varepsilon_n(p,q)}{\varepsilon_n}\geq M \tag{3.6}$$

and

$$\forall p\quad\forall M>0\quad\exists q\geq p\qquad d_n\left(U_q,U_p\right)\leq e^{-M\varepsilon_n}\qquad\forall n\in\mathbb{N} \tag{3.7}$$

Let take a $(x_n)_{n\in\mathbb{N}}\in\Delta(\Lambda_\infty(\varepsilon))$, then there exists a $S>0$ such that

$$\sup_{n\in\mathbb{N}}|x_n|\,e^{-S\varepsilon_n}<+\infty \tag{3.8}$$

which means that there exists a $C>0$ such that for every $n\in\mathbb{N}$

$$|x_n|\leq Ce^{S\varepsilon_n}. \tag{3.9}$$

Now, for a fixed p and the number S, from (3.6) we can find a $q\geq p$ such that for every $n\in\mathbb{N}$

$$|x_n|\,d_n\left(U_q,U_p\right)\leq Ce^{S\varepsilon_n}e^{-S\varepsilon_n}=C. \tag{3.10}$$

Then, $(x_n)_{n\in\mathbb{N}} \in \Delta(E)$ and so $\Delta(\Lambda_\infty(\varepsilon)) \subseteq \Delta(E)$. Since we always have $\Delta(E) \subseteq \Delta(\Lambda_\infty(\varepsilon))$, we obtain $\Delta(E) = \Delta(\Lambda_\infty(\varepsilon))$.

To prove the sufficiency part, assume $\Delta(E) = \Delta(\Lambda_\infty(\varepsilon))$ and $\delta(E) \neq \delta(\Lambda_\infty(\varepsilon))$.

$$\begin{aligned}
&\delta(E) \neq \delta(\Lambda_\infty(\varepsilon)) \;\Leftrightarrow\; \exists\, p \;\sup_{q\geq p}\liminf_{n\in\mathbb{N}} \frac{\varepsilon_n(p,q)}{\varepsilon_n} < +\infty \\
&\Leftrightarrow \exists\, p \;\; \exists M>0 \;\; \sup_{q\geq p}\liminf_{n\in\mathbb{N}} \frac{\varepsilon_n(p,q)}{\varepsilon_n} \leq M \\
&\Leftrightarrow \exists\, p \;\; \exists M>0 \;\; \forall q\geq p \;\; \liminf_{n\in\mathbb{N}} \frac{\varepsilon_n(p,q)}{\varepsilon_n} \leq M \\
&\Leftrightarrow \exists\, p \;\; \exists M>0 \;\; \forall q\geq p \;\; \exists I_q \subseteq \mathbb{N} \;\; d_n\left(U_q,U_p\right) \geq e^{-M\varepsilon_n} \qquad \forall n \in I_q
\end{aligned} \tag{3.11}$$

Now since $\Delta(E) = \Delta(\Lambda_\infty(\varepsilon)) = \Lambda_\infty(\varepsilon)' = \left\{ (x_n)_{n\in\mathbb{N}} : \exists R>0 \;\; \sup_{n\in\mathbb{N}} |x_n|\, e^{-R\varepsilon_n} < +\infty \right\}$, for every $R>0$, we have $e^{R\varepsilon_n} \in \Lambda_\infty(\varepsilon)' = \Delta(E)$. Therefore, for the above p, we can find a $\tilde{q} > p$, such that

$$\sup_{n\in\mathbb{N}} e^{R\varepsilon_n} d_n\left(U_{\tilde{q}},U_p\right) < +\infty. \tag{3.12}$$

Then for every $n \in I_{\tilde{q}}$, we obtain

$$e^{(R-M)\,\varepsilon_n} \leq e^{R\varepsilon_n} d_n\left(U_{\tilde{q}},U_p\right) \leq \sup_{n\in\mathbb{N}} e^{R\varepsilon_n} d_n\left(U_{\tilde{q}},U_p\right) < +\infty. \tag{3.13}$$

But then if we choose $R > M$, we have a contradiction. Hence $\Delta(E) = \Delta(\Lambda_\infty(\varepsilon))$ implies $\delta(E) = \delta(\Lambda_\infty(\varepsilon))$, as desired. □

We end this section with the following result which gives a relation between having the property D_1 and its diametral dimension of a nuclear Fréchet space with the properties $\underline{DN}$ and Ω.

Proposition 3.1.4 *Let E be a nuclear Fréchet space with the properties $\underline{DN}$ and Ω and ε be the associated exponent sequence of E. Then, $\Delta(E) = \Delta(\Lambda_\infty(\varepsilon))$ implies that E has the property D_1.*

Proof. Let us assume that $\Delta(E) = \Delta(\Lambda_\infty(\varepsilon))$. From Lemma 3.1.3 and Theorem 3.1.1, we have

$$\inf_{p\in\mathbb{N}} \sup_{q>p} \liminf_{n\in\mathbb{N}} \frac{\varepsilon_n(p,q)}{\varepsilon_n} = +\infty \tag{3.14}$$

where $\varepsilon_n(p,q) = -\log d_n\left(U_q, U_p\right)$. Then,

$$\begin{aligned}
\Delta(E) = \Delta(\Lambda_\infty(\varepsilon)) \quad &\Leftrightarrow \quad \inf_{p\in\mathbb{N}} \sup_{q>p} \liminf_{n\in\mathbb{N}} \frac{\varepsilon_n(p,q)}{\varepsilon_n} = +\infty \\
&\Leftrightarrow \quad \forall M>0 \ \forall p \ \exists q \quad \liminf_{n\in\mathbb{N}} \frac{\varepsilon_n(p,q)}{\varepsilon_n} \geq M \\
&\Leftrightarrow \quad \forall M>0 \ \forall p \ \exists q \ \exists n_0\in\mathbb{N} \ \forall n\geq n_0 \quad d_n(U_q,U_p) \leq d_n(U_2,U_1)^M
\end{aligned} \tag{3.15}$$

Now we fix $p \in \mathbb{N}$. From Corollary 2.3.4, there exists a $C > 0$ and $0 < j < 1$ so that

$$d_n(U_2,U_1) \leq C\, d_n(U_p,U_1)^j \tag{3.16}$$

for all $n \in \mathbb{N}$. We also choose a M_0 which is greater than $\frac{2}{j}$, and from (3.14), there exists a $q > p$ and $n_0 \in \mathbb{N}$ such that

$$d_n(U_q,U_p) \leq d_n(U_2,U_1)^{M_0} \leq C^{M_0}\, d_n(U_p,U_1)^{M_0 j} \tag{3.17}$$

for all $n \geq n_0$. This gives us that

$$\frac{d_n(U_q,U_p)}{d_n(U_p,U_1)} \leq C^{M_0}\, d_n(U_p,U_1)^{M_0 j-1} \leq C^{M_0}\, d_n(U_p,U_1) \tag{3.18}$$

for all $n \geq n_0$. Since E nuclear, we can assume $\lim_{n\to\infty} d_n(U_p,U_1) = 0$. Then we have that

$$\forall p \quad \exists q \qquad \lim_{n\to\infty} \frac{d_n(U_q,U_p)}{d_n(U_p,U_1)} = 0. \tag{3.19}$$

This means that E has the property D_1, as requested. □

3.2 Results for the Case of Power Series Space of Finite Type

In this section, we turn our attention to the finite type power series case for Question 1.0.2. First of all, we show that $\delta(E) = \delta(\Lambda_1(\varepsilon))$ implies $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$ which supports Question 1.0.2 in one direction. Furthermore, we give certain conditions on Kolmogorov diameters of E for which Question 1.0.2 verifies in the other direction. Then in main result of this section, we prove that if E is a nuclear Fréchet space with the properties $\underline{\mathrm{DN}}$ and Ω, then $\delta(E) = \delta(\Lambda_1(\varepsilon))$ if and only if E has a prominent bounded set and $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$. However, the answer of Question 1.0.2 in finite type case in general is negative as seen in the fourth chapter.

We begin this section by giving the following proposition which answers Question 1.0.2 in one direction.

Proposition 3.2.1 *Let E be a nuclear Fréchet space with properties $\underline{DN}$ and Ω and $\varepsilon = (\varepsilon_n)_{n\in\mathbb{N}}$ be the associated exponent sequence of E. Then $\delta(E) = \delta(\Lambda_1(\varepsilon))$ implies $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$.*

Proof. Let us assume that $\delta(E) = \delta(\Lambda_1(\varepsilon))$. From Proposition 3.1.2, we have

$$\begin{aligned}\delta(E) = \delta(\Lambda_1(\varepsilon)) \quad &\Leftrightarrow \quad \inf_p \sup_{q\geq p} \limsup_{n\in\mathbb{N}} \frac{\varepsilon_n(p,q)}{\varepsilon_n} = 0 \\ &\Leftrightarrow \forall r>0 \ \ \exists p \ \ \forall q\geq p \ \ \limsup_{n\in\mathbb{N}} \frac{\varepsilon_n(p,q)}{\varepsilon_n} \leq r \\ &\Leftrightarrow \forall r>0 \ \ \exists p \ \ \forall q\geq p \ \ \exists n_0\in\mathbb{N} \ \ \forall n\geq n_0 \ \ d_n\left(U_q,U_p\right) \geq e^{-r\varepsilon_n}. \end{aligned} \tag{3.20}$$

Now, we take $(x_n)_{n\in\mathbb{N}} \in \Delta(E)$ and for the above p, we find a $\tilde{q} > p$ such that

$$\sup_{n\in\mathbb{N}} |x_n|\, d_n\left(U_{\tilde{q}},U_p\right) < +\infty \tag{3.21}$$

and from the above inequality, we obtain

$$|x_n|\, e^{-r\varepsilon_n} \leq \sup_{n\in\mathbb{N}} |x_n|\, d_n\left(U_{\tilde{q}},U_p\right) \tag{3.22}$$

for sufficienty large n, this means that $(x_n)_{n\in\mathbb{N}} \in \Delta(\Lambda_1(\varepsilon))$ and so $\Delta(E) \subseteq \Delta(\Lambda_1(\varepsilon))$. But then since $\Delta(E) \supseteq \Delta(\Lambda_1(\varepsilon))$, we have $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$. □

One can prove the converse of Proposition 3.2.1 under some assumption on the canonical topology of the diametral dimension as shown in the following theorem:

Theorem 3.2.2 *Let E be a nuclear Fréchet space with properties $\underline{DN}$ and Ω and $\varepsilon = (\varepsilon_n)_{n\in\mathbb{N}}$ be the associated exponent sequence of E. If $\Delta(E)$, with the canonical topology, is barrelled, then $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$ if and only if $\delta(E) = \delta(\Lambda_1(\varepsilon))$.*

Proof. The proof of the necessity part follows from Proposition 3.2.1. To prove the sufficiency part, let $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$ and assume that $\Delta(E)$ with the canonical topology be barrelled. Then since the convergence in $\Delta(E)$ implies the coordinate-wise convergence, the inclusion $\Delta(E) \hookrightarrow \Lambda_1(\varepsilon)$ has a closed graph. But then since $\Delta(E)$ is barrelled, the inclusion map $\Delta(E) \hookrightarrow \Lambda_1(\varepsilon)$ is continuous by [37, Theorem 5]. Taking into account that $\Delta(E)$ is the projective limit of inductive limits of Banach spaces,

$\bigcap_{p\in\mathbb{N}} \bigcup_{q\geq p+1} \Delta\left(U_q,U_p\right)$, the continuity of the inclusion map $\bigcap_{p\in\mathbb{N}} \bigcup_{q\geq p+1} \Delta\left(U_q,U_p\right) \hookrightarrow \Lambda_1(\varepsilon)$ gives us

$$\forall t>0 \;\; \exists p \;\; \forall q>p \;\; \exists C>0 \;\; \forall n\in\mathbb{N} \qquad e^{-t\varepsilon_n} \leq C\, d_n\left(U_q,U_p\right). \tag{3.23}$$

This implies $\inf_{p\in\mathbb{N}} \sup_{q>p} \limsup_{n\in\mathbb{N}} \frac{\varepsilon_n(p,q)}{\varepsilon_n} = 0$, so $\delta(E)=\delta(\Lambda_1(\varepsilon))$, as desired. □

It is worth noting that, by Theorem 2.1.5, the barrelledness of the canonical topology of $\Delta(E)$ is equivalent to the following condition

$$\begin{aligned} &(wQ):\; \forall\; N \;\; \exists\; M,n \;\; \forall\; K,m \;\; \exists\; k,S>0: \\ &\qquad \min\left(d_i\left(U_n,U_N\right), d_i\left(U_k,U_K\right)\right) \leq S\, d_i\left(U_m,U_M\right) \qquad \forall i\in\mathbb{N}. \end{aligned} \tag{3.24}$$

However, determining the barrelledness of $\Delta(E)$ is not easy in practice. In the following proposition, by posing below condition on diameters, we eliminate the barrelledness condition of Theorem 3.2.2.

Condition $\mathbb{A}$: $\forall p$, $\forall q>p$, $\exists s>q$, $\forall k>s$, $\exists C>0$ $\;d_n\left(U_q,U_p\right) \leq C d_n\left(U_k,U_s\right)$ $\forall n\in\mathbb{N}$.

Proposition 3.2.3 *Let E be a nuclear Fréchet space with the properties $\underline{DN}$ and Ω and ε be the associated exponent sequence of E. If E satisfies the condition $\mathbb{A}$ and $\Delta(E)=\Delta(\Lambda_1(\varepsilon))$, then $\delta(E)=\delta(\Lambda_1(\varepsilon))$.*

Proof. Suppose that E satisfies the condition $\mathbb{A}$ and $\Delta(E)=\Delta(\Lambda_1(\varepsilon))$. If $\delta(E)\neq\delta(\Lambda_1(\varepsilon))$, then by Proposition 3.1.2, we have $\inf_{p} \sup_{q\geq p} \limsup_{n\in\mathbb{N}} \frac{\varepsilon_n(p,q)}{\varepsilon_n} \neq 0$. and this gives the following condition:

$$\exists M>0 \;\; \forall p \;\; \exists q_p>p,\; I_p\subseteq\mathbb{N} \quad d_n\left(U_q,U_p\right) < e^{-M\varepsilon_n} \qquad \forall n\in I_p \tag{3.25}$$

For p=1, there exists a number q_1 and an infinite subset I_1 so that for all $n\in I_1$

$$d_n\left(U_{q_1},U_p\right) < e^{-M\varepsilon_n}, \tag{3.26}$$

and so it follows from the condition $\mathbb{A}$ that we have a number q_2 such that for all $k\geq q_2$ there exists a $C>0$ so that for all $n\in\mathbb{N}$

$$d_n\left(U_{q_1},U_1\right) \leq C d_n\left(U_k,U_{q_2}\right) \tag{3.27}$$

holds. Then, by the inequality 3.25, there exists a number q_3 and an infinite subset I_2 so that for all $n \in I_2$

$$d_n\left(U_{q_3},U_{q_2}\right) < e^{-M\varepsilon_n}. \tag{3.28}$$

It follows that there exists a $C_1 > 0$ so that for all $n \in \mathbb{N}$

$$d_n\left(U_{q_1},U_1\right) \leq C_1 d_n\left(U_{q_3},U_{q_2}\right) \tag{3.29}$$

holds. Now applying the same process for q_2 and q_3, we can find q_4, q_5 and $C_2 > 0$ such that

$$d_n\left(U_{q_3},U_{q_2}\right) \leq C_2 d_n\left(U_{q_5},U_{q_4}\right), \tag{3.30}$$

for all $n \in \mathbb{N}$. Continuing in this way, we can find the sequences $\{q_k\}_{k\in\mathbb{N}}$ and $\{C_k\}_{k\in\mathbb{N}}$ satisfying

$$d_n\left(U_{q_1},U_1\right) \leq C_1 d_n\left(U_{q_3},U_{q_2}\right) \leq C_2 d_n\left(U_{q_5},U_{q_4}\right) \leq \cdots \leq C_k d_n\left(U_{q_{2k+1}},U_{q_{2k}}\right) \leq \cdots \tag{3.31}$$

for all $n \in \mathbb{N}$. Moreover, for each $k \in \mathbb{N}$, there exists a $I_k \subseteq \mathbb{N}$ such that

$$d_n\left(U_{q_{2k+1}},U_{q_{2k}}\right) < e^{-M\varepsilon_n} \tag{3.32}$$

for all $n \in I_k$.

