ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY

FINITE-TIME CONTROL OF SWITCHED LINEAR SYSTEMS WITH TIME-DELAY

Ph.D. THESIS

Gökhan GÖKSU

Department of Mathematical Engineering

Mathematical Engineering Programme

JANUARY 2020

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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

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OCAK 2020

Gökhan GÖKSU, a Ph.D. student of ITU Graduate School of Science Engineering and Technology 509132052 successfully defended the thesis entitled "FINITE-TIME CONTROL OF SWITCHED LINEAR SYSTEMS WITH TIME-DELAY", which he/she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

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To the dearest one, family, friends and pure joy of mathematics,

FOREWORD

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ABBREVIATIONS

ADT	**:** Average-Dwell Time
ASIL	**:** Asymptotically Stable in terms of Lyapunov
DT	**:** Dwell Time
ESIL	**:** Exponentially Stable in terms of Lyapunov
FT	**:** Finite-Time
LMI	**:** Linear Matrix Inequality
SIL	**:** Stable in terms of Lyapunov

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FINITE-TIME CONTROL OF
SWITCHED LINEAR SYSTEMS WITH TIME-DELAY

SUMMARY

Control theory is a branch of engineering and mathematics that examines the system's behavior by adjusting the input of the dynamic system according to its output. The systems examined can be discrete or continuous according to time, in some cases the behavior of the dynamical system may consist of a combination of continuous and discrete events. Such systems are called hybrid systems.

A certain class of hybrid systems is called switched systems. The switched systems are continuous systems with discrete and instant switching events. At the analysis stage, switched systems and hybrid systems differ by neglecting the details of the discrete behavior and instead considering all possible switching patterns from a certain class. Many works related to the switched systems, asymptotic stability is examined. However, in most practical applications, finite-time (FT) stability/boundedness is the main concern, i.e., the behavior of the system is kept at certain boundaries in FT. Asymptotically stable systems may not be FT stable/bounded and FT stable/bounded systems may not be asymptotically stable. Another study area for switching systems is the dwell time (DT) or average-dwell time (ADT). DT is the minimum time difference between successive switching instants whereas the average time difference between successive switching instants is called ADT.

Some dynamical systems in engineering may depend on the past status of the system. Such systems are called time-delayed systems, and a time delay can cause poor performance or system instability.

In this thesis, switched systems with completely unstable and mixed stable subsystems are considered. FT stability/boundedness and H_∞ FT boundedness of switched systems with interval time-delay and disturbances are examined. In the beginning, the difference between FT stability and asymptotic stability are shown on the examples and a sufficient condition for FT stability of the switched system, which is composed of linear time invariant subsystems having non-Hurwitz system matrices is derived. New sufficient conditions on the existence of observer-based controller for FT boundedness and H_∞-control of switched linear systems with time-varying interval delay and exogenous disturbances are obtained by using Lyapunov-Krasovskii functional. The observer-based controller is designed without any matrix decomposition and new ADT bounds are introduced for switched systems with both completely unstable and mixed stable subsystems, seperately. These bounds contain some unknown constants which depend on nonlinear terms. These terms are composed of the matrices from the solution of the sufficient conditions. An algorithm is presented for the calculation of unknown constants in the ADT bounds in terms of well-known cone complementarity linearization method. Similar work is achieved for the state feedback design.

In the first chapter, a system with a control process is briefly introduced. The studies on hybrid systems and switching systems are summarized. On the other hand, studies

on FT stability and studies on ADT are mentioned. Thereafter, the references on time-delay systems are given. Latter, literature overview is completed by marking the open parts of the switched systems with time delay.

In the second chapter, the basic definitions and background used in this thesis are introduced. First of all, the sufficient conditions for the existence and uniqueness of the solutions of the systems of differential equations are given. Hybrid systems are introduced by an example in engineering with vehicle gear dynamics. State dependent switching and time dependent switching are discussed in detail. Constrained switching concepts DT and ADT are introduced. FT stability and boundedness definitions are given by comparing with Lyapunov stability definitions, conceptual differences between these two stability types are presented and an example is given on switched systems. In the given example, it is shown that two stable subsystems are observed to be stable or unstable depending on different periods of switching. The notation used in the thesis, concepts of vector norm and matrix norm to be used in the third section, the Schur complement lemma, Grönwall's lemma and Jensen inequality to be used in the following sections are presented.

In the third chapter, FT stability of switched linear systems with stable, unstable and mixed stable subsystems are examined by using vector and matrix norms. FT stability conditions related to the eigenvalues and the condition numbers composed by the (generalized) eigenvectors of the subsystem matrices are obtained. Possible activation numbers of the subsystems are also deduced from these conditions. New ADT bounds to ensure FT stability of the switching system having negative, positive and mixed spectral norm bounds are proposed. Finally, several numerical examples are provided to demonstrate the effectiveness of the theoretical results.

In the fourth chapter, the FT boundedness analysis of switched systems with interval time-delay using state feedback is considered. ADT is obtained with sufficient conditions. Since there are non-convex terms in these conditions, a cone complementarity linearization method and algorithm that converts these terms into LMI conditions is presented. Finally, a numerical example is given.

In the fifth chapter, observer-based FT boundedness of switched systems with time-delay is examined. Two theorems are stated in the case that all of the subsystem matrices of the state vectors are unstable and mixed stable. In both cases, new sufficient conditions and ADT bounds are found with the presence of the observer. A cone complementarity linearization method and algorithm for the calculation of unknown eigenvalues over ADT bound is shown. Finally, a comparative example examining the unstable and mixed stable cases are given.