Now, for each $k \in \mathbb{N}$, we define

$$B_k = \left\{x = (x_n) : \sup_{n\in\mathbb{N}} C_k |x_n| d_n\left(U_{q_{2k+1}},U_{q_{2k}}\right) < +\infty\right\}, \tag{3.33}$$

where B_k is a Banach space under the norm $\|x\|_k = \sup_{n\in\mathbb{N}} C_k|x_n| d_n\left(U_{q_{2k+1}},U_{q_{2k}}\right)$ for all $k \in \mathbb{N}$. By the inequality 3.31, we have $B_{k+1} \subseteq B_k$ and $\|\cdot\|_k \leq \|\cdot\|_{k+1}$ for all $k \in \mathbb{N}$. Since $(q_k)_{k\in\mathbb{N}}$ is strictly increasing and unbounded, for all $p \in \mathbb{N}$, there exists a $k_0 \in \mathbb{N}$ such that $q_{2k_0} > p$ and this gives us $U_{q_{2k_0}} \subseteq U_p$. For all $n \in \mathbb{N}$

$$d_n\left(U_{q_{2k_0+1}},U_p\right) \leq d_n\left(U_{q_{2k_0+1}},U_{q_{2k_0}}\right), \tag{3.34}$$

which means that $\bigcap_k B_k \subseteq \Delta(E)$. Moreover, the equality $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$ yields a continuous imbedding of the projective limit $\bigcap_k B_k$ into $\Lambda_1(\varepsilon)$. Then since $\bigcap_k B_k$ and

$\Lambda_1(\varepsilon)$ are Fréchet spaces and the imbedding map has a closed graph, by [37, Theorem 5], this map is continuous and so

$$\begin{aligned}
&\bigcap_k B_k \hookrightarrow \Delta(\Lambda_1(\varepsilon)) \text{ is continuous} \\
\Leftrightarrow\ & \forall t>0\ \exists k,C \quad \sup_n |x_n| e^{-t\varepsilon_n} \le C \sup_{n\in\mathbb{N}} |x_n| d_n\left(U_{q_{2k+1}}, U_{q_{2k}}\right) \qquad \forall (x_n) \in \bigcap_p B_p \\
\Leftrightarrow\ & \forall t>0\ \exists k,C \quad \forall n\in\mathbb{N} \quad e^{-t\varepsilon_n} \le C\, d_n\left(U_{q_{2k+1}}, U_{q_{2k}}\right).
\end{aligned} \tag{3.35}$$

But, this contradics to the inequality 3.28. Therefore, $\delta(E) = \delta(\Lambda_1(\varepsilon))$ holds when $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$ and the condition $\mathbb{A}$ holds. □

There could be other diameter conditions as above which yields the same conclusion in Proposition 3.2.3. For example, by introducing

Condition $\mathbb{B}$: $\forall p \ \ \forall q_1, q_2, ..., q_p,\ \exists s \le p, \exists C > 0 \quad \max_{1\le i\le q} d_n\left(U_{q_i}, U_i\right) \le C d_n\left(U_{q_s}, U_s\right)$ $\forall n \in \mathbb{N}$.

we get the following result:

Proposition 3.2.4 *Let E be a nuclear Fréchet space with the properties* DN *and* Ω *and* ε *be the associated exponent sequence of E. If E satisfies the condition* $\mathbb{B}$ *and* $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$*, then* $\delta(E) = \delta(\Lambda_1(\varepsilon))$.

The proof is similar to Proposition 3.2.3 except that the projective limit will be replaced by $\bigcap_k D_k$, where $D_k = \left\{ x = (x_n) : \sup_{n\in\mathbb{N}} |x_n| \max_{1\le i\le p} d_n\left(U_{q_i}, U_i\right) < +\infty \right\}$.

In [38], T. Terzioğlu defined the notion *prominent bounded subset* in order to show that the diametral dimension of some Fréchet spaces is determined by a single bounded set:

Definition 3.2.5 *Let E be a Fréchet space. A bounded set B is said to* ***prominent*** *if*

$$\Delta(E) = \left\{ (x_n)_{n\in\mathbb{N}} \ :\ \lim_{n\to+\infty} x_n\, d_n(B, U_p) = 0 \ \ \forall p \right\}. \tag{3.36}$$

In this case one can introduce a natural Fréchet space topology on $\Delta(E)$ as well as its canonical topology which is not Fréchet.

T. Terzioğlu also gave a necessary and sufficient condition for a bounded subset to be prominent [Proposition 3, [39]], namely, B is a prominent set if and only if for each p there is a q and $C > 0$ such that

$$d_n\left(U_q, U_p\right) \leq C d_n\left(B, U_q\right) \tag{3.37}$$

holds for all $n \in \mathbb{N}$.

In the following proposition, we prove that having a prominent bounded subset is closely related to Bessaga's condition D_2:

$$D_2: \quad \forall p \quad \exists q \quad \forall k \qquad \lim_{n\to\infty} \frac{d_n\left(U_q, U_p\right)}{d_n\left(U_k, U_q\right)} = 0. \tag{3.38}$$

Proposition 3.2.6 *Let E be a nuclear Fréchet space. The following are equivalent:*

1. *E has a prominent bounded set B.*
2. *E has the property D_2.*
3. *For every p there exists $q > p$ such that $\sup_{l\geq q} \lim \sup_{n\in\mathbb{N}} \frac{\varepsilon_n(q,l)}{\varepsilon_n(p,q)} < 1$ holds.*

We need the following lemma for the proof of Proposition 3.2.6. As usual, we assume that all semi-norms are Hilbertian.

Lemma 3.2.7 *Let E be a nuclear Fréchet space. Then for all $p, q > p$, there is a $s > q$ such that*

$$\lim_{n\to+\infty} \frac{d_n\left(U_s, U_p\right)}{d_n\left(U_q, U_p\right)} = 0. \tag{3.39}$$

Proof. Let E be a nuclear Fréchet space and $p, q > p$. Since nuclear Fréchet spaces are Schwartz, there is a number $k > p$ such that U_k is precompact with respect to U_p and so the canonical inclusion map $i_p^k : E_q \to E_p$ is compact.

Now assume that $k \geq q$. Then $U_k \subseteq U_q$ and there exists a $s \geq k$ so that $i_k^s : E_s \to E_k$ is compact. Thus it follows from [13, Proposition 1.2] given by Demeulenaere et al.

$$\lim_{n\to+\infty}\frac{d_n(U_s,U_p)}{d_n(U_k,U_p)}=0 \tag{3.40}$$

and since $U_k\subseteq U_q$, $d_n(U_k,U_p)\le d_n\left(U_q,U_p\right)$ for all $n\in\mathbb{N}$. Hence we get

$$\lim_{n\to+\infty}\frac{d_n(U_s,U_p)}{d_n\left(U_q,U_p\right)}=0. \tag{3.41}$$

If now $q\ge k$, then the map $i_p^q:E_q\to E_p$ is compact since $i_p^q=i_k^q\circ i_p^k$ and i_p^k is compact. On the other hand, there exists a $\bar{s}$ satisfying $\bar{s}>q$ and $i_q^{\bar{s}}:E_{\bar{s}}\to E_q$ is compact. Again from [13, Proposition 1.2], we get

$$\lim_{n\to+\infty}\frac{d_n(U_{\bar{s}},U_p)}{d_n\left(U_q,U_p\right)}=0. \tag{3.42}$$

Therefore,

$$\forall p,\ q>p\quad \exists s>q,\qquad \lim_{n\to+\infty}\frac{d_n(U_s,U_p)}{d_n\left(U_q,U_p\right)}=0, \tag{3.43}$$

as desired. □

It is worth noting that, by using Lemma 4.6, the condition D_2 can also be stated as follows:

$$D_2:\qquad \forall p\quad \exists q\quad \forall k\qquad \sup_{n\in\mathbb{N}}\frac{d_n\left(U_q,U_p\right)}{d_n\left(U_k,U_q\right)}<+\infty. \tag{3.44}$$

We are now ready to give the proof of Proposition 3.2.6.

Proof of Proposition 3.2.6 $1\Rightarrow 2$: This follows immediately from Lemma 4.6 and the definition of D_2.

$2\Rightarrow 3$: Suppose E has the condition D_2. Then, for every p, there exists a $q>p$ such that for all $k>q$

$$\begin{aligned}\sup_{n\in\mathbb{N}}\frac{d_n\left(U_q,U_p\right)}{d_n\left(U_k,U_q\right)}<+\infty\quad&\Leftrightarrow\quad \exists M>0\quad \forall n\in\mathbb{N}\qquad \frac{d_n\left(U_q,U_p\right)}{d_n\left(U_k,U_q\right)}<M\\&\Leftrightarrow\quad \exists M>0\quad \forall n\in\mathbb{N}\qquad \varepsilon_n(p,q)>-\ln M+\varepsilon_n(q,k)\\&\Leftrightarrow\quad \limsup_{n\in\mathbb{N}}\frac{\varepsilon_n(q,k)}{\varepsilon_n(p,q)}\le 1.\end{aligned} \tag{3.45}$$

Further by Corollary 2.3.4, we have a condition on Kolmogorov diameters for this q

$$\exists C > 0,\, 0 < \lambda < 1 \qquad d_n(U_{q+2}, U_q) \leq C\, d_n(U_{q+1}, U_q)^{\frac{1}{1-\lambda}} \qquad \forall n \in \mathbb{N}. \tag{3.46}$$

Then, for every $k > q+1$, there exists a $\tilde{C} > 0$ such that the inequality

$$\varepsilon_n(q+1,k) \leq \tilde{C} + (1-\lambda)\varepsilon_n(q,k) \qquad \forall n \in \mathbb{N} \tag{3.47}$$

holds. This gives

$$\frac{\varepsilon_n(q+1,k)}{\varepsilon_n(p,q+1)} = \frac{\varepsilon_n(q+1,k)}{\varepsilon_n(p,q+1)} \frac{\varepsilon_n(q,k)}{\varepsilon_n(q,k)} \leq \frac{\varepsilon_n(q+1,k)}{\varepsilon_n(q,k)} \frac{\varepsilon_n(q+1,k)}{\varepsilon_n(p,q)} \tag{3.48}$$

and

$$\limsup_{n\in\mathbb{N}} \frac{\varepsilon_n(q+1,k)}{\varepsilon_n(p,q+1)} \leq \limsup_{n\in\mathbb{N}} \frac{\varepsilon_n(q+1,k)}{\varepsilon_n(q,k)} \frac{\varepsilon_n(q+1,k)}{\varepsilon_n(p,q)} \leq (1-\lambda) < 1 \tag{3.49}$$

Hence, for every p there exists a $\tilde{q} > p$ such that

$$\sup_{l\geq q} \limsup_{n\in\mathbb{N}} \frac{\varepsilon_n(q,l)}{\varepsilon_n(p,\tilde{q})} < 1. \tag{3.50}$$

$3 \Rightarrow 1$: Now we assume that the condition in 3 is satisfied. We fix a $p \in \mathbb{N}$ and choose a q satisfying $\sup_{k\geq q} \limsup_{n\in\mathbb{N}} \frac{\varepsilon_n(q,k)}{\varepsilon_n(p,q)} < 1$. We also fix an $\varepsilon > 0$ satisfying $\sup_{k\geq q} \limsup_{n\in\mathbb{N}} \frac{\varepsilon_n(q,k)}{\varepsilon_n(p,q)} \leq 1-\varepsilon$. Then, for each $k \geq q$ there exists $N(k)$ so that

$$\varepsilon_n(q,k) \leq (1-\varepsilon)\,\varepsilon_n(p,q) \qquad \forall n \geq N(k). \tag{3.51}$$

Without loss of generality, we can choose the sequence $(N(k))_{k\in\mathbb{N}}$ to be increasing. Let $\delta_1 := \frac{1}{2} e^{-(1-\varepsilon)\varepsilon_{N(q+1)}(p,q)}$. Since E is in particular Schwartz, we can assume that U_{q+1} is precompact with respect to U_q. Therefore, we can find a finite subset $Z^1_{(q,1)} \subseteq U_{q+1}$ satisfying

$$U_{q+1} \subseteq Z^1_{(q,1)} + \delta_1 U_q. \tag{3.52}$$

Thus, we can write

$$d_n(U_{q+1}, U_q) \leq d_n(Z^1_{(q,1)}, U_q) + \delta_1 \qquad \forall n \in \mathbb{N} \tag{3.53}$$

and

$$2\delta_1 = e^{-(1-\varepsilon)\varepsilon_{N(q+1)}(p,q)} \leq e^{-\varepsilon_{N(q+1)}(p,q+1)} = d_{N(q+1)}(U_{q+1},U_q) \\ \leq d_{N(q+1)}(Z^1_{(q,1)},U_q) + \delta_1. \tag{3.54}$$

This gives us that

$$d_{N(q+1)}(Z^1_{(q,1)},U_q) \geq \delta_1 = \frac{1}{2}\, e^{-(1-\varepsilon)\varepsilon_{N(q+1)}(p,q)}. \tag{3.55}$$

For each $N(q+1) \leq n \leq N(q+2)$, using the above argument, we can get a finite subset $Z^1_{(q,n)} \subseteq U_q$ with

$$d_n(Z^1_{(q,n)},U_q) \geq \frac{1}{2}\, e^{-(1-\varepsilon)\varepsilon_n(p,q)} \tag{3.56}$$

Let $Z^1_q = \bigcup\limits_{N(q+1)\leq n\leq N(q+2)} Z^1_{(q,n)}$. Then,

$$d_n(Z^1_q,U_q) \geq \frac{1}{2}\, e^{-(1-\varepsilon)\varepsilon_n(p,q)} \qquad \forall\ N(q+1) \leq n \leq N(q+2). \tag{3.57}$$

We proceed to get a finite set $Z^s_q \subseteq U_{q+s}$ so that

$$d_n(Z^s_q,U_q) \geq \frac{1}{2}\, e^{-(1-\varepsilon)\varepsilon_n(p,q)} \qquad \forall\ N(q+s) \leq n \leq N(q+s+1). \tag{3.58}$$

Let $Z_q = \bigcup\limits_{s=1}^{\infty} Z^s_q$. Since $Z^s_q \subseteq U_{q+s}$ for all s, then Z_q is bounded and

$$d_n(Z_q,U_q) \geq \frac{1}{2}\, e^{-(1-\varepsilon)\varepsilon_n(p,q)} \qquad \forall n \geq N(q+1). \tag{3.59}$$

Without loss of generality, we can assume $q = p+1$. Then, we can write

$$d_n(Z_p,U_{p+1}) \geq \frac{1}{2}\, e^{-(1-\varepsilon)\varepsilon_n(p,p+1)} \qquad \forall n \geq N(p+1). \tag{3.60}$$

Now we choose a sequence $(\lambda_k)_{k\in\mathbb{N}}$, $\lambda_k > 0$ such that $Z = \bigcup\limits_{k=1}^{\infty} \lambda_k Z_k$ is bounded. Then, we find that for every p,

$$d_n(Z,U_{p+1}) \geq \lambda_p d_n(Z_p,U_{p+1}) \geq \frac{\lambda_p}{2} d_n(U_{p+1},U_p)^{(1-\varepsilon)} \geq \frac{\lambda_p}{2} d_n(U_{p+1},U_p) \tag{3.61}$$

for all $n \in \mathbb{N}$. This completes the proof, as desired. □

As an easy consequence of Proposition 3.2.6 and Proposition 3.1.2, we obtain the following result which gives a relation between having prominent bounded subset and its approximate diametral dimension of a nuclear Fréchet space with the properties $\underline{DN}$ and Ω:

Corollary 3.2.8 *Let E be a nuclear Fréchet space with the properties $\underline{DN}$ and Ω and the associated exponent sequence ε. Then $\delta(E) = \delta(\Lambda_1(\varepsilon))$ implies that E has a prominent bounded subset.*

The following theorem is the main result of this section which says that Question 1.0.2 holds true provided E has a prominent bounded subset:

Theorem 3.2.9 *Let E be a nuclear Fréchet space with the properties $\underline{DN}$ and Ω and ε the associated exponent sequence. $\delta(E) = \delta(\Lambda_1(\varepsilon))$ if and only if E has a prominent bounded set and $\Delta(E) = \Lambda_1(\varepsilon)$.*

Proof. Let E be a nuclear Fréchet space with a prominent bounded subset B. Then E satisfies condition D_2

$$\forall p \quad \exists q \quad \forall k \qquad \lim_{n\to\infty} \frac{d_n(U_q, U_p)}{d_n(U_k, U_q)} = 0 \tag{3.62}$$

and, in particular, if we take $N = p$, $M = n = q$ and $m = k$, we get

$$\forall N \ \exists \ M, n \ \forall m, \ \exists \ S > 0: \ d_n(U_n, U_N) \leq S d_n(U_m, U_M) \qquad \forall n \in \mathbb{N}. \tag{3.63}$$

which means that E satisfy the condition (wQ) given in Theorem 2.1.5 and so $\Delta(E)$ is barrelled with respect to the canonical topology. Hence the result follows from Theorem 3.2.2. □

In the final part of this section we examine the conditions for which the converse of Corollary 3.2.8 also holds.