In the last chapter, an observer-based controller is designed for H_∞ FT boundedness of switched systems with time-delay. The reason that H_∞ FT boundedness is investigated is the presence of the disturbance. In this section, a numerical example is given to illustrate the effectiveness and validity of the proposed conditions for the mixed stable case described in the fifth chapter.

As a future work, it is envisaged to expand the results to mode-dependent stabilization analysis and robust stability.

ZAMAN GECİKMELİ VE ANAHTARLAMALI DOĞRUSAL SİSTEMLERİN SONLU ZAMAN DENETİMİ

ÖZET

Denetim kuramı dinamik sistemin girdisini, çıktısına göre ayarlamak suretiyle sistemin belirli bir davranışı sergilemesini inceleyen bir mühendislik ve matematik dalıdır. İncelenen sistemler zamana göre ayrık veya sürekli olabildiği gibi, bazı durumlarda dinamik sistemin davranışı sürekli ve ayrık olayların birleşiminden de oluşabilir. Bu tip sistemlere melez (hybrid) sistemler adı verilir. Melez sistemler konusunda sürekli sistemlerin ayrık ve anlık olaylarla değiştiği sistemler olan anahtarlamalı sistemler konusu yaygın olarak çalışılmaktadır.

Anahtarlamalı sistemlerle ilgili çalışmalarda genellikle sistemin asimptotik kararlı olması durumu incelenmiştir. Halbuki bir çok pratik uygulamada sonlu zaman kararlı/sınırlı olması durumu, yani sistemin davranışının sonlu zamanda belli sınırlarda tutulması durumu önem arz etmektedir. Asimptotik olarak denge noktasına giden asimptotik kararlı sistemler, sonlu zaman kararlı/sınırlı olmayabilir; bazı sonlu zaman kararlı/sınırlı sistemler asimptotik kararlı olmayabilir.

Anahtarlamalı sistemlerle ilgili ana çalışma alanı ise yaşam süresi veya ortalama yaşam süresidir. Yaşam süresi ardışık anahtarlama zamanlarının farkının belli bir yaşam süresinden fazla olması; ortalama yaşam süresi ise ardışık anahtarlama zamanlarının farkının ortalamasının belli bir ortalama yaşam süresinden fazla olmasıdır.

Mühendislikte ve matematikte incelenen bazı dinamik sistemler; sistemin o andaki durumunun yanında, sistemin geçmişteki durumuna da bağlı olabilir. Bu tip sistemler zaman gecikmeli sistemler olarak adlandırılır ve zaman gecikmesi kötü performansa veya sistem kararsızlığına neden olabilir.

Bu çalışmada, anahtarlamalı sistemlerin alt sistemlerinin kararsız ve karışık kararlı olması durumu ele alınmıştır. Anahtarlamalı ve aralık zaman gecikmeli sistemlerin bozucu etkisinde sonlu zaman kararlı/sınırlı ve H_∞ sınırlı olma durumları incelenmiştir. Öncelikle, sonlu zaman kararlılığı ile asimptotik kararlılık arasındaki farklar örnekler üzerinde gösterilmiş, sistem matrisleri Hurwitz kararlı olmayan ve zamana bağlı olmayan doğrusal sistemlerin sonlu zaman kararlılığı için yeter koşul elde edilmiştir. Sonlu zaman sınırlılığı ve H_∞ denetimi sağlayacak gözlemci tabanlı denetimcinin varlığı için Lyapunov-Krasovskii fonksiyoneli kullanılarak yeni yeter koşullar elde edilmiştir. Herhangi bir matris ayrıştırımına ihtiyaç olmadan gözlemci tabanlı denetimci tasarlanarak, alt sistemlerin kararsız ve karışık kararlı olduğu durumlar için ortalama yaşam süresi sınırları bulunmuştur. Bu sınırlarda doğrusal olmayan terimlere bağlı olan bazı sabitler içerdiğinden ve bu terimler de yeter koşullardaki matrislerden oluştuğundan dolayı; ortalama yaşam süresindeki bu sabitlerin çözümü için koni tamamlayıcı bir algoritma sunulmuştur. Tüm bu çalışmalar durum geri beslemesi için de uygulanmıştır.

Çalışmanın birinci bölümü olan giriş bölümünde kontrol süreci gösterilmiştir. Melez sistemler ve anahtarlamalı sistemler konusundaki çalışmalar özetlenmiş, sonlu zaman kararlılığı konusunda yapılan çalışmalar ile ortalama yaşam süresi konusunda yapılan çalışmalardan bahsedilmiştir. Tezde ele alınan problemlerden anahtarlamalı ve zaman gecikmeli sistemlerde yapılan çalışmalarda eksik olan kısımlar özetlenerek literatür özeti tamamlanmıştır.

İkinci bölümde, bu tezde kullanılan temel tanımlar ve bilgiler tanıtılmıştır. Öncelikle diferansiyel denklem sistemlerinin çözümlerinin varlığı ve tekliği için yeter koşullar verilmiştir. Melez sistemler, bir mühendislik örneği olan araçların vites dinamiği ile tanıtılarak, anahtarlamalı sistemlerin ne tarz durumlarda ortaya çıkabileceği gösterilmiş; duruma bağlı anahtarlama ve zamana bağlı anahtarlama durumları ayrıntılarıyla ele alınmıştır. Kısıtlamalı anahtarlama altında anahtarlama durumlarına bağlı yaşam süresi ve ortalama yaşam süresi kavramları tanıtılarak zaman gecikmeli sistemler ile ilgili temel bilgiler verilmiştir. Sonlu zaman kararlılığı ve sınırlılığı, Lyapunov kararlılık tanımları verilerek, bu iki kararlılık tanımları arasındaki kavram farkılıkları ortaya konmuş ve anahtarlamalı sistemler üzerinde örnek verilmiştir. Verilen örnekte kararlı iki alt sistemin periyodik anahtarlama altında periyoda bağlı kararlı veya kararsız olma durumlarının gözlemlendiği gösterilmiştir. Daha sonraki bölümlerde kullanılacak olan; vektör normu ve matris normu kavramları, Schur yardımcı teoremi, Grönwall yardımcı teoremi ve Jensen eşitsizliği sunulmuş ve tezde kullanılan notasyonlar belirtilmiştir.

Üçüncü bölümde; kararlı, kararsız ve karışık kararlı alt sistemlere sahip doğrusal anahtarlamalı sistemlerin vektör ve matris normları kullanılarak sonlu zaman kararlılık analizi yapılmıştır. Alt sistem matrislerinin özdeğerleri ve koşullandırma sayılarına bağlı sonlu zaman kararlılık koşulları ve bu alt sistemlerin olası aktivasyon sayıları elde edilmiştir. Anahtarlamalı sistemin sonlu zaman kararlılığının sağlanması için yeni ortalama yaşam süresi önerilmiştir. Son olarak da sayısal örneklerle teorik sonuçlar açıklanmıştır.

Dördüncü bölümde, anahtarlamalı ve aralık zaman gecikmeli sistemlerin durum geri beslemesi altındaki sonlu zaman sınırlılığı ele alınmıştır. Yeter koşullarla birlikte ortalama yaşam süresi elde edilmiştir. Bu koşullarda dışbükey olmayan terimler olduğu için bu terimleri doğrusal matris eşitsizliği koşullarına çeviren bir koni tamamlayıcı doğrusallaştırma yöntemi ve algoritması kullanılmıştır. Son olarak da sayısal bir örnek verilmiştir.

Beşinci bölümde, anahtarlamalı ve aralık zaman gecikmeli sistemlerin gözlemci tabanlı sonlu zaman sınırlılığı durum vektörlerinin başındaki alt sistem matrislerinin tamamının kararsız ve karışık kararlı (yani bir kısmı kararlı bir kısmı kararsız) olması durumlarına göre incelenmiştir. Bu iki durumda da gözlemcinin varlığı için yeni yeter koşullar ve ortalama yaşam süresi tanıtılmıştır. Ortalama yaşam süresindeki parametrelerin hesabı için koni tamamlayıcı doğrusallaştırma yöntemi ve algoritması gösterilmiştir. Son olarak da literatürdeki durum vektörlerinin başındaki alt sistem matrislerinin tamamının kararsız olma durumunu inceleyen karşılaştırmalı bir örnek ile bu matrislerin karışık kararlı olma durumunu inceleyen sayısal örnekler verilmiştir.

Altıncı bölümde, anahtarlamalı ve aralık zaman gecikmeli sistemlerin H_∞ sonlu zaman sınırlılığı için bir gözlemci tabanlı denetimci tasarlanmıştır. H_∞ sonlu zaman sınırlılığı incelenen sisteme bozucu etki etmesinden dolayı incelenmiştir. Bu bölümde durum vektörlerinin başındaki alt sistem matrislerinin karışık kararlı olması durumu için koşullar elde edilip, önerilen koşulların etkinliği ve geçerliliği sayısal bir örnek üzerinde gösterilmiştir.

Gelecek çalışmalarda, moda bağımlı kararlılaştırma analizi ve gürbüz kararlılık ele alınarak şu ana kadar yapılan çalışmaların genişletilmesi düşünülmektedir.

1. INTRODUCTION

Control theory is a branch of engineering and mathematics which aims to change the behavior of the dynamical systems by manipulating the system input according to the system output. A brief demonstration of a system with a control process is shown in Figure 1.1.

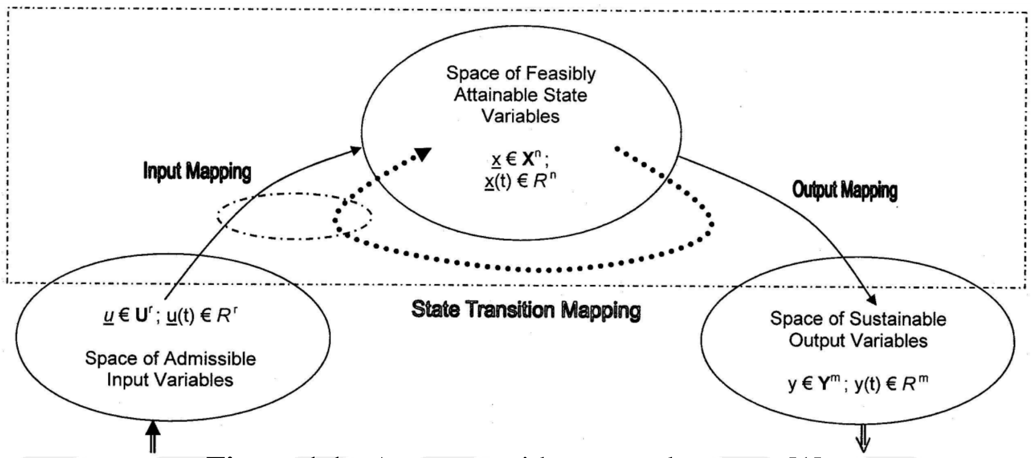

Figure 1.1 : A system with a control process [1].