For this, we define

$$\blacktriangle(E) := \left\{ (t_n)_{n\in\mathbb{N}} : \ \forall p, \ \forall 0 < \varepsilon < 1, \ \exists q > p \ \lim_{n\to+\infty} t_n d_n(U_q, U_p)^{\varepsilon} = 0 \right\}. \tag{3.64}$$

The next result provides a condition that implies $\delta(E) = \delta(\Lambda_1(\varepsilon))$ when E has a prominent subset.

Proposition 3.2.10 *Let E be a nuclear Fréchet space with the properties $\underline{DN}$ and Ω, its associated exponent sequence $\varepsilon = (\varepsilon_n)_{n\in\mathbb{N}}$. If E has a prominent bounded subset B and $\Delta(E) = \blacktriangle(E)$, then $\delta(E) = \delta(\Lambda_1(\varepsilon))$.*

Proof. Let B be a prominent bounded subset of E. Then, for all $p \in \mathbb{N}$, there exists a $q > p$ and a $C > 0$ so that for every $n \in \mathbb{N}$

$$d_n\left(U_q, U_p\right) \leq C d_n\left(B, U_q\right) \tag{3.65}$$

holds. Also, since B is bounded and ε_n is the associated exponent sequence, then there exist $C_1, C_2, D_1, D_2 > 0$ and $a_1, a_2 > 0$ satisfying

$$D_1 e^{-a_1 \varepsilon_n} \leq C_1 d_n\left(U_q, U_p\right) \leq d_n\left(B, U_q\right) \leq C_2 d_n\left(U_{q+1}, U_q\right) \leq D_2 e^{-a_2 \varepsilon_n} \tag{3.66}$$

for every $n \in \mathbb{N}$. On the other hand,

$$\Delta(E) = \left\{ (x_n)_{n\in\mathbb{N}} : \ \forall p \ \lim_{n\to+\infty} |x_n|\, d_n(B, U_p) = 0 \right\} \tag{3.67}$$

is a Fréchet space since B is a prominent bounded set. Fix p, $q > p$ and ε. Consider the Banach space

$$B_{p,\varepsilon,q} = \left\{ t = (t_n)_{n\in\mathbb{N}} : \ \lim_{n\to+\infty} |t_n|\, d_n\left(U_q, U_p\right)^{\varepsilon} = 0 \right\}. \tag{3.68}$$

Since $U_{q+1} \subseteq U_q$, we have $d_n\left(U_{q+1}, U_p\right) \leq d_n\left(U_q, U_p\right)$ for every $n \in \mathbb{N}$ and $B_{p,\varepsilon,q} \subseteq B_{p,\varepsilon,q+1}$. Then we endowe the space $\mathbb{\Delta}(E) = \bigcap_{(p,\varepsilon)} \bigcup_{q>p} B_{p,\varepsilon,q}$ with the projective limit of inductive limits of Banach spaces $B_{p,\varepsilon,q}$. In view of Grothendieck Factorization theorem ([10], p.225), for all p, $0 < \varepsilon < 1$ there exists a $q > p$ such that $\Delta(E) \hookrightarrow B_{p,\varepsilon,q}$ is continous

$$\forall p,\ 0 < \varepsilon < 1, \quad \exists q > p,\ C > 0 \qquad d_n\left(U_q, U_p\right)^{\varepsilon} \leq d_n\left(B, U_q\right) \qquad \forall n \in \mathbb{N}. \tag{3.69}$$

Now take $\delta > 0$. Then, for a given p, we choose $0 < \varepsilon < 1$ so that $0 < \varepsilon < \dfrac{\delta}{a_1}$. Then there exists a $\overline{C} > 0$ so that for all $n \in \mathbb{N}$,

$$\overline{C} e^{-\varepsilon a_1 \varepsilon_n} \leq C d_n(B, U_p)^{\varepsilon} \leq d_n\left(U_q, U_p\right)^{\varepsilon} \quad \Leftrightarrow \quad \overline{C} e^{-\delta\varepsilon_n} \leq d_n\left(U_q, U_p\right)^{\varepsilon} \leq C d_n\left(B, U_q\right)$$

$$\Leftrightarrow \quad \ln \overline{C} - \delta\varepsilon_n \leq \ln C + \ln d_n\left(B, U_q\right) \leq \ln C + \ln d_n\left(U_l, U_q\right)$$

$$\Rightarrow \quad -\ln d_n\left(U_l, U_q\right) \leq \left(\ln C - \ln \overline{C}\right) + \delta\varepsilon_n$$

$$\Rightarrow \quad \limsup_n \frac{\varepsilon_n(q,l)}{\varepsilon_n} \leq \delta.$$

Hence, we obtain that for all $\delta > 0$ there is a q such that

$$\sup_{l>q} \limsup_n \frac{\varepsilon_n(q,l)}{\varepsilon_n} \leq \delta \qquad \text{and} \qquad \inf_q \sup_{l>q} \limsup_n \frac{\varepsilon_n(q,l)}{\varepsilon_n} = 0, \tag{3.70}$$

which means that $\delta(E) = \delta(\Lambda_1(\varepsilon))$. □

Below we observe that the reverse implication in Proposition 3.2.10 is also true. Tndeed:

Note that $\blacktriangle(E)$ is always an algebra under pointwise multiplication. If $(t_n)_{n\in\mathbb{N}} \in \blacktriangle(E)$, then for all p, $0<\varepsilon<1$, we can choose $q>p$ such that

$$\lim_{n\to\infty} t_n d_n\left(U_q, U_p\right)^{\frac{\varepsilon}{2}} = 0, \tag{3.71}$$

which means $\left(t_n^2\right) \in \blacktriangle(E)$. Thus for any $(t_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}} \in \blacktriangle(E)$, we have that $(t_n s_n)_{n\in\mathbb{N}} \in \blacktriangle(E)$ as $|t_n s_n| \leq \dfrac{|t_n|^2}{2} + \dfrac{|s_n|^2}{2}$ for all $n\in\mathbb{N}$.

However $\Delta(E)$ need not to be an algebra under pointwise multiplication. If it does, then $\Delta(E)$ satisfies the condition "$(t_n)\in\Delta(E)$ implies $\left(t_n^2\right)\in\Delta(E)$", vice versa. This condition gives that $\left(t_n^{2^m}\right)\in\Delta(E)$ for all $m\in\mathbb{N}$. Now, for a p and $\varepsilon>0$, we can choose sufficiently large $m\in\mathbb{N}$ such that $\dfrac{1}{2^m}\leq\varepsilon$ and find a q so that

$$\lim_{n\to\infty} t_n^{2^m} d_n\left(U_q, U_p\right) = 0 \quad \text{ and } \quad \lim_{n\to\infty} t_n d_n\left(U_q, U_p\right)^{\varepsilon} = 0 \tag{3.72}$$

which gives that $\Delta(E)\subseteq\blacktriangle(E)$. Since the inclusion $\blacktriangle(E)\subseteq\Delta(E)$ always holds, we have $\blacktriangle(E)=\Delta(E)$. Hence we have proved the following:

Proposition 3.2.11 *Let E be a nuclear Fréchet space with the properties* <u>DN</u> *and* Ω*. The following conditions are equivalent:*

1. $\Delta(E)=\blacktriangle(E)$
2. $\Delta(E)$ *is an algebra under pointwise multiplication.*
3. $(t_n)\in\Delta(E)$ *implies* $\left(t_n^2\right)\in\Delta(E)$.

As a consequence we get the following result completing Proposition 3.2.10 with which we end this section:

Corollary 3.2.12 *Let E be a nuclaer Fréchet space with the properties* <u>DN</u> *and* Ω*, its associated exponent sequence* $\varepsilon=(\varepsilon_n)_{n\in\mathbb{N}}$*. Then E has a prominent bounded subset and* $\Delta(E)$ *is an algebra under pointwise multiplication if and only if* $\delta(E)=\delta(\Lambda_1(\varepsilon))$.

4. POWER SERIES SUBSPACES OF A NUCLEAR KÖTHE SPACE WITH THE PROPERTIES $\underline{DN}$ AND Ω

This chapter centered around the Question 1.0.1 stated in Introduction that is about the existence power series subspaces of a nuclear Köthe space with the properties $\underline{DN}$ and Ω such that its diametral and/or approximate diametral dimension coincides with that of a power series space.

In the first section, we show that a regular nuclear Köthe space with the properties $\underline{DN}$ and Ω is itself a power series space if its diametral dimension coincides with that of a power series space of infinite type or its approximate diametral dimension coincides with that of a power series space of finite type.

In the second section, we construct a family $\mathcal{K}$ of nuclear Köthe spaces with the properties $\underline{DN}$ and Ω whose elements parameterized by an exponent sequence α. Motivated by our finding in the third section, we obtain that Question 1.0.2 has a negative answer for some element of $\mathcal{K}$ with certain parameter α. Then, we show that for certain parameter α, an element of $\mathcal{K}$ does not have a subspace which is isomorphic to power series space of finite type generated by its associated exponent sequence. Hence the Question 1.0.1 is answered negatively in general.

4.1 Regular Köthe Spaces Whose Diametral and/or Approximate Diametral Dimension Coincides with That of A Power Series Space.

The aim of this section is to prove the following results:

Proposition 4.1.1 *Let $K(a_{k,n})$ be a regular nuclear Köthe space with the properties $\underline{DN}$ and Ω and ε be the associated exponent sequence of $K(a_{k,n})$. Then, $\Delta(K(a_{k,n})) = \Delta(\Lambda_\infty(\varepsilon))$ if and only if $K(a_{k,n})$ is isomorphic to $\Lambda_\infty(\varepsilon)$.*

Proposition 4.1.2 *Let $K(a_{k,n})$ be a regular nuclear Köthe space with the properties $\underline{DN}$ and Ω and ε be the associated exponent sequence of $K(a_{k,n})$. Then, $\delta(K(a_{k,n})) = \delta(\Lambda_1(\varepsilon))$ if and only if $K(a_{k,n})$ is isomorphic to $\Lambda_1(\varepsilon)$.*

Before giving the proofs of the above propositions, we need to mention two significant results:

Proposition 4.1.3 *If E is a nuclar Köthe space with the property d_1 and d_4, then E is isomorphic to a power series space of infinite type.*

Proof. [27, 1.4.2 Proposition] □

Proposition 4.1.4 *If E is a nuclar Köthe space with the property d_2 and d_5, then E is isomorphic to a power series space of finite type.*

Proof. [27, 1.4.3 Proposition] □

Proof of Proposition 4.1.1 The proof of the necessity part is clear. For the sufficiency part, assume that $\Delta(K(a_{k,n})) = \Delta(\Lambda_\infty(\varepsilon))$ for a regular nuclear Köthe space with the properties $\underline{DN}$ and Ω. Proposition 3.1.4 gives us that $K(a_{k,n})$ has the property D_1 which is equivalent the properties d_1 and DN. Then, $K(a_{k,n})$ is isomorphic to a power series space of infinite type by Proposition 4.1.3. Since $\Delta(K(a_{k,n})) = \Delta(\Lambda_\infty(\varepsilon))$, it follows that $K(a_{k,n})$ and $\Lambda_\infty(\varepsilon)$ are isomorphic, as desired. □

Proof of Propostion 4.1.2 The proof of the necessity part is clear. For the sufficiency part, assume that $\delta(K(a_{k,n})) = \delta(\Lambda_\infty(\varepsilon))$ for a regular nuclear Köthe space with the properties $\underline{DN}$ and Ω. Proposition 3.2.6 and Corollary 3.2.8 give us that $K(a_{k,n})$ has the property D_2 which is equivalent the properties d_2 and $\overline{\Omega}$. Then, $K(a_{k,n})$ is isomorphic to a power series space of finite type from Proposition 4.1.4. Since $\Delta(K(a_{k,n})) = \Delta(\Lambda_1(\varepsilon))$, it follows that $K(a_{k,n})$ and $\Lambda_1(\varepsilon)$ are isomorphic, as desired. □

4.2 $\mathcal{K}_\alpha$-Spaces

In this section, we will construct a family of nuclear Köthe spaces with the properties $\underline{DN}$ and Ω and parameterized by a finitely nuclear sequence α and show that a subfamily of these Köthe spaces satisfied that $\Delta\left(K\left(a_{k,n}\right)\right) = \Delta(\Lambda_1(\varepsilon))$ and $\delta\left(K\left(a_{k,n}\right)\right) \neq \delta(\Lambda_1(\varepsilon))$ for its associated exponent sequence ε. This shows that Question 1.0.2 has a negative answer.

We proceed as follows: First, we will divide natural numbers $\mathbb{N}$ into infinite disjoint union of infinite subsets. For this purpose, we order the elements of $\mathbb{N}^2$ by matching them with the elements of $\mathbb{N}$ such that any element $(x,y) \in \mathbb{N}^2$ corresponds to the element $\frac{(x+1)(x+2)}{2} + y(x+1) + \frac{y(y-1)}{2} \in \mathbb{N}$. One can visualize this ordering as shown in the following graphic:

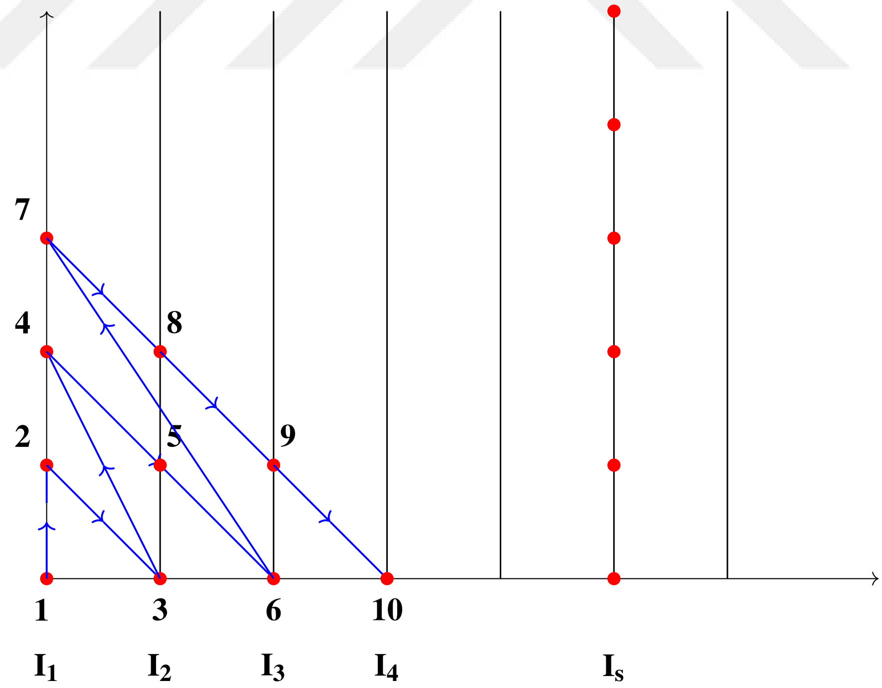

Figure 4.1 : Graphic 1

As shown in the above figure, each vertical line I_s has infinitely many elements and $\mathbb{N}$ can be expressed as an infinite disjoint union of I_s, that is, $N = \bigcup_{s\in\mathbb{N}} I_s$.