The state of the system may change in discrete or continuous set of points in time. In some cases, the behavior of the dynamical system may be a combination of both continuous and discrete events. Such kind of systems are called the hybrid systems. The majority of the recent studies in hybrid systems are the continuous systems having discrete switching events. These type of systems are classified into a special type of systems called switched systems. Many scholars are interested in the stability analysis and control design of such kind of systems, because the switched systems even in general the hybrid systems may behave different even the continuous or discrete dynamics have solely different characteristics.

Stability is one of the basic research topic for switched systems, which has attracted most of the attention in recent years, [4–10]. Most of the studies related to stability of switched systems focus on Lyapunov asymptotic stability, which is defined over an infinite time interval. However, in many practical applications, finite-time (FT) stability of a system is the main concern, which means keeping the system behavior/state within specified bounds in a fixed FT interval. FT stability for switched systems is an emerging concept in recent years, [11–14].

Average-dwell time (ADT) is the major research topic for switched systems. ADT means that the number of switching instants in a finite interval is bounded and the average time between consecutive switching instants is not less than a constant. In literature, there are plenty of works considering suitable Lyapunov functional to obtain an ADT bound as small as possible for the stability and the stabilization of switched systems, [15–25].

The behavior of some dynamical systems may depend to the behavior of the system in the past. Their behavior may also depend even to a distributed interval of a time. Such kind of systems are called time-delay systems in general and the delay may cause bad performance or instability of the system.

Time-delay systems have been widely studied in last decades [3, 26–28], and the references therein. The current methods of stability analysis are divided into two categories: delay-dependent and delay-independent. Results in delay dependent case does not include any information on the size of delay but delay dependent solutions include such information. Many works for time-delay systems consider only the upper bound for delay. If both upper and lower bounds on time-delay exist, such systems are called interval time-delay systems, [29–35].

In literature, the vast majority of the recent studies for the stabilization of switched time-delay systems are dealing with state feedback, [36–39]. Stabilization of time-delay switched systems by observer-based controller is examined only in [40]. In that study, interval time-delay is not considered and the calculation of the observer gain matrix depends on the decomposition of one of the solution matrix obtained by the linear matrix inequalities (LMIs) given in the sufficient condition. There is also no implicit explanation about the calculations of the constants in ADT which depend on the inverse matrices. Besides that, the system matrices of the state vector are chosen Hurwitz stable and switching among unstable and mixed stable subsystems are not considered. In this dissertation, the FT stability/boundedness and H_∞ FT boundedness of the switched systems with interval time-delay and disturbances are investigated under state feedback and observer-based controller.

2. BASIC DEFINITIONS AND BACKGROUND

In this chapter, the notations, basic definitions and background are introduced. The conditions that guarantee the existence and uniqueness of the solutions of system of differential equations are introduced. Hybrid and switched systems are presented, [2]. Time-delay systems are summarized, [3]. Vector and matrix norms are recalled, [41, 42]. FT stability and boundedness concepts are defined and lemmas that will be used in this dissertation are shown.

2.1 Notation

The notation used in this dissertation is fairly standard. “*" in a matrix means to be the symmetric term of the corresponding upper triangular element and $\lambda_{max}(A)$ (respectively $\lambda_{min}(A)$) represents the maximum (minimum) eigenvalue of A. Matrices, if not stated, are assumed to have compatible dimensions for algebraic operations. Throughout the paper; for vectors x and y with compatible dimensions and a positive definite symmetric matrix P, $x^T Py + y^T Px$ is written in short as $2x^T Py$ or $2y^T Px$.

2.2 Solutions of System of Differential Equations

Let us consider the system of differential equations

$$\dot{x}(t) = f(t,x), \quad x \in \mathbf{R}^n. \tag{2.1}$$

The system (2.1) has a unique solution for the initial condition (t_0,x_0), if the function f is continuous in t and locally Lipschitz in x. Being Lipschitz means that for every pair (t_0,x_0) there exists a positive constant L such that the condition

$$|f(t,x) - f(t,y)| \leq L|x-y| \tag{2.2}$$

holds for all (t,x) and (t,y) in the neighborhood of (t_0,x_0) in $[t_0,\infty) \times \mathbf{R}$. So, the solution exists on the maximal interval $[t_0,T_{max})$. Unless not stated, the initial time is taken as $t_0 = 0$ among the thesis.

2.3 Hybrid&Switched Systems

As it is said in the introduction part, some dynamical systems consists as a combination of both continuous and discrete events. Let q be a state which takes values from a finite set Q and x be the continuous state variable. The hybrid system with well-defined interactions u and v are demonstrated in Figure 2.1.

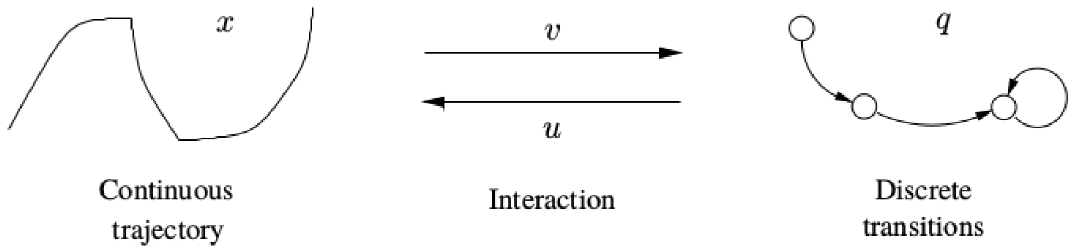

Figure 2.1 : A hybrid system, [2].

Switched systems arise in many engineering applications. Here, there is a motivational example for a car transmission system.

Example 1. *Let x_1 be the position, x_2 be the velocity, $a \geq 0$ be the acceleration input and $q \in \{1,2,3,4,5,-1,0\}$ be the gear shift position of an automobile. The generalized dynamics of a car will be*

$$\begin{aligned} \dot{x_1} &= x_2, \\ \dot{x_2} &= f(a,q) \end{aligned} \tag{2.3}$$

where the function f

- *is a decreasing function in a and takes negative values when $q = -1$,*
- *is an increasing function in a and takes negative values when $q = 0$,*
- *is an increasing function in a and takes positive values for sufficiently large a when $q > 0$.*

Here, x_1 and x_2 are the continuous states whereas q is the discrete state, [2].

Switching events can be classified into

- state-dependent,
- time-dependent

in context of dependency to the switching events.

2.3.1 State-dependent switching

In such kind of switching, the state space is divided into subspaces or regions. A continous-time dynamical system is acting on each of these regions. When the system trajectory hits the boundary of these regions, the dynamics of the system state is changed. This is called as reset map. Note that the system trajectory may lose its differentiability at these switching instants. A simple generalized visualization of state-dependent switching is shown in Figure 2.2.

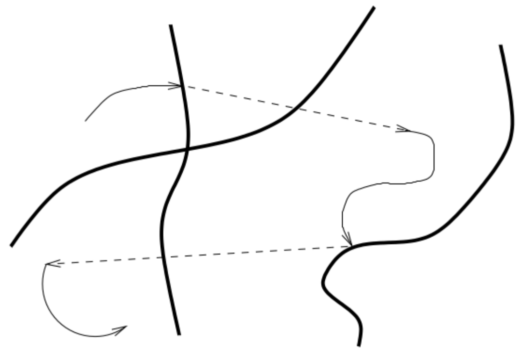

Figure 2.2 : State-dependent switching, [2].

2.3.2 Time-dependent switching

For a given family of functions f_i, $i \in \mathscr{I} = \{1, 2, ..., N\}$ from $\mathbf{R}^n$ to $\mathbf{R}^n$ where $\mathscr{I}$ is an index set. The functions f_i are all assumed to be locally Lipschitz. So, this corresponds to a family of systems

$$\dot{x} = f_i(t, x), \quad i \in \mathscr{I}. \tag{2.4}$$

To define a switched system generated by the above family, we need to define the switching signal. The switching signal is a piecewise constant function $\sigma : [0, \infty) \rightarrow \mathscr{I}$ having finite or denumerably infinite number of discontinuities, called switching times, and the function σ takes a value from $\mathscr{I}$ on every time interval between two consecutive switching instant which can be seen in Figure 2.3.

The time-dependent switched system is defined as

$$\dot{x}(t) = f_{\sigma(t)}(t, x), \; x(0) = x_0 \tag{2.5}$$

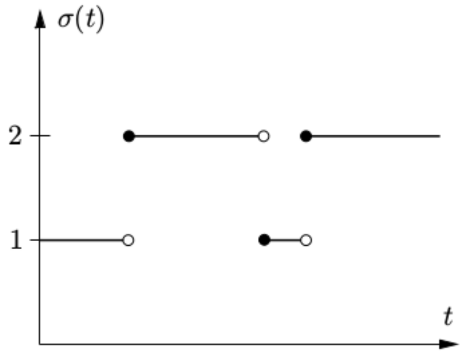

Figure 2.3 : A switching signal, [2].

with the switching signal σ defined above. The switched linear system in particular is defined as follows

$$\dot{x}(t) = A_{\sigma(t)}x(t),\ x(0) = x_0. \tag{2.6}$$

Here in this dissertation, the switching signal is defined on a finite time interval as $\sigma(t) : [0, T_f] \to \mathscr{I}$ with the switching sequence $\Sigma = \{(i_0, t_0 = 0), (i_1, t_1), ..., (i_n, t_n)\}$, i.e. ${i_k}^{th}$ system is activated when $t \in [t_k, t_{k+1})$. As a consequence, ${i_n}^{th}$ system is activated when $t \in [t_n, T_f)$. The switching signal may be constrained by adjusting the switching instants. Here are two examples of constrained switching.

2.3.2.1 Dwell-time

If the switching signal is restricted to satisfy $t_k - t_{k-1} \geq \tau_d$ for all switching instants, the number $\tau_d > 0$ is called the dwell time which means that each subsystem is activated at least τ_d units of time. So the constrained set of switching signals are stated as

$$\mathscr{S} = \mathscr{S}_{dwell}[\tau_d] = \left\{\sigma \in \Sigma \mid t_k - t_{k-1} > \tau\right\}. \tag{2.7}$$

2.3.2.2 Average dwell-time

Considering dwell time constraint may be seen as a strict constraint, because the next switching should wait until τ_d units of time is passed. However, this context may be subvented by adjusting the switching instants by average-dwell time so that the activation time may compensate the emergent switching by adjusting the consecutive switch instants. The definition is as follows.

Definition 1. *Let $N_{\sigma(t)}(t,T)$ denotes the switching number of the switching signal σ for the interval $0 \leq t \leq T$. N_0 is the chatter bound. Then the following inequality holds*

$$N_{\sigma(t)}(t,T) \leq N_0 + (T-t)/\tau_a$$

for so called ADT τ_a, [15].