Let $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ be a strictly increasing, positive, finitely nuclear sequence. We define a matrix $\left(a_{k,n}\right)_{k,n\in\mathbb{N}}$ by setting

$$a_{k,n} = \begin{cases} e^{-\frac{1}{k}\alpha_n}, & \text{if } k \le s \\ e^{\left(-\frac{1}{k}+1\right)\alpha_n}, & \text{if } k \ge s+1. \end{cases} \tag{4.1}$$

where $n \in I_s$, $s \in \mathbb{N}$.

Infact, $\left(a_{k,n}\right)_{k,n\in\mathbb{N}}$ is a Köthe matrix, since for every $n,k \in \mathbb{N}$, $0 < a_{k,n} \le a_{k+1,n}$. We denote the Köthe space generated by a matrix $(a_{k,n})_{k,n\in\mathbb{N}}$ as in 4.1 by $\mathcal{K}_\alpha$. We say that the space $\mathcal{K}_\alpha$ is parameterized by the sequence α. We denote the family of all Köthe space $\mathcal{K}_\alpha$ by $\mathcal{K}$.

Now, we show that each element of the family $\mathcal{K}$ is nuclear and satisfies the properties $\underline{DN}$ and Ω:

Lemma 4.2.1 *Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ parametrized by $\alpha = (\alpha_n)_{n\in\mathbb{N}}$. Then, $\mathcal{K}_\alpha$ is nuclear and has the properties $\underline{DN}$ and Ω.*

Proof. In order to prove the **nuclearity** of $\mathcal{K}_\alpha$, we show that the series $\sum_{n=1}^{\infty} \frac{a_{k,n}}{a_{k+1,n}}$ is convergent for each $k \in \mathbb{N}$: Since

$$\frac{a_{k,n}}{a_{k+1,n}} \le e^{\left(-\frac{1}{k}+\frac{1}{k+1}\right)\alpha_n} \tag{4.2}$$

for every $k,n \in \mathbb{N}$ and $\Lambda_1(\alpha)$ is nuclear, then the series $\sum_{n=1}^{\infty} \frac{a_{k,n}}{a_{k+1,n}}$ is convergent and so $\mathcal{K}_\alpha$ is nuclear, as asserted.

We now prove that $\mathcal{K}_\alpha$ has the **$\underline{\text{DN}}$ property** by using Proposition 2.3.2. We will show that for all $p \in \mathbb{N}$ there exists a $0 < \lambda < 1$ such that the inequality

$$a_{p,n} \le (a_{1,n})^{\lambda} \left(a_{p+1,n}\right)^{1-\lambda} \tag{4.3}$$

is satisfied for all $n \in \mathbb{N}$.

Let $p,n \in \mathbb{N}$ and assume $n \in I_s$, $s \in \mathbb{N}$. There are two cases for p and s: $p \le s$ or $p > s$.

First we assume that $p \le s$: In this case,

$$a_{1,n} = e^{-\alpha_n}, \quad a_{p,n} = e^{-\frac{1}{p}\alpha_n} \quad \text{and} \quad a_{p+1,n} \geq e^{-\frac{1}{p+1}\alpha_n}. \tag{4.4}$$

Then, the inequality 4.3 is satisfied for any $\lambda < \dfrac{\frac{1}{p} - \frac{1}{p+1}}{1 - \frac{1}{p+1}}$.

Second we assume that $s < p$: In this case,

$$a_{1,n} = e^{-\alpha_n}, \quad a_{p,n} = e^{\left(-\frac{1}{p}+1\right)\alpha_n} \quad \text{and} \quad a_{p+1,n} = e^{\left(-\frac{1}{p+1}+1\right)\alpha_n}. \tag{4.5}$$

But then the inequality 4.3 is satisfied for any $\lambda < \dfrac{\frac{1}{p} - \frac{1}{p+1}}{2 - \frac{1}{p+1}}$.

Hence, if we choose a $\lambda > 0$ satisfying

$$\lambda < \min\left\{\frac{\frac{1}{p} - \frac{1}{p+1}}{1 - \frac{1}{p+1}}, \frac{\frac{1}{p} - \frac{1}{p+1}}{2 - \frac{1}{p+1}}\right\} = \frac{\frac{1}{p} - \frac{1}{p+1}}{2 - \frac{1}{p+1}} \tag{4.6}$$

then inequality 4.3 holds in general and so $\mathcal{K}_\alpha$ has the property DN, as claimed.

We now prove that $\mathcal{K}_\alpha$ has **Ω -property** by using Proposition 2.3.1. We will show that for all $p \in \mathbb{N}$ and for $k > p$ there exists a $j > 0$ such the inequality

$$(a_{p,n})^j a_{k,n} \leq (a_{p+1,n})^{j+1} \tag{4.7}$$

is satisfied for all $n \in \mathbb{N}$.

Let $p, n \in \mathbb{N}$ and assume $n \in I_s$, $s \in \mathbb{N}$. There are two case for p and s: $p \leq s$ or $p > s$.

First we assume that $p \leq s$: In this case,

$$a_{p,n} = e^{-\frac{1}{p}\alpha_n}, \quad a_{p+1,n} \geq e^{-\frac{1}{p+1}\alpha_n} \quad \text{and} \quad a_{k,n} \leq e^{\left(-\frac{1}{k}+1\right)\alpha_n} \tag{4.8}$$

for all $k \geq q$. Then, the inequality 4.7 is satisfied for any $j \geq \dfrac{\frac{1}{p+1} - \frac{1}{k} + 1}{\frac{1}{p} - \frac{1}{p+1}}$.

Second we assume that $s < p$: In this case,

$$a_{p,n} = e^{\left(-\frac{1}{p}+1\right)\alpha_n}, \quad a_{p+1,n} = e^{\left(-\frac{1}{p+1}+1\right)\alpha_n} \quad \text{and} \quad a_{k,n} = e^{\left(-\frac{1}{k}+1\right)\alpha_n} \tag{4.9}$$

for all $k \geq q$. Therefore, the inequality 4.7 is satisfied for any $j \geq \dfrac{\frac{1}{p+1} - \frac{1}{k} + 1}{\frac{1}{p} - \frac{1}{p+1}}$.

Now, we choose a $j > 0$ satisfying

$$j \geq \max\left(\frac{\frac{1}{p+1} - \frac{1}{k} + 1}{\frac{1}{p} - \frac{1}{p+1}}, \frac{\frac{1}{p+1} - \frac{1}{k}}{\frac{1}{p} - \frac{1}{p+1}}\right) = \frac{\frac{1}{p+1} - \frac{1}{k} + 1}{\frac{1}{p} - \frac{1}{p+1}} \tag{4.10}$$

and so that the inequality 4.7 is satisfied for all $n \in \mathbb{N}$. Hence $\mathcal{K}_\alpha$ has the property Ω, as claimed. □

Remark 4.2.2 *It is worth noting that any element $\mathcal{K}_\alpha$ of the family $\mathcal{K}$ does not have the property (d_2),*

$$(d_2): \qquad \forall k \quad \exists j \quad \forall l \qquad \sup_n \frac{a_{kn}\, a_{ln}}{(a_{jn})^2} < +\infty. \tag{4.11}$$

Since for all $j \in \mathbb{N}$, $n \in I_j$,

$$a_{1,n} = e^{-\alpha_n} \qquad a_{jn} = e^{-\frac{1}{j}\alpha_n} \qquad a_{j+1,n} = e^{(-\frac{1}{j+1}+1)\alpha_n} \tag{4.12}$$

and

$$\frac{a_{1,n}\, a_{j+1,n}}{(a_{jn})^2} = e^{\frac{j+2}{j(j+1)}\alpha_n} \qquad \Rightarrow \qquad \sup_{n\in I_j} \frac{a_{1,n}\, a_{j+1,n}}{(a_{jn})^2} = \sup_{n\in\mathbb{N}} \frac{a_{1,n}\, a_{j+1,n}}{(a_{jn})^2} = +\infty \tag{4.13}$$

then, $\mathcal{K}_\alpha$ does not have the property (d_2). So the family $\mathcal{K}$ does not contain a power series space of finite type.

4.2.1 Kolmogorov diameters of an element $\mathcal{K}_\alpha$ of the family $\mathcal{K}$

In this subsection, we calculate Kolmogorov diameters of an element $\mathcal{K}_\alpha$ of the family $\mathcal{K}$. In order to determine n^{th}-Kolmogorov diameter of a Köthe space $\mathcal{K}_\alpha$, we will rerwrite the sequence $\left(\frac{a_{p,n}}{a_{q,n}}\right)_{n\in\mathbb{N}}$ in descending order. We know from Remark 2.2.4 that the n^{th}-Kolmogorov diameter of the space $\mathcal{K}_\alpha$ is the $n+1^{th}$-term of this descending sequence.

Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ parameterized by an exponent sequence α. Let us take a p, a $q > p$ and an $n \in I_s$, $s \in \mathbb{N}$. Then, we can write

$$\frac{a_{p,n}}{a_{q,n}} = \begin{cases} e^{c_{pq}\,\alpha_n}, & s \geq q \text{ or } s < p \\ e^{\left(c_{pq}-1\right)\alpha_n}, & p \leq s < q \end{cases} \tag{4.14}$$

where c_{pq} is the negative number $= -\dfrac{1}{p} + \dfrac{1}{q}$.

We define the set $\mathbf{I} = \bigcup\limits_{\mathbf{p}\leq\mathbf{s}<\mathbf{q}} \mathbf{I_s}$ with the elements $(n_i)_{\in\mathbb{N}}$ ordered increasingly, namely, $n_i \leq n_{i+1}$ for all $i \in \mathbb{N}$. We also denote the index of the element of I_p on the line with the equation $x+y = q+k-2$ by $\mathbf{s_k}$ for each $k = 0,1,2,...$, as seen from the following graphic. Since every a line with the equation $x+y = q+k-2$ has $q-p$ elements of I, then $\boxed{s_{k+1} - s_k = q-p}$ for every $k = 0,1,2,....$

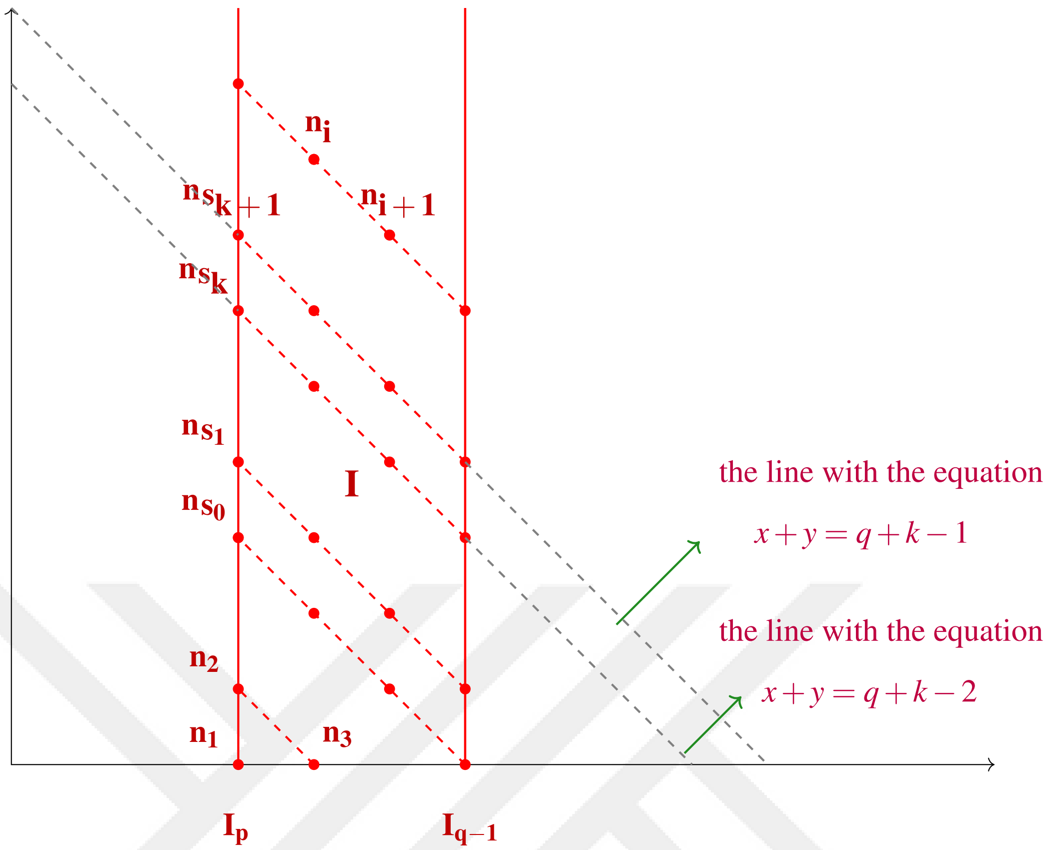

Figure 4.2 : Graphic 2

Considering the above graphic, the elements of I are placed in $\mathbb{N}$ as follows:

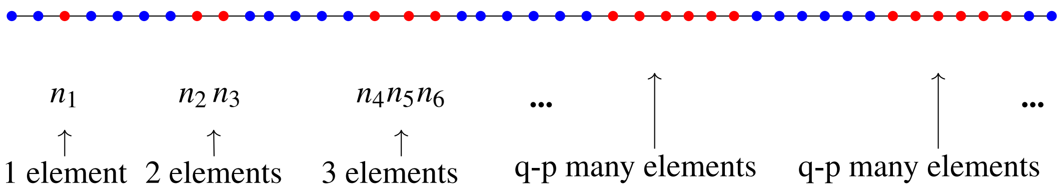

Figure 4.3 : Graphic 3

Now we assume that the terms $e^{c_{pq}\alpha_m}$, $m \in \mathbb{N} - I$, are on the blue points and the terms $e^{(c_{pq}-1)\alpha_{n_i}}$, $n_i \in I$, are on the red points at this line.

Before sorting the terms of the sequence $\left(\frac{a_{p,n}}{a_{q,n}}\right)_{n\in\mathbb{N}}$, we note that the terms of the sequences $\left(e^{c_{pq}\alpha_m}\right)_{m\in\mathbb{N}-I}$ and $\left(e^{(c_{pq}-1)\alpha_{n_i}}\right)_{i\in\mathbb{N}}$ have decreasing order in themselves.

We are now ready to order the terms of the sequence $\left(\frac{a_{p,n}}{a_{q,n}}\right)_{n\in\mathbb{N}}$ decreasing and read Kolmogorov diameters $d_n\left(U_q,U_p\right)$ for all $n=0,1,2,\ldots$.

At first, we take into account the part of $\left(\frac{a_{p,n}}{a_{q,n}}\right)_{n\in\mathbb{N}}$ including the first n_1-1 terms:

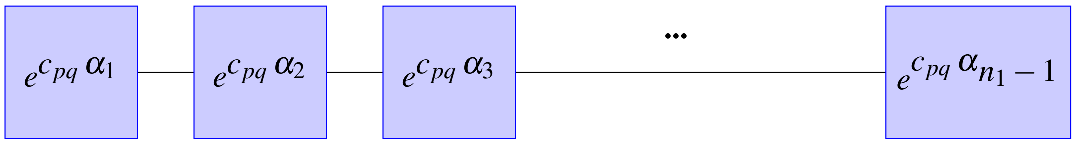

Figure 4.4 : Diagram 1

Since α is increasing, this part has decreasing order and all terms in this part is greater than the terms corresponding to the elements of I. Then, having decreasing order, this part remains the same. However, we write this part by shifting to the left taking into account the zero indices for Kolmogorov diameter.