2.4 Time-Delay Systems

Time-delay systems are the systems of differential equations, whose behavior depends on events in the past. This type of systems are also known as a special type of functional differential equations with deviating arguments of delay or delay differential equations with retarded type. So, the time-delay system can be represented as

$$\dot{x}(t) = f(t, x(t), x_t). \tag{2.8}$$

Here, $x_t(s) = x(t-s)$, $s \in [h_1, h_2]$, for $h_1, h_2 > 0$ represents the history of the solution as it may be seen in Figure 2.4.

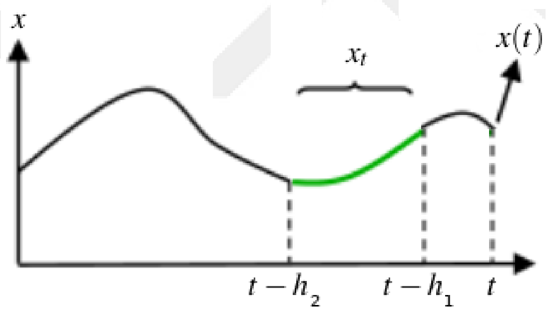

Figure 2.4 : The history of the solution.

In linear case, (2.8) will be

$$\dot{x}(t) = Ax(t) + A_d x(t-h(t)). \tag{2.9}$$

Note that, in order to construct the solution for $t > 0$, the value of $x(t)$ should be known in $[-h_2, 0]$. So there is a need of an initial condition $x(t) = \phi(t)$ where $t \in [-h_2, 0]$ defined over the function space $\mathscr{C}([-h_2,0], \mathbf{R}^n)$. Here, $\mathscr{C}([-h_2,0], \mathbf{R}^n)$ is the set of all continuous functions equipped with the supremum norm $\|\phi\| := \sup_{\tau\in[-h_2,0]} |\phi(\tau)|$.

Note that, $\mathscr{C}([-h_2,0], \mathbf{R}^n)$ is not a Hilbert space, because it does not satisfy parallelogram law

$$\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2). \tag{2.10}$$

for all $f,g \in \mathscr{C}([-h_2,0],\mathbf{R}^n)$. See the following example.

Example 2. *Consider the two functions $f(t) = 1$ and $g(t) = t$ where $f,g \in \mathscr{C}([-1,0],\mathbf{R})$. Then, from the supremum defined above, we have $\|f\|^2 = \|g\|^2 = 1$, $\|f+g\|^2 = 1$ and $\|f-g\|^2 = 4$. Thus, the parallelogram law is violated.*

2.5 Finite-Time Stability and Boundedness

Consider a nonlinear dynamical system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0 \tag{2.11}$$

where $x(t) \in \mathscr{D} \subseteq \mathbf{R}^n$ is the system state vector and $f : \mathscr{D} \to \mathbf{R}^n$ is the vector field. If $f(x_e) = 0$, then the point $x_e \in \mathscr{D}$ is said to be the equilibrium point of the operating system. Note that $x_e = 0$ for linear systems. Before the statement of FT stability and boundedness definitions, let us consider the Lyapunov stability definitions.

Definition 2. *The equilibrium point of the system* (2.11) *is said to be stable in terms of Lyapunov (SIL), if for every given $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $\|x(0) - x_e\| < \delta$, then for every $t \geq 0$ we have $\|x(t) - x_e\| < \varepsilon$.*

Definition 3. *The equilibrium point of the system* (2.11) *is said to be asymptotically stable in terms of Lyapunov (ASIL), if it is SIL and there exists $\delta > 0$ such that, if $\|x(0) - x_e\| < \delta$, then $\lim_{t\to\infty} \|x(t) - x_e\| < \varepsilon$.*

Definition 4. *The equilibrium point of the system* (2.11) *is said to be exponentially stable in terms of Lyapunov (ESIL), if it is ASIL and there exists $\alpha > 0$, β, $\delta > 0$ such that, if $\|x(0) - x_e\| < \delta$, then $\|x(t) - x_e\| \leq \alpha e^{-\beta t} \|x(0) - x_e\|$, $\forall t \geq 0$.*

In contrast to the above Lyapunov stability definitions, in some engineering applications finite time stability and/or boundedness is the main concern, which means that the system state is bounded within specific bounds in FT. Formal definition of FT stability is as follows.

Definition 5. *Consider scalar $T_f > 0$ and a matrix $R > 0$ with appropriate dimensions, the system* (2.11) *is said to be FT stable with respect to $(\delta, \varepsilon, T_f, R)$, if for every given $\varepsilon > 0$, there exists a $\delta > 0$ with $\varepsilon > \delta$ such that, if $x_0^T R x_0 < \delta$, then $x^T(t) R x(t) < \varepsilon$, $\forall t \in [0, T_f]$.*

In order to demonstrate the distinction between Lyapunov stability and FT stability definitions, the following example is given.

Example 3. *Consider the following system*

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0 \tag{2.12}$$

where $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ *with initial condition* $x_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. *The solution of* (2.12) *is*

$$x(t) = e^{At}x_0. \tag{2.13}$$

Note that the eigenvalues of the matrix A are $\lambda_{1,2} = 1 \pm j$, *where* $j = \sqrt{-1}$ *which means that the system* (2.12) *is unstable in terms of Lyapunov in all sense. However, the system is FT stable with respect to* $(\varepsilon, \delta, T_f, R) = (16, 4, 1, I)$, *since*

$$x^T(t)Rx(t) = x_0^T e^{A^T t} R e^{At} x_0 \leq x_0^T e^{A^T T_f} R e^{AT_f} x_0 \approx 14.7781 < 16 \tag{2.14}$$

which can also be seen in Figure 2.5.

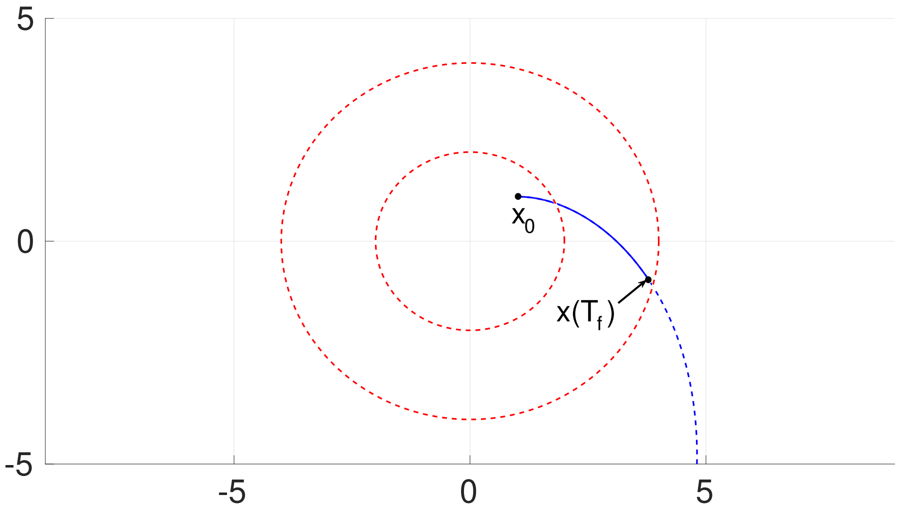

Figure 2.5 : Phase portrait of the system (2.12).

The motivating phenomena of switched systems is the dependence of the stability to the switching law. It is a fact that the switching between subsystems, even if they are all stable (in sense of Lyapunov), may cause instability of the whole system. Similarly, the switching law effects the finite-time stability of switched systems, which may be seen below.

Example 4. *(Motivating Example) [12] A switched linear system* (2.6) *with subsystems is given as follows*

$$A_1 = \begin{bmatrix} 0 & 10 \\ -30 & -2 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & -30 \\ 10 & 0 \end{bmatrix} \tag{2.15}$$

Choose $\delta = 8$*,* $R = I$*,* $T = 10$*,* $\varepsilon = 25$*, the initial state* $x_0 = \begin{bmatrix} 2 & -2 \end{bmatrix}^T$ *satisfying the initial condition* $x_0^T x_0 \leq 8$*. Then, the simulation results for each subsystem are given in Figure 2.6 and 2.7.*

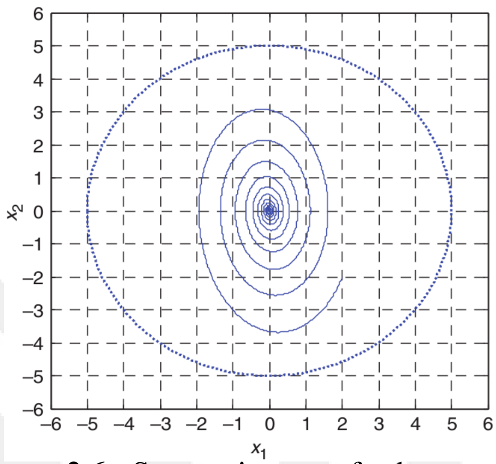

Figure 2.6 : State trajectory of subsystem 1.

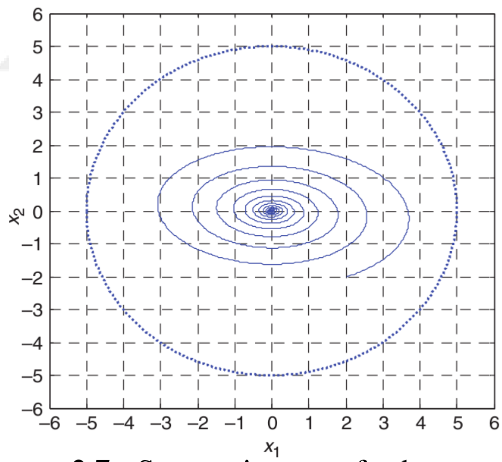

Figure 2.7 : State trajectory of subsystem 2.

By the simulation results, it is easy to see that the state trajectories are both bounded in $x^T(t)x(t) \leq 25$*.*

Then, two periodical switching signals S_1 *and* S_2 *are defined as follows:*

- S_1 *is a periodical switching signal, where the system switches from one subsystem to another every 1 s.*
- S_2 *is a periodical switching signal, where the system switches from one subsystem to another every 0.3 s.*

Both two switching signals are initialized to start operating with the first subsystem and the initial state is chosen as $x_0 = \begin{bmatrix} 2 & -2 \end{bmatrix}^T$*. Then, the simulation results are shown in Figure 2.8 and 2.9.*

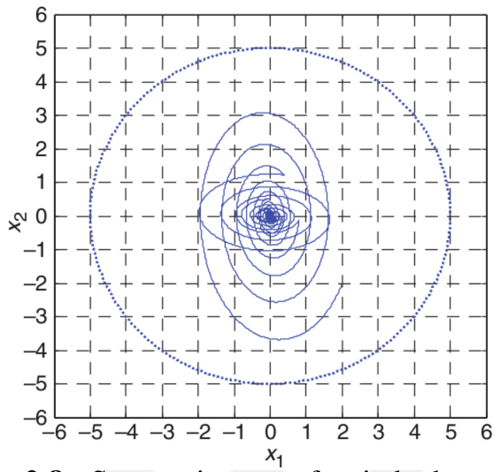

Figure 2.8 : State trajectory of switched system S_1.