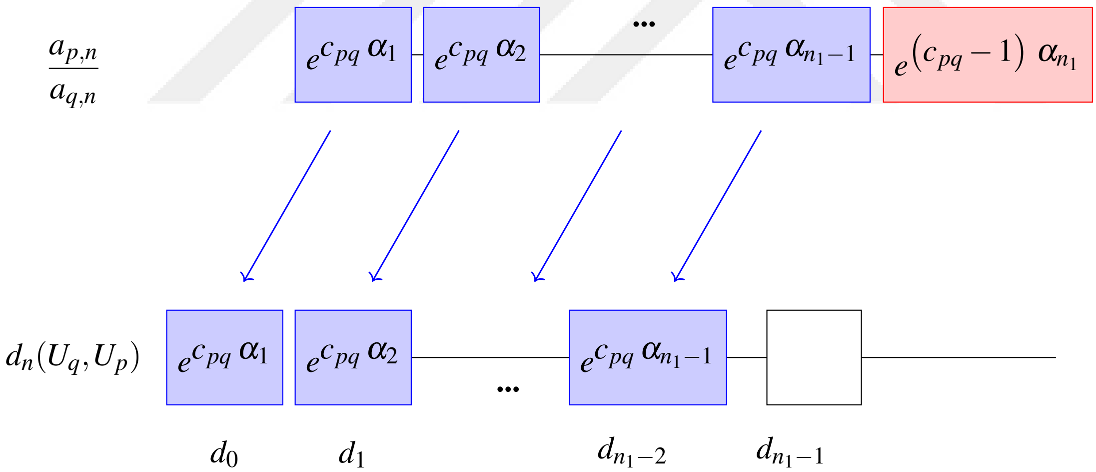

Figure 4.5 : Diagram 2

So, for every $\mathbf{0\leq n\leq n_1-2}$,

$$\mathbf{d_n(U_q,U_p)=e^{c_{pq}\,\alpha_{n+1}}}. \tag{4.15}$$

In order to find the diameter $d_{n_1-1}(U_q,U_p)$, we will compare the term $e^{\left(c_{pq}-1\right)\,\alpha_{n_1}}$ with the terms $e^{c_{pq}\,\alpha_m}$, $m\in\mathbb{N}-I$, $m>n_1$, and the greatest term gives the diameter $d_{n_1-1}(U_q,U_p)$:

$$e^{(c_{pq}-1)\alpha_{n_1}} \leq e^{c_{pq}\,\alpha_m} \quad \Leftrightarrow \quad \alpha_m \leq A_{pq}\alpha_{n_1} \tag{4.16}$$

where $A_{pq} = 1 + \dfrac{pq}{q-p}$. Then, the terms $e^{c_{pq}\,\alpha_m}$, $m \in \mathbb{N} - I$, $m > n_1$, satisfying $\alpha_m \leq A_{pq}\alpha_{n_1}$ is greater than the term $e^{\left(c_{pq}-1\right)\,\alpha_{n_1}}$. So we must write the terms $e^{c_{pq}\,\alpha_m}$, $m \in \mathbb{N} - I$, $m > n_1$, satisfying $\alpha_m \leq A_{pq}\alpha_{n_1}$ before the term $e^{\left(c_{pq}-1\right)\,\alpha_{n_1}}$ in decreasing order.

We call the greatest element $m \in \mathbb{N} - I$ satisfying $\alpha_m \leq A_{pq}\alpha_{n_1}$ as $\mathbf{i_1}$. As shown in the following figure, we can assume that there exists a $k_1 > 0$ so that the inequality

$$n_{s_{k_1}} < i_1 < n_{s_{(k_1+1)}} \tag{4.17}$$

holds.

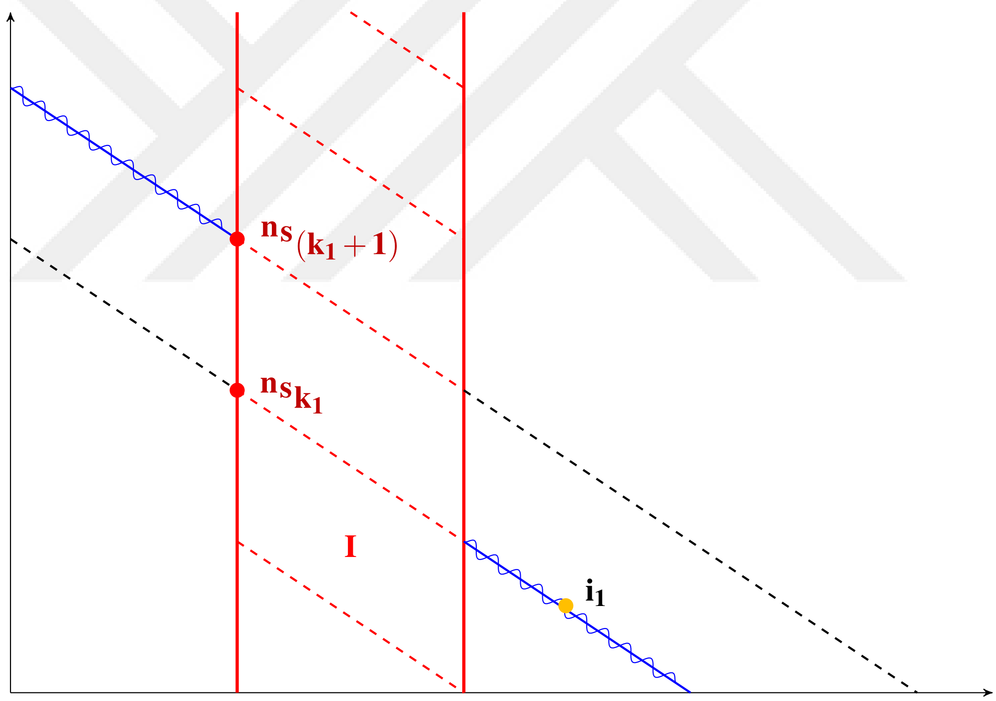

Figure 4.6 : Graphic 4

The above figure can also be visualized as follows:

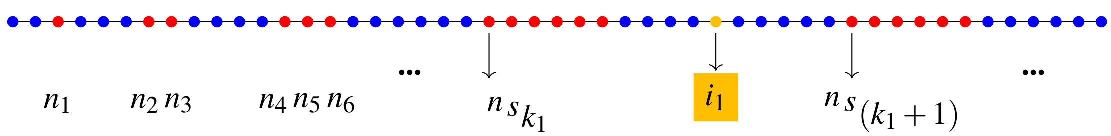

Figure 4.7 : Graphic 5

This means that the number of elements of I which is less than i_1 is $\mathbf{s_{(k_1+1)}-1}$. So, before the term $e^{(c_{pq}-1)\,\alpha_{n_1}}$, we will write $i_1-[s_{(k_1+1)}-1]$ many $e^{c_{pq}\,\alpha_m}$, $m \in \mathbb{N}-I$, $m \leq i_1$, terms in decreasing order. Furthermore, while writing these terms in decreasing order, every term $e^{(c_{pq}-1)\,\alpha_{n_a}}$, $1 \leq a \leq s_{(k_1+1)}-1$ shifts to the right and every term $e^{c_{pq}\alpha_m}$, $m \in \mathbb{N}-I$, $m \leq i_1$, shifts to the left, as shown in **Figure 4.9**. In order to find n_1-1^{th} Kolmogorov diameter, we shift the term corresponding to the first element n_1 of I. Considering also that we shift the terms to the left for $d_0(U_q,U_p)$, we find that for every $\mathbf{n_1-1 \leq n \leq n_2-3}$,

$$\mathbf{d_n(U_q,U_p)=e^{c_{pq}\,\alpha_{n+2}}}. \tag{4.18}$$

So, we found the Kolmogorov diameters until the indices n_2-2. Now, we also shift the terms corresponding to the element n_2 and n_3 of I. Up till now, we shift the terms to the left four-indices, then we find that for every $\mathbf{n_2-2 \leq n \leq n_4-5}$

$$\mathbf{d_n(U_q,U_p)=e^{c_{pq}\,\alpha_{n+4}}}. \tag{4.19}$$

We would like to point out that the endpoints of the intervals in which we determine Kolmogorov diameters are generally represented by the elements of I_p. Because the terms corresponding to the elements of I that we shift to the right and the terms corresponding to the elements of $\mathbb{N}-I$ that we shift to the left are between the two elements of I_p, as seen in the following figure.

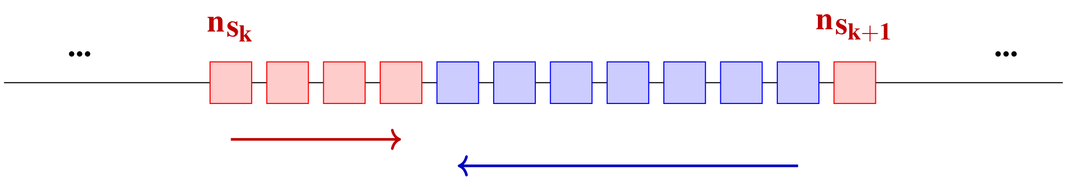

Figure 4.8 : Graphic 6

Another significant point in writing the endpoints of the intervals in which we determine the diameters are to find out how many elements, the terms corresponding to the elements of I, we shift to the right.

Let's continue to calculate the diameters with this perspective.

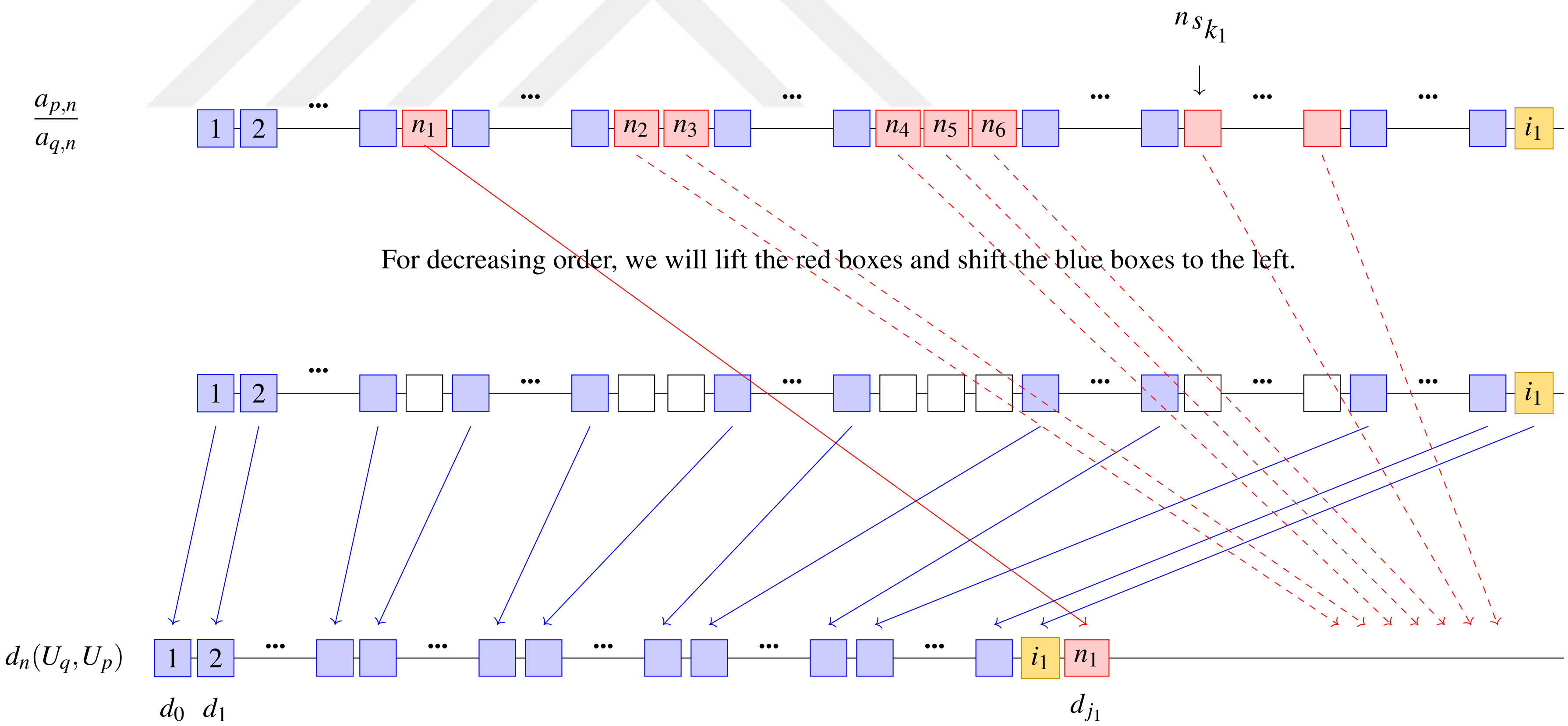

Figure 4.9 : Diagram 3

Let us assume that we replaced $n_{s_0} - [s_0 + 1]$ terms in decreasing order. In order to find $n_{s_0} - s_0$-th Kolmogorov diameter, we shift s_1 terms corresponding to the elements of I in total, for every $\mathbf{n_{s_0} - s_0 \leq n \leq n_{s_1} - [s_1 + 1]}$, we have

$$\mathbf{d_n(U_q, U_p) = e^{c_{pq}\alpha_{n+s_1}}}. \tag{4.20}$$

Considering the terms that we shift to the right in each step, we can write for every $0 \leq k < k_1$ and for every $\mathbf{n_{s_k} - s_k \leq n \leq n_{s_{(k+1)}} - [s_{(k+1)} + 1]}$

$$\mathbf{d_n\left(U_q, U_p\right) = e^{c_{pq}\,\alpha_{n+s_{(k+1)}}}} \tag{4.21}$$

and for all $\mathbf{n_{s_{k_1}} - s_{k_1} \leq n \leq i_1 - s_{(k_1+1)}}$

$$\mathbf{d_n\left(U_q, U_p\right) = e^{c_{pq}\,\alpha_{n+s_{(k_1+1)}}}}. \tag{4.22}$$

Therefore, we shift $i_1 - [s_{(k_1+1)} - 1]$ many terms $e^{c_{pq}\,\alpha_m}$, $m \in \mathbb{N} - I$, $m \leq i_1$ to left, namely, we sort all terms which is greater than $e^{(c_{pq}-1)\alpha_{n_1}}$. Hence, the term $e^{(c_{pq}-1)\alpha_{n_1}}$ is replaced at the indices $\boxed{\mathbf{j_1 = i_1 - s_{(k_1+1)} + 1}}$, namely,

$$\boxed{\mathbf{d_{j_1}(U_q, U_p) = e^{(c_{pq}-1)\alpha_{n_1}}}}. \tag{4.23}$$

Now assume that the first $a-1$, $(a \geq 1)$ terms corresponding to the elements of I are placed in decreasing order.

Before the term $e^{(c_{pq}-1)\alpha_{n_a}}$, we must write the terms $e^{c_{pq}\,\alpha_m}$, $m \in \mathbb{N} - I$ which is greater than $e^{(c_{pq}-1)\alpha_{n_a}}$, satisfying the inequality $\alpha_m \leq A_{pq}\alpha_{n_a}$. We call the greatest element of $m \in \mathbb{N}$ satisfying $\alpha_m \leq A_{pq}\alpha_{n_a}$ as $\mathbf{i_a}$. We can assume that there exists a $k_a \in \mathbb{N}$ so that

$$n_{s_{k_a}} < i_a < n_{s_{(k_a+1)}} \tag{4.24}$$

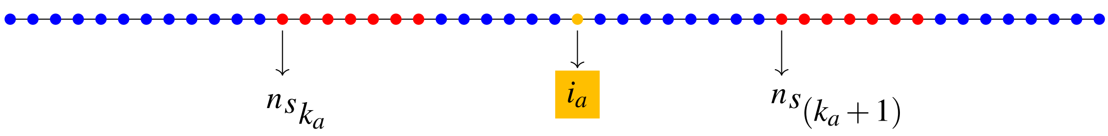

Figure 4.10 : Graphic 7

This means that the number of elements of I which is less than i_a is $\mathbf{s_{(k_a+1)}-1}$. So, before the term $e^{(c_{pq}-1)\,\alpha n_a}$, we write $i_a-(s_{(k_a+1)}-1)$ many $e^{c_{pq}\,\alpha_m}$, $m\in\mathbb{N}-I$, $m\le i_a$, terms in decreasing order. Since we assume that the first $a-1$ terms corresponding to the elements of I is placed in decreasing order, then the term $e^{(c_{pq}-1)\alpha n_a}$ are replaced at the indices $\boxed{\mathbf{j_a=i_a-s_{(k_a+1)}+a}}$, namely,

$$\boxed{\mathbf{d_{j_a}(U_q,U_p)=e^{(c_{pq}-1)\alpha n_a}.}} \tag{4.25}$$

Now, we determine Kolmogorov diameters between the indices j_a and j_{a+1} for every $a\ge 1$.

Starting the index j_a, we must compare the term $e^{(c_{pq}-1)\,\alpha n_{a+1}}$ with the terms $e^{c_{pq}\,\alpha_m}$, $m\in\mathbb{N}-I$ and for every $m\in\mathbb{N}-I$ satisfying $\alpha_m\le A_{pq}\alpha n_{a+1}$, we write the terms $e^{c_{pq}\alpha_m}$, before the term $e^{(c_{pq}-1)\,\alpha n_{a+1}}$. Again, we call the largest element of $\mathbb{N}-I$ satisfying above inequality as i_{a+1} for which there is $k_{a+1}\in\mathbb{N}$ satisfying

$$ns_{k_{a+1}}<i_{a+1}<ns_{(k_{a+1}+1)}. \tag{4.26}$$

Let us continue to decreasing order from j_a+1:

For all $\mathbf{j_a+1\le n\le ns_{(k_a+1)}-s_{(k_a+1)}+a-1}$

$$\mathbf{d_n(U_q,U_p)=e^{c_{pq}\alpha n+s_{(k_a+1)}-a}}. \tag{4.27}$$

If any, for every $k_a+1\le k\le k_{a+1}-1$ and for every $\mathbf{ns_k-s_k+a\le n\le ns_{(k+1)}-s_{(k+1)}+a-1}$

$$\mathbf{d_n(U_q,U_p)=e^{c_{pq}\alpha n+s_{(k+1)}-a}} \tag{4.28}$$

and for every $\mathbf{ns_{k_{a+1}}-s_{k_{a+1}}+a\le n\le i_{a+1}-s_{(k_{a+1}+1)}+a}$

$$\mathbf{d_n(U_q,U_p)=e^{c_{pq}\alpha n+s_{(k_{a+1}+1)}-a}}. \tag{4.29}$$

We sort all terms which is greater than $e^{(c_{pq}-1)\alpha n_{a+1}}$. Then, the term $e^{(c_{pq}-1)\alpha n_{a+1}}$ is replaced at the indices $\boxed{\mathbf{j_{a+1}=i_{a+1}-s_{(k_{a+1}+1)}+a+1}}$, namely,

$$\boxed{\mathbf{d_{j_{a+1}}(U_q,U_p)=e^{(c_{pq}-1)\alpha n_{a+1}}}}. \tag{4.30}$$

Hence, we determine all Kolmogorov diameters between the terms $e^{(c_{pq}-1)\alpha n_a}$ and $e^{(c_{pq}-1)\alpha n_{a+1}}$ for every $a \geq 1$. Therefore, we can calculate all Kolmogorov diameters by following the above observation, and finally we can write:

1. Let $J := \{j_a : a \in \mathbb{N}\}$ where $j_a = i_a - s_{(k_a+1)} + a$. For all $a \in \mathbb{N}$,

$$\boxed{d_{j_a}(U_q, U_p) = e^{(c_{pq}-1)\,\alpha n_a}.} \tag{4.31}$$

2. For $a, k \in \mathbb{N}$, we define

$$I_{a,k} = \left[n s_k - s_k + a, n s_{k+1} - s_{k+1} + a - 1\right] \tag{4.32}$$

and

$$K = \bigcup_{a \in \mathbb{N}} \bigcup_{k_a+1 \leq k \leq k_{a+1}-1} I_{a,k}. \tag{4.33}$$

For every $n \in K$, there is an $a \in \mathbb{N}$ and a $k \in \mathbb{N}$ satisfying $k_a + 1 \leq k \leq k_{a+1} - 1$ such that

$$\boxed{d_n(U_q, U_p) = e^{c_{pq}\alpha n + s_{k+1} - a}.} \tag{4.34}$$

3. Let $L = \bigcup_{a \in \mathbb{N}} \left[j_a + 1, n s_{(k_a+1)} - s_{(k_a+1)} + a - 1\right]$. For every $n \in L$, there is an $a \in \mathbb{N}$ such that

$$\boxed{d_n(U_q, U_p) = e^{c_{pq}\alpha n + s_{(k_a+1)} - a}.} \tag{4.35}$$

4. Let $M = \bigcup_{a \in \mathbb{N}} \left[n s_{k_a} - s_{k_a} + a - 1, j_a - 1\right]$. For every $n \in M$, there is an $a \in \mathbb{N}$ such that

$$\boxed{d_n(U_q, U_p) = e^{c_{pq}\alpha n + s_{(k_a+1)} - (a-1)}.} \tag{4.36}$$

All Kolmogorov diameters in the light of above observation are found since $\mathbb{N} = \{0, 1, ..., n_1 - 2\} \cup J \cup K \cup L \cup M$. This completes the determination of the diameters.

Now, we give an estimation for Kolmogorov diameters of an element $\mathcal{K}_\alpha$ of the family $\mathcal{K}$ which is parameterized by α.

Theorem 4.2.3 *Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ with the parameter α. For every p, $q > p$ there exists a $N \in \mathbb{N}$ such that*

$$e^{c_{pq}\alpha 4n} \leq d_n(U_q, U_p) \leq e^{c_{pq}\alpha n} \tag{4.37}$$

for every $n \geq N$.

Proof. Let $p \in \mathbb{N}$ and $q > p$. Above we obtained Kolmogorov diameters $d_n(U_q, U_p)$ on each subsets $\{0, 1, ..., n_1 - 1\}$, J, K, L and M of $\mathbb{N}$. We will show that the inequality 4.37 holds for sufficiently large elements of each subsets J, K, L, and M of $\mathbb{N}$.

Primarily, we will show that $2j_a > i_a$ for sufficiently large $a \in \mathbb{N}$. We know that for every i_a there exists a $k_a \in \mathbb{N}$ satisfying $n s_{k_a} < i_a < n s_{(k_a+1)}$.

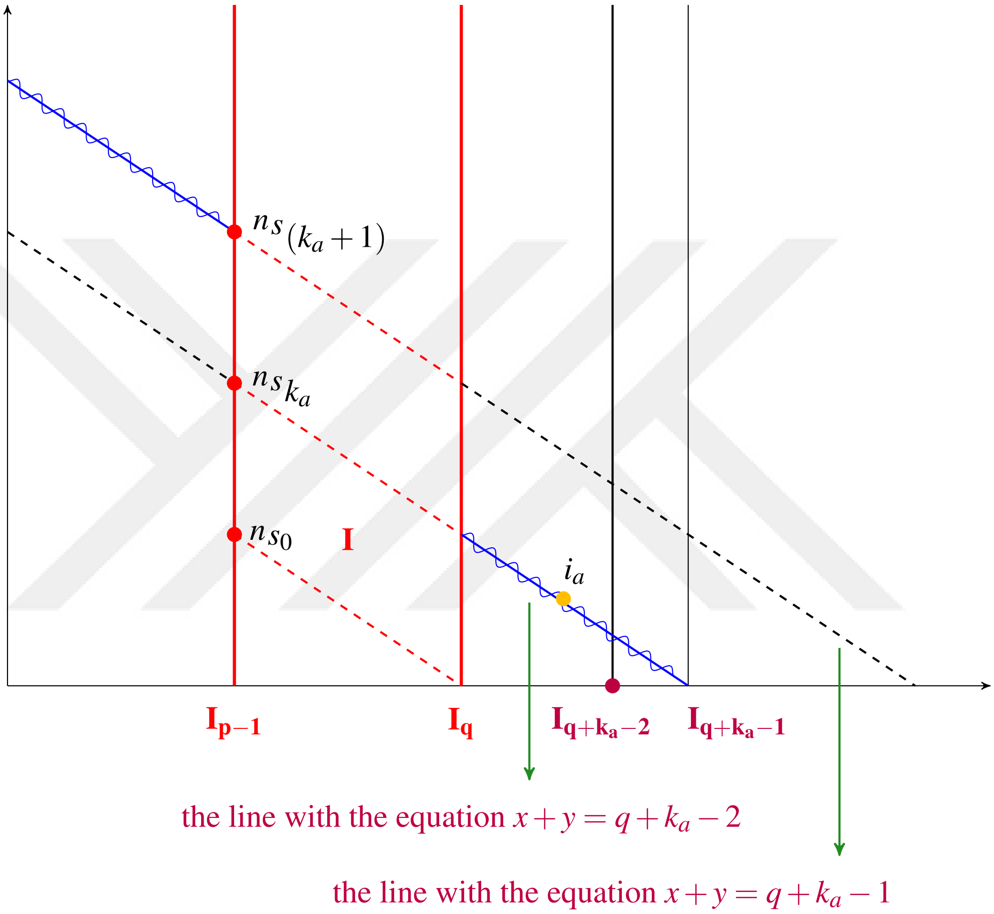

Figure 4.11 : Graphic 8

As shown in the above figure, $n s_{k_a}$ is on the line which has the equation $x + y = q + k_a - 2$. Since the first element of I_{q+k_a-2} is less than $n s_{k_a}$, we can write

$$n s_{k_a} \geq \frac{(q+k_a-2)(q+k_a-1)}{2} = \frac{(q-2)(q-1)}{2} + (q-1)k_a + \frac{k_a(k_a-1)}{2}. \quad (4.38)$$

Also as $n s_{k_a} \leq i_a$, then we can write

$$i_a \geq \frac{(q-2)(q-1)}{2} + (q-1)k_a + \frac{k_a(k_a-1)}{2}. \quad (4.39)$$

Since $\lim_{a\to+\infty} k_a = +\infty$, we can assume that $\frac{k_a(k_a-1)}{4} \geq (k_a+1)(q-p)$ and $\frac{(q-1)k_a}{2} \geq s_0$ for sufficiently large $a \in \mathbb{N}$. Hence we can write

$$\frac{i_a}{2} \geq s_0 + (k_a+1)(q-p) = s_{(k_a+1)} \tag{4.40}$$

and we find

$$j_a = i_a - s_{(k_a+1)} + a > i_a - \frac{i_a}{2} = \frac{i_a}{2} \quad \Rightarrow \quad \boxed{2j_a > i_a}. \tag{4.41}$$

Now, we will show that the inequality 4.37 is satisfied for a sufficiently large element of J. Let take an $a \in \mathbb{N}$ satisfiying $2j_a > i_a$. There exist two cases for $2j_a$: $2j_a \in \mathbb{N} - I$ or $2j_a \in I$.

We know that i_a is the greatest element of $m \in \mathbb{N} - I$ satisfying $e^{(c_{pq}-1)\alpha_{n_a}} \leq e^{c_{pq}\alpha_m}$, then we can write

$$e^{c_{pq}\alpha_k} < e^{(c_{pq}-1)\alpha_{n_a}} \tag{4.42}$$

for every $k > i_a$, $k \in \mathbb{N} - I$. If $2j_a \in \mathbb{N} - I$, then

$$e^{c_{pq}\alpha_{4j_a}} \leq e^{c_{pq}\alpha_{2j_a}} < e^{(c_{pq}-1)\alpha_{n_a}}. \tag{4.43}$$

If $2j_a \in I$, then $2j_a + (q-p) \in \mathbb{N} - I$ and $2j_a + (q-p) \leq 4j_a$ is satisfied for a sufficiently large a and we find

$$e^{c_{pq}\alpha_{4j_a}} \leq e^{c_{pq}\alpha_{2j_a+(q-p)}} < e^{(c_{pq}-1)\alpha_{n_a}}. \tag{4.44}$$

Also, we know that $i_a \geq j_a$ for every $a \in \mathbb{N}$, thus we can write

$$d_{j_a}(U_q, U_p) = e^{(c_{pq}-1)\alpha_{n_a}} \leq e^{c_{pq}\alpha_{i_a}} \leq e^{c_{pq}\alpha_{j_a}}. \tag{4.45}$$

The above inequalites give us that

$$\boxed{e^{c_{pq}\alpha_{4j_a}} \leq d_{j_a}(U_q, U_p) = e^{(c_{pq}-1)\alpha_{n_a}} \leq e^{c_{pq}\alpha_{j_a}}.} \tag{4.46}$$

Then, the inequality 4.37 is satisfied for sufficiently large element of J.

We now prove that the inequality 4.37 is satisfied for sufficiently large elements of K, L and M. In order to see this, we first show that

$$n_{s_k} \geq 2s_k \tag{4.47}$$

for sufficently large $k \in \mathbb{N}$.

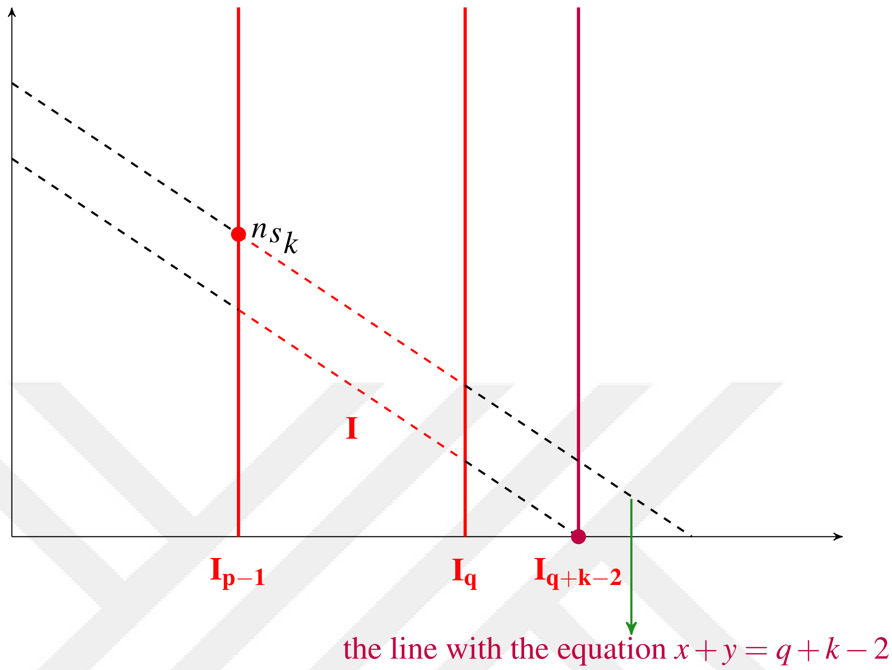

Figure 4.12 : Graphic 9

We know that n_{s_k} is on the line which has equation $x+y=q+k-2$ for every $k=0,1\ldots.$ Since the first element of I_{q+k-2} is less than n_{s_k}, then we can write

$$n_{s_k} \geq \frac{(q+k-2)(q+k-1)}{2} = \frac{(q-2)(q-1)}{2} + (q-1)k + \frac{k(k-1)}{2}. \tag{4.48}$$

The inequalities

$$\frac{k.(k-1)}{4} \geq k(q-p) \quad \text{and} \quad (q-1)k \geq 2(s_0+1) \tag{4.49}$$

hold for a sufficiently large k. Then we find

$$\boxed{n_{s_k} \geq 2(s_0+k(q-p)+1) = 2s_k} \tag{4.50}$$

for a sufficiently large $k \in \mathbb{N}$.

Now we show that the inequality 4.37 is satisfied for sufficiently large element of K. Let take an $n \in K$. Then, there exist a $a \in \mathbb{N}$ and a $k \in \mathbb{N}$ satisfying $k_a+1 \leq k \leq k_{a+1}-1$ such that $n_{s_k} - s_k + a \leq n \leq n_{s_{k+1}} - s_{k+1} + a - 1$ and

$$d_n\left(U_q,U_p\right)=e^{c_{pq}\alpha_{n+s_{k+1}-a}}. \tag{4.51}$$

Since $n_{s_k} \geq 2s_k$ for a sufficiently large $k \in \mathbb{N}$, we can write

$$s_k \leq n_{s_k} - s_k + a \leq n \quad \Rightarrow \quad n + s_{k+1} - a \leq 2n. \tag{4.52}$$

for sufficiently large a. Then, we obtain

$$d_n\left(U_q,U_p\right)=e^{c_{pq}\alpha_{n+s_{k+1}-a}} \geq e^{c_{pq}\alpha_{2n}} \geq e^{c_{pq}\alpha_{4n}} \tag{4.53}$$

and always we have

$$d_n\left(U_q,U_p\right)=e^{c_{pq}\alpha_{n+s_{k+1}-a}} \leq e^{c_{pq}\alpha_n} \tag{4.54}$$

since α is increasing.