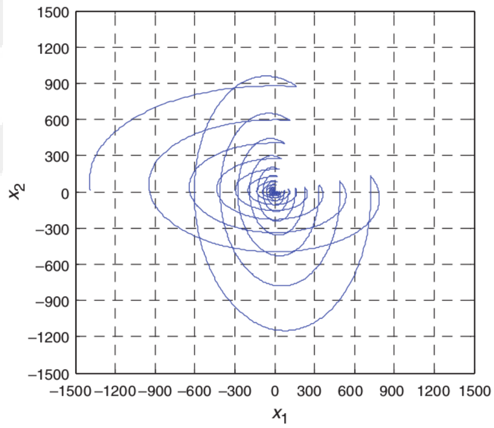

Figure 2.9 : State trajectory of switched system S_2.

From Figures 2.8 and 2.9, it is seen that the state trajectory is bounded with respect to bound $\varepsilon = 25$ *under the switching signal* S_1*, as the state trajectory does not satisfy* $x^T(t)x(t) \leq 25$*,* $\forall t \in [0, 10]$ *under switching signal* S_2*.*

Now, let us consider the system

$$\dot{x}(t) = f(x(t), w(t)), \quad x(0) = x_0 \tag{2.16}$$

with an exogeneous disturbance $w(t)$. The presence of the disturbance leads us to the definition of FT boundedness.

Definition 6. *Consider scalar* $T_f > 0$ *and a matrix* $R > 0$ *with appropriate dimensions, the system* (2.16) *is said to be FT bounded with respect to* $(\delta, \varepsilon, T_f, d, R)$*, if for every* *given* $\varepsilon > 0$*, there exists a* $\delta > 0$ *with* $\varepsilon > \delta$ *such that, if* $x_0^T R x_0 < \delta$*, then* $x^T(t)Rx(t) < \varepsilon$*,* $\forall t \in [0, T_f]$ *and* $\forall w(t)$ *satisfying* $\int_0^{T_f} w^T(t)w(t)dt < d$*.*

This definition can be revised for the time-delay system with an exogeneous disturbance

$$\dot{x}(t) = f(t, x(t), x_t, w(t)), \quad x(t) = \phi(t), t \in [-h_2, 0] \tag{2.17}$$

for $x_t = x(t - h(t))$, $h(t) \in [h_1, h_2]$ and $h_1, h_2 > 0$.

Definition 7. *Consider scalar* $T_f > 0$ *and a matrix* $R > 0$ *with appropriate dimensions, the system* (2.17) *is said to be FT bounded with respect to* $(\delta, \varepsilon, T_f, d, R)$*, if for every given* $\varepsilon > 0$*, there exists a* $\delta > 0$ *with* $\varepsilon > \delta$ *such that, if* $\sup_{s \in [-h_2, 0]} \left\{ x^T(s)Rx(s) \right\} < \delta$*, then* $x^T(t)Rx(t) < \varepsilon$*,* $\forall t \in [0, T_f]$ *and* $\forall w(t)$ *satisfying* $\int_0^{T_f} w^T(t)w(t)dt < d$*.*

Now, consider the time-delay system with exogeneous disturbance

$$\begin{aligned} \dot{x}(t) =& f_1(t, x(t), x_t, w(t)), \quad x(t) = \phi(t),\ t \in [-h_2, 0], \\ z(t) =& f_2(x(t)) \end{aligned} \tag{2.18}$$

with output $z(t)$. This leads us to the definition of H_∞ FT boundedness.

Definition 8. *The system* (2.18) *is said to be* H_∞ *FT bounded with respect to* $(\delta, \varepsilon, T_f, d, R)$ *if the following conditions are satisfied:*

1) The system (2.18) *is FT bounded.*

2) $\int_0^{T_f} z^T(t)z(t)dt < \gamma^2 \int_0^{T_f} w^T(t)w(t)dt$ *under zero-initial condition* $\phi(t) = 0$*,* $\forall t \in [-h_2, 0]$*, where* $\gamma > 0$*,* $0 \le \delta < \varepsilon$*,* $d \ge 0$ *and* $R > 0$*.*

2.6 Norm

In this section, vector and matrix norms are defined.

2.6.1 Vector norms

A vector norm is a function $\|\cdot\| : F^n \to \mathbf{R}$ defined over a field of real or complex numbers F satisfying the following properties $\forall \alpha \in F,\ u, v \in F^{m \times n}$:

- $\|\alpha v\| = |\alpha|\|v\|$ (absolute homogeneity),
- $\|u + v\| \le \|u\| + \|v\|$ (subadditivity or triangle inequality)

- $||v|| > 0$ if $x \neq 0$ and $||v|| = 0$ only if $v = 0$ (positive definite property).

Here are some vector norm definitions in the literature.

Let $x_1 = \begin{bmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,n} \end{bmatrix}^T, x_2 = \begin{bmatrix} x_{2,1} & x_{2,2} & \dots & x_{2,n} \end{bmatrix}^T \in \mathbf{R}^n$ and $z_1 = \begin{bmatrix} z_{1,1} & z_{1,2} & \dots & z_{1,n} \end{bmatrix}^T, z_2 = \begin{bmatrix} z_{2,1} & z_{2,2} & \dots & z_{2,n} \end{bmatrix}^T \in \mathbf{C}^n$. For both Euclidean