Therefore, the inequality 4.37 is satisfied for sufficiently large elemets of K.

Now, we will show that the inequality 4.37 is satisfied for a sufficiently large element of L. Let us take a $n \in L$. Then, there is an $a \in \mathbb{N}$ such that

$$j_a + 1 \leq n \leq n_{s_{(k_a+1)}} - s_{(k_a+1)} + a - 1 \tag{4.55}$$

and

$$d_n\left(U_q,U_p\right)=e^{c_{pq}\alpha_{n+s_{(k_a+1)}-a}}. \tag{4.56}$$

Since $s_{k_a} \leq n_{s_{k_a}} - s_{k_a} + a \leq j_a + 1 \leq n$ and $n + s_{(k_a+1)} - a \leq 2n$ for a sufficiently large n, then we find

$$d_n\left(U_q,U_p\right)=e^{c_{pq}\alpha_{n+s_{(k_a+1)}-a}} \geq e^{c_{pq}\alpha_{2n}} \geq e^{c_{pq}\alpha_{4n}}, \tag{4.57}$$

and always we have

$$d_n\left(U_q,U_p\right)=e^{c_{pq}\alpha_{n+s_{k_a}-a}} \leq e^{c_{pq}\alpha_n} \tag{4.58}$$

since α is increasing. Therefore, the inequality 4.37 is satisfied for sufficiently large element of L.

Now we will show that the inequality 4.37 is satisfied for a sufficiently large element of M. If $n \in M$, then there is an $a \in \mathbb{N}$

$$n_{s_{k_a}} - s_{k_a} + a \leq n \leq j_a - 1(4.59)$$

and

$$d_n\left(U_q,U_p\right)=e^{c_{pq}\alpha_{n+s_{(k_a+1)}-(a-1)}}. \tag{4.60}$$

Again we can write $s_{k_a}\leq n_{s_{k_a}}-s_{k_a}+a\leq n$ and $n_{s_{k_a}}+s_{(k_a+1)}-a+1\leq 2n$ for a sufficiently large a. Hence we find

$$d_n\left(U_q,U_p\right)=e^{c_{pq}\alpha_{n+s_{(k_a+1)}-(a-1)}}\geq e^{c_{pq}\alpha_{2n}}\geq e^{c_{pq}\alpha_{4n}} \tag{4.61}$$

and always we have

$$d_n\left(U_q,U_p\right)=e^{c_{pq}\alpha_{n+s_{(k_a+1)}-a}}\leq e^{c_{pq}\alpha_n} \tag{4.62}$$

since α is increasing. Therefore, the inequality 4.37 is satisfied for a sufficiently large element of M. This completes the proof. □

4.2.2 The diametral dimension and the approximate diametral dimension of an element of the family $\mathcal{K}$ parameterized by a stable sequence α

As a consequence of Theorem 4.2.3, we will compute the diametral dimension and the approximate diametral dimension of an element $\mathcal{K}_\alpha$ of the family $\mathcal{K}$ which is parameterized by a stable sequence α.

Corollary 4.2.4 *Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ which is parameterized by a stable sequence α. Then, $\Delta(\mathcal{K}_\alpha)=\Delta(\Lambda_1(\alpha_n))$ and $\delta(\mathcal{K}_\alpha)=\delta(\Lambda_1(\alpha_n))$.*

Proof. From Proposition 4.2.3, we have

$$\Delta(\Lambda_1(\alpha_n))\subseteq\Delta(\mathcal{K}_\alpha)\subseteq\Delta(\Lambda_1(\alpha_{4n})) \tag{4.63}$$

and

$$\delta(\Lambda_1(\alpha_{4n}))\subseteq\delta(\mathcal{K}_\alpha)\subseteq\delta(\Lambda_1(\alpha_n)). \tag{4.64}$$

On the other hand, $\Lambda_1(\alpha_n)\cong\Lambda_1(\alpha_{4n})$ since α is stable. Then $\Delta(\mathcal{K}_\alpha)=\Delta(\Lambda_1(\alpha_n))$ and $\delta(\mathcal{K}_\alpha)=\delta(\Lambda_1(\alpha_n))$. □

4.2.3 The diametral dimension and the approximate diametral dimension of an element of the family $\mathcal{K}$ parameterized by an unstable sequence α

In this subsection, we will prove that $\Delta(\mathcal{K}_\alpha) = \Delta(\Lambda_1(\alpha_{n+1}))$ and $\delta(\mathcal{K}_\alpha) \neq \delta(\Lambda_1(\alpha_{n+1}))$ for an element $\mathcal{K}_\alpha$ of the family $\mathcal{K}$ which is parameterized by an unstable sequence α. Besides, we will show that all regular elements of the family $\mathcal{K}$ are parameterized by an unstable sequence α.

Proposition 4.2.5 *Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ which is parameterized by an unstable sequence α. Then, $\Delta(\mathcal{K}_\alpha) = \Delta(\Lambda_1(\alpha_{n+1}))$.*

Proof. We can calculate Kolmogorov diameters as in the previous determined for every p and $q > p$. Since α is unstable, then there exists an $a_0 \in \mathbb{N}$ such that for all $a \geq a_0$, there is no $m > n_a$, $m \in \mathbb{N}$ satisfying

$$\alpha_m \leq A_{pq}\,\alpha_{n_a} \tag{4.65}$$

Now, we examine closely the indices replaced the term $e^{(c_{pq}-1)\,\alpha_{n_{a_0}}}$. We know that

$$d_{j_{(a_0-1)}}(U_q,U_p) = e^{(c_{pq}-1)\alpha_{n_{(a_0-1)}}} \tag{4.66}$$

where $j_{(a_0-1)} := i_{(a_0-1)} - s_{(a_0-1)} + a_0 - 2$. Since $\alpha_{i_{(a_0-1)}} \leq A_{pq}\,\alpha_{n_{(a_0-1)}}$ and there is no $m > n_{a_0}$ satisfying $\alpha_m \leq A_{pq}\,\alpha_{n_{a_0}}$, then we find $i_{(a_0-1)} < n_{a_0}$. Therefore, the following figure below is valid:

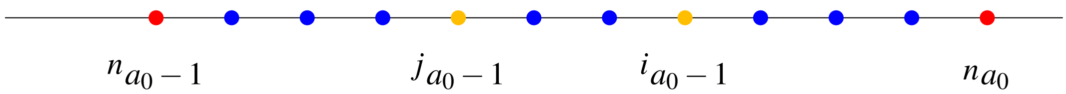

Figure 4.13 : Graphic 10

This gives that for all $j_{(a_0-1)} \leq n \leq n_{a_0} - 2$,

$$d_n(U_q,U_p) = e^{c_{pq}\,\alpha_{n+1}}. \tag{4.67}$$

Besides, we obtain that the sequence $\left(\frac{a_{p,n}}{a_{q,n}}\right)_{n\in\mathbb{N}}$ has decreasing order starting from the indices $j_{(a_0-1)}+1$, since for every $a \geq a_0$, there is no $n > n_{a_0}$ satisfying $\alpha_n \leq A_{pq}\,\alpha_{n_a}$. Then, we have for all $a \geq a_0$

$$d_{n_a-1}(U_q,U_p) = e^{(c_{pq}-1)\alpha_{n_a}} \tag{4.68}$$

and for all $m \geq j_{(a_0-1)}$, $m \in \mathbb{N} - I$

$$d_m(U_q, U_p) = e^{c_{pq}\,\alpha_{m+1}}. \tag{4.69}$$

Since $d_n(U_q, U_p) \leq e^{c_{pq}\alpha_{n+1}}$ for every $n \in \mathbb{N}$, then we find $\Delta(\mathcal{K}_\alpha) \supseteq \Delta(\Lambda_1(\alpha_{n+1}))$.
For the other direction, let us take a sequence $(x_n)_{n\in\mathbb{N}} \in \Delta(\mathcal{K}_\alpha)$, an $\varepsilon > 0$ and a $p \in \mathbb{N}$ satisfying $\frac{1}{p} < \varepsilon$. We will show that

$$\sup_{n\in\mathbb{N}} |x_n|\, e^{-\varepsilon\,\alpha_{n+1}} < +\infty. \tag{4.70}$$

Since $(x_n)_{n\in\mathbb{N}} \in \Delta(\mathcal{K}_\alpha)$, there exist a $q > p$ and $M_1 > 0$ satisfying

$$\sup_{n\in\mathbb{N}} |x_n|\, d_n\left(U_p, U_q\right) < M_1. \tag{4.71}$$

Let us define $I = \bigcup_{p\leq s<q} I_s$. For sufficiently large $n \in \mathbb{N} - I$, we can write

$$|x_n|\, e^{-\varepsilon\,\alpha_{n+1}} \leq |x_n|\, d_n\left(U_q, U_p\right) = e^{c_{pq}\,\alpha_{n+1}} \leq M_1 \tag{4.72}$$

since $c_{pq} \geq -\varepsilon$. Therefore, the sequence $|x_n|\, e^{-\varepsilon\alpha_{n+1}}$ is bounded on the set $\mathbb{N} - I$. If we show that $|x_n|\, e^{-\varepsilon\,\alpha_{n+1}}$ is also bounded on I, then we will find that $(x_n)_{n\in\mathbb{N}} \in \Delta\left(\Lambda_1\left(\alpha_{n+1}\right)\right)$.
Let take another $p_0 > q$, then there exist a q_0 and $M_2 > 0$ such that

$$\sup_{n\in\mathbb{N}} |x_n|\, d_n\left(U_{q_0}, U_{p_0}\right) < M_2. \tag{4.73}$$

Let us define $J = \bigcup_{p_0\leq s<q_0} I_s$. Since $c_{p_0,q_0} \geq -\varepsilon$, we find

$$|x_n|\, e^{-\varepsilon\,\alpha_{n+1}} \leq |x_n|\, d_n\left(U_{q_0}, U_{p_0}\right) = e^{c_{p_0,q_0}\alpha_{n+1}} \leq M_2. \tag{4.74}$$

for sufficently large $n+1 \in \mathbb{N} - J$. Also, it is easy to see that $I \subset \mathbb{N} - J$. Then, the above inequalities give us that

$$|x_n|\, e^{-\varepsilon\,\alpha_{n+1}} \leq M_2. \tag{4.75}$$

for all $n \in I$. Hence, the sequence $|x_n|\, e^{-\varepsilon\,\alpha_{n+1}}$ is also bounded on I. Therefore, we find

$$\sup_{n\in\mathbb{N}} |x_n|\, e^{-\varepsilon\,\alpha_{n+1}} < +\infty \tag{4.76}$$

and $(x_n)_{n\in\mathbb{N}} \in \Delta\left(\Lambda_1\left(\alpha_{n+1}\right)\right)$. This says that $\Delta(\mathcal{K}_\alpha) = \Delta(\Lambda_1\left(\alpha_{n+1}\right))$. $\square$

Proposition 4.2.6 *Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ which is parameterized by a stable sequence α. Then, $\delta(\mathcal{K}_\alpha) \neq \delta(\Lambda_1(\alpha_{n+1}))$.*

Proof. In the proof of the previous proposition, we show that if α is unstable, then for all $p \in \mathbb{N}$ and $q > p$, there is a $a_0 \in \mathbb{N}$ such that for all $a \geq a_0$

$$d_{n_a - 1}(U_q, U_p) = e^{(c_{p,q} - 1)\alpha_{n_a}}, \tag{4.77}$$

so the last equality holds except for finitely many numbers of elements of I. Then we have

$$\frac{\varepsilon_{n_a-1}(p,q)}{\alpha_{n_a}} = 1 - c_{p,q} \tag{4.78}$$

and

$$\limsup_{a\in\mathbb{N}} \frac{\varepsilon_{n_a-1}(p,q)}{\alpha_{n_a}} = 1 - c_{p,q} \qquad \Rightarrow \qquad \inf_p \sup_q \limsup_{n\in\mathbb{N}} \frac{\varepsilon_n(p,q)}{\alpha_{n+1}} > 0. \tag{4.79}$$

By Proposition 3.1.2, we have $\delta\left(K\left(a_{k,n}\right)\right) \neq \delta\left(\Lambda_1\left(\alpha_{n+1}\right)\right)$. □

Remark 4.2.7 *Proposition 4.2.5 and Proposition 4.2.6 shows that Question 1.0.2 has a negative answer for the elements of the family $\mathcal{K}$ which is parametrized by an unstable exponent sequence.*

Now, we will show that all regular elements of the family $\mathcal{K}$ are parameterized by an unstable sequence α. First, we will give a condition for the regularity of the elements of the family $\mathcal{K}$.

Recall that a Köthe space generated by the matrix $\left(a_{k,n}\right)_{k,n\in\mathbb{N}}$ is called regular if the inequality

$$\frac{a_{k+1,n}}{a_{k,n}} \leq \frac{a_{k+1,n+1}}{a_{k,n+1}} \tag{4.80}$$

is satisfied for all $k,n \in \mathbb{N}$ see Definition 2.2.3.

Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ parameterized by an exponent sequence α and $n \in I_s$, $s \in \mathbb{N}$. Then, there exist two cases for $n+1$: $n+1 \in I_{s+1}$ or $n+1 \in I_1$.

- First we assume $n+1 \in I_{s+1}$: For this case, $n+1 \geq \dfrac{(s+1)(s+2)}{2} \geq s+1$.

i) For $k+1 \leq s$, we have

$$a_{k,n} = e^{-\frac{1}{k}\alpha_n}, \qquad a_{k+1,n} = e^{-\frac{1}{k+1}\alpha_n},$$
$$a_{k,n+1} = e^{-\frac{1}{k}\alpha_{n+1}}, \qquad a_{k+1,n+1} = e^{-\frac{1}{k+1}\alpha_{n+1}}. \tag{4.81}$$

Since α is increasing, the inequality

$$\frac{a_{k+1,n}}{a_{k,n}} = e^{\left(\frac{1}{k} - \frac{1}{k+1}\right)\alpha_n} \leq e^{\left(\frac{1}{k} - \frac{1}{k+1}\right)\alpha_{n+1}} = \frac{a_{k+1,n+1}}{a_{k,n+1}} \tag{4.82}$$

holds in this case.

ii) For $k \geq s+1$, we have

$$a_{k,n} = e^{\left(-\frac{1}{k}+1\right)\alpha_n}, \qquad a_{k+1,n} = e^{\left(-\frac{1}{k+1}+1\right)\alpha_n},$$
$$a_{k,n+1} = e^{\left(-\frac{1}{k}+1\right)\alpha_{n+1}}, \qquad a_{k+1,n+1} = e^{\left(-\frac{1}{k+1}+1\right)\alpha_{n+1}}. \tag{4.83}$$

Since α is increasing, the inequality

$$\frac{a_{k+1,n}}{a_{k,n}} = e^{\left(\frac{1}{k} - \frac{1}{k+1}\right)\alpha_n} \leq e^{\left(\frac{1}{k} - \frac{1}{k+1}\right)\alpha_{n+1}} = \frac{a_{k+1,n+1}}{a_{k,n+1}}. \tag{4.84}$$

holds in this case.

iii) For $k = s$, we have

$$a_{k,n} = e^{-\frac{1}{k}\alpha_n}, \qquad a_{k+1,n} = e^{\left(-\frac{1}{k+1}+1\right)\alpha_n},$$
$$a_{k,n+1} = e^{-\frac{1}{k}\alpha_{n+1}}, \qquad a_{k+1,n+1} = e^{-\frac{1}{k+1}\alpha_{n+1}}. \tag{4.85}$$

Then, these give that

$$\frac{a_{k+1,n}}{a_{k,n}} = e^{\left(\frac{1}{k} - \frac{1}{k+1}+1\right)\alpha_n} \quad \text{and} \quad \frac{a_{k+1,n+1}}{a_{k,n+1}} = e^{\left(\frac{1}{k} - \frac{1}{k+1}\right)\alpha_{n+1}}. \tag{4.86}$$

In this case, the regularity condition $\frac{a_{k+1,n}}{a_{k,n}} \leq \frac{a_{k+1,n+1}}{a_{k,n+1}}$ is equivalent to the following inequality:

$$(1+k(k+1))\alpha_n \leq \alpha_{n+1} \qquad \forall\, n \in I_k,\ k \in \mathbb{N} \tag{4.87}$$

- Next, we assume $n+1 \in I_1$: For this case, $n = \dfrac{s(s+1)}{2} \geq s$.

i) For $k+1 \leq s$, we have

$$a_{k,n} = e^{-\frac{1}{k}\alpha_n}, \qquad a_{k+1,n} = e^{-\frac{1}{k+1}\alpha_n},$$
$$a_{k,n+1} \leq e^{\left(-\frac{1}{k}+1\right)\alpha_{n+1}}, \qquad a_{k+1,n+1} = e^{\left(-\frac{1}{k+1}+1\right)\alpha_{n+1}}. \tag{4.88}$$

Since α is increasing,

$$\frac{a_{k+1,n}}{a_{k,n}} = e^{\left(\frac{1}{k} - \frac{1}{k+1}\right)\alpha_n} \leq e^{\left(\frac{1}{k} - \frac{1}{k+1}\right)\alpha_{n+1}} \leq \frac{a_{k+1,n+1}}{a_{k,n+1}}. \tag{4.89}$$

ii) For $k \geq s+1$, we have

$$a_{k,n} = e^{\left(-\frac{1}{k}+1\right)\alpha_n}, \qquad a_{k+1,n} = e^{\left(-\frac{1}{k+1}+1\right)\alpha_n},$$
$$a_{k,n+1} = e^{\left(-\frac{1}{k}+1\right)\alpha_{n+1}}, \qquad a_{k+1,n+1} = e^{\left(-\frac{1}{k+1}+1\right)\alpha_{n+1}}. \tag{4.90}$$

Since α is increasing,

$$\frac{a_{k+1,n}}{a_{k,n}} = e^{\left(\frac{1}{k}-\frac{1}{k+1}\right)\alpha_n} \leq e^{\left(\frac{1}{k}-\frac{1}{k+1}\right)\alpha_{n+1}} = \frac{a_{k+1,n+1}}{a_{k,n+1}}. \tag{4.91}$$

iii) For $k = s$, we have

$$a_{k,n} = e^{-\frac{1}{k}\alpha_n}, \qquad a_{k+1,n} = e^{\left(-\frac{1}{k+1}+1\right)\alpha_n},$$
$$a_{k,n+1} \leq e^{\left(-\frac{1}{k}+1\right)\alpha_{n+1}}, \qquad a_{k+1,n+1} = e^{\left(-\frac{1}{k+1}+1\right)\alpha_{n+1}}. \tag{4.92}$$

Then, these gives that

$$\frac{a_{k+1,n}}{a_{k,n}} = e^{\left(\frac{1}{k}-\frac{1}{k+1}+1\right)\alpha_n} \quad \text{and} \quad e^{\left(\frac{1}{k}-\frac{1}{k+1}\right)\alpha_{n+1}} \leq \frac{a_{k+1,n+1}}{a_{k,n+1}}. \tag{4.93}$$

In this case, the regularity condition $\frac{a_{k+1,n}}{a_{k,n}} \leq \frac{a_{k+1,n+1}}{a_{k,n+1}}$ is equivalent to the inequality (4.5).

Hence, we have obtained a regularity condition for a Köthe space $\mathcal{K}_\alpha$ from above observation:

Proposition 4.2.8 *Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ parameterized by the sequence α. Then, $\mathcal{K}_\alpha$ is regular if and only if the inequality*

$$(1+s(s+1))\alpha_n \leq \alpha_{n+1} \tag{4.94}$$

is satisfied for all $n \in I_s$ and $s \in \mathbb{N}$.

We also note that the sequence $(\alpha_n)_{n\in\mathbb{N}} = \left(\prod_{i=0}^{n-1}(1+i(i+1))\right)_{n\in\mathbb{N}}$ satisfies the condition of Proposition 4.2.8 since

$$\frac{\alpha_{n+1}}{\alpha_n} = (1+n(n+1)) \geq (1+s(s+1)) \tag{4.95}$$

for all $n \in I_s$, $s \in \mathbb{N}$.

As a consequence of Proposition 4.2.8, we obtain the following result:

Corollary 4.2.9 *Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ parameterized by the sequence α. If $\mathcal{K}_\alpha$ is regular, then the sequence α is unstable.*

Proof. Let $\mathcal{K}_\alpha$ be a regular Köthe space generated by the matrix $(a_{kn})_{k,n\in\mathbb{N}}$ given in 4.1 and assume α is not unstable, that is, $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} \neq +\infty$. Then, there exist a $M > 0$ and a non-decreasing sequence $(n_k)_{k\in\mathbb{N}}$ so that $\sup_{k\in\mathbb{N}} \frac{\alpha_{n_k+1}}{\alpha_{n_k}} < M$. Since $(n_k)_{k\in\mathbb{N}}$ is non-decrasing and $\mathcal{K}_\alpha$ is regular, we can write

$$\frac{\alpha_{k+1}}{\alpha_k} \leq \frac{\alpha_{n_k+1}}{\alpha_{n_k}} \leq M \tag{4.96}$$

for all $k \in \mathbb{N}$ and from Proposition 4.2.8, we find that

$$(1+s(s+1)) \leq \frac{\alpha_{k+1}}{\alpha_k} \leq M \tag{4.97}$$

for all $k \in I_S$, $s \in \mathbb{N}$. This is a contradiction, therefore α must be unstable, as desired. □

Remark 4.2.10 *Being unstable is not sufficient for regularity of Köthe space $\mathcal{K}_\alpha$. For instance, the sequence $(\alpha_n)_{n\in\mathbb{N}} = ((n-1)!)_{n\in\mathbb{N}}$ does not satisfy the condition of Proposition 4.2.8. Indeed, for every $s \in \mathbb{N}$, $n = \frac{s(s+1)}{2} \in I_s$ and*

$$\frac{\alpha_{n+1}}{\alpha_n} = n = \frac{s(s+1)}{2} < 1+s(s+1). \tag{4.98}$$

Remark 4.2.11 *As a corollary of Proposition 4.2.5, Proposition 4.2.6 and Corollary 4.2.9, we can obtain that $\Delta(\mathcal{K}_\alpha)) = \Delta(\Lambda_1(\alpha_{n+1}))$ and $\delta(\mathcal{K}_\alpha) \neq \delta(\Lambda_1(\alpha_{n+1}))$ for a regular element $\mathcal{K}_\alpha$ of the family $\mathcal{K}$ which is parameterized by an exponent sequence α.*

4.3 Inferences Acquired From the Family $\mathcal{K}$

In this section, we compile some additional information for the family $\mathcal{K}$.

We have shown that an element $\mathcal{K}_\alpha$ of the family $\mathcal{K}$ which is parametrized by an unstable sequence α constitutes a counterexample to Question 1.0.2. An element $\mathcal{K}_\alpha$ of the family $\mathcal{K}$ which is parametrized by an unstable sequence α is crucial for Question 1.0.1, as well:

Theorem 4.3.1 *There exists a nuclear Fréchet space E with the properties $\underline{DN}$ and Ω satisfying $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$, for its associated exponent sequence ε, with the property that there is no subspace of E which is isomorphic to $\Lambda_1(\varepsilon)$.*

Proof. Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ which is parametrized by an unstable sequence α. We proved that $\Delta(\mathcal{K}_\alpha) = \Delta(\Lambda_1(\alpha_{n+1}))$ in Proposition 4.2.5. Therefore, the sequence $(\alpha_{n+1})_{n\in\mathbb{N}}$ is the associated exponent sequence of $\mathcal{K}_\alpha$. Besides, we showed that $\delta(\mathcal{K}_\alpha) \neq \delta(\Lambda_1(\alpha_{n+1}))$ in Proposition 4.2.6.
Assume that there exists a subspace of $\mathcal{K}_\alpha$ which is isomorphic to $\Lambda_1(\alpha_{n+1})$. This gives us that $\delta(\Lambda_1(\alpha_{n+1})) \subseteq \delta(\mathcal{K}_\alpha)$ by Proposition 2.1.4. Since always $\delta(\mathcal{K}_\alpha) \subseteq \delta(\Lambda_1(\alpha_{n+1}))$, we conclude that $\delta(\mathcal{K}_\alpha) = \delta(\Lambda_1(\alpha_{n+1}))$. But this is a contradiction. Hence, there is no subspace of $\mathcal{K}_\alpha$ which is isomorphic to $\Lambda_1(\alpha_{n+1})$. $\square$

Remark 4.3.2 *The above theorem indicates that Question 1.0.1 has a negative answer. It is worth mentioning that we can even find even a nuclear regular Köthe space with the properties listed in Theorem 4.3.1.*

In the third chapter, we gave conditions confirming an affirmative answer for Question 1.0.2. Obviously, these conditions are not valid for an element $\mathcal{K}_\alpha$ of the family $\mathcal{K}$ which is parameterized by an unstable sequence α. For instance, Theorem 3.2.2 says that Question 1.0.2 has positive answer provided that $\Delta(E)$ is barrelled with respect to the canonical topology. Therefore, we obtain the following:

Proposition 4.3.3 *Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ parameterized by an unstable sequence α. Then $\Delta(\mathcal{K}_\alpha)$, with the canonical topology, is neither barrelled nor ultrabornological.*

We actually wanted the barrelledness in Theorem 3.2.2 to be able to use a closed graph type theorem, [37, Theorem 5, Pg. 40] which says that a linear map f from a barrelled space X into a Fréchet space Y is continuous provided that the graph of f is closed in $X \times Y$.

Since $\delta(\mathcal{K}_\alpha) \neq \delta(\Lambda_1(\alpha_{n+1}))$ and $\Delta(\mathcal{K}_\alpha) = \Lambda_1(\alpha_{n+1})$, the technique used in the proof of Theorem 3.2.2 is not valid for an element $\mathcal{K}_\alpha$ of the family $\mathcal{K}$ parameterized by an unstable sequence α. Hence, this gives us that the identity mapping from $\Delta(\mathcal{K}_\alpha)$ into $\Lambda_1(\alpha_{n+1})$ is not continuous although it has a closed graph:

Theorem 4.3.4 *Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ parameterized by an unstable sequence α. Then $\Delta(\mathcal{K}_\alpha) = \Delta(\Lambda_1(\alpha_{n+1}))$ and the identity map from $\Delta(\mathcal{K}_\alpha)$ into $\Lambda_1(\alpha_{n+1})$ is not continuous although it has a closed graph.*

Again since $\Delta(\mathcal{K}_\alpha) = \Delta(\Lambda_1(\alpha_{n+1}))$ and $\delta(\mathcal{K}_\alpha) \neq \delta(\Lambda_1(\alpha_{n+1}))$ for an element $\mathcal{K}_\alpha$ of the family $\mathcal{K}$ parameterized by an unstable sequence α, as a corollary of Proposition 3.2.6 and Corollary 3.2.8, we have the following:

Theorem 4.3.5 *There exists a nuclear Fréchet space E with the properties $\underline{DN}$ and Ω satisfying $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$ for its associated exponent sequence ε such that there is no prominent bounded set of E. Hence, this space also does not satisfy D_2-property.*

Remark 4.3.6 *It is worth to note that as a consequence of Corollary 4.2.4, Proposition 3.2.6 and Corollary 3.2.8, an element $\mathcal{K}_\alpha$ of the family $\mathcal{K}$ parameterized by a stable sequence α has the property D_2, whereas, we showed that this space does not have the property d_2 in Remark 4.2.2.*

A nuclear Fréchet space E with an increasing sequence of seminorms $(\|.\|_k)_{k\in\mathbb{N}}$ is called *tame* if there exists an increasing function $\sigma : \mathbb{N} \to \mathbb{N}$, such that for every continuous linear operator $T : E \to E$ there exists a $n_0 \in \mathbb{N}$ and $C > 0$ so that

$$\|T(x)\|_k \leq C\,\|x\|_{\sigma(k)} \qquad \forall x \in E. \tag{4.99}$$

In [2, Theorem 2.3], A. Aytuna proved that a nuclear Fréchet space E with the properties $\underline{DN}$ and Ω and stable associated exponent sequence ε is isomorphic to a power series space of finite type if and only if E is tame and $\delta(E) = \delta(\Lambda_1(\varepsilon))$. As a consequence of this result and Remark 4.3.6, we have the following:

Proposition 4.3.7 *Let $\mathcal{K}_\alpha$ be an element of the family $\mathcal{K}$ parameterized by a stable sequence α. Then, $\mathcal{K}_\alpha$ is not tame.*

5. CONCLUSIONS

In this thesis we have investigated the structure of nuclear Fréchet spaces with the properties $\underline{DN}$ and Ω by studying the relations between their topological invariants. Diametral dimension and/or approximate diametral dimension of a nuclear Fréchet space E with the properties $\underline{DN}$ and Ω is closely related to those of a designated power series space. In this thesis, we focused our attention to nuclear Fréchet spaces E with the properties $\underline{DN}$ and Ω whose diametral and/or approximate diametral dimension coincides with that of a designated power series space.

In the first chapter, we mention some significant studies in the theory of nuclear Fréchet spaces and give the aim of this thesis. In the second chapter, we give preliminary materials and essential theorems.

In the third chapter, we showed that Question 1.0.2 has an affirmative answer when power series space is of infinite type. Then we searched an answer for the Question 1.0.2 in the finite type case and, in this regard, we first prove that the condition $\delta(E) = \delta(\Lambda_1(\varepsilon))$ always implies $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$. For the other direction, existence of a prominent bounded subset in the nuclear Fréchet space E plays a decisive role for the answer of Question 1.0.2. Among other things, we prove that $\delta(E) = \delta(\Lambda_1(\varepsilon))$ if and only if E has a prominent bounded set and $\Delta(E) = \Delta(\Lambda_1(\varepsilon))$.

In the first section of the fourth chapter, we showed that a regular nuclear Köthe space with the properties $\underline{DN}$ and Ω is a power series space if its diametral dimension coincides with that of a power series space of infinite type or its approximate diametral dimension coincides with that of a power series space of finite type.

In the second section of the fourth chapter, we constructed a family $\mathcal{K}$ of nuclear

Köthe spaces $\mathcal{K}_\alpha$ with the properties $\underline{DN}$ and Ω. First we showed that for an element of the family of $\mathcal{K}$ which is parameterized by a stable sequence α, we have $\Delta(\mathcal{K}_\alpha) = \Delta(\Lambda_1(\alpha))$ and $\delta(\mathcal{K}_\alpha) = \delta(\Lambda_1(\alpha))$. Next, we proved that for an element of the family of $\mathcal{K}$ which is parameterized by an unstable sequence α, we have $\Delta(\mathcal{K}_\alpha) = \Delta(\Lambda_1(\varepsilon))$ and $\delta(\mathcal{K}_\alpha) \neq \delta(\Lambda_1(\varepsilon))$ for its associated exponent sequence ε. This showed that the second question has a negative answer for power series space of finite type. Furthermore, we proved in Theorem 4.3.1 that the first question has a negative answer, that is, $\Lambda_1(\varepsilon)$ is not isomorphic to any subspace of these Köthe spaces $\mathcal{K}_\alpha$, let alone is isomorphic to a complemented subspace, though the condition $\Delta(\mathcal{K}_\alpha) = \Delta(\Lambda_1(\varepsilon))$ is satisfied. In the third section of fourth chapter, motivated by our finding in the third section, we compiled some additional information. For instance, for an element E of the family $\mathcal{K}$ parameterized by an unstable sequence:

- E does not have a prominent bounded set.
- $\Delta(E)$, with respect to the canonical topology, is not barrelled, hence, not ultrabornological.
- Although the equality $\Delta(E) = \Lambda_1(\varepsilon)$ is satisfied and the canonical imbedding from $\Delta(E)$ into $\Lambda_1(\varepsilon)$ has a closed graph, the canonical imbedding from $\Delta(E)$ into $\Lambda_1(\varepsilon)$ is not continuous.

In the future, we would like to search the conditions which will give us an imbedding from a power series space of finite type to a nuclear Fréchet spaces with the properties $\underline{\text{DN}}$ and Ω. In this context we propose to give special emphasize to the elements of the family $\mathcal{K}$ parameterized by a stable sequence.

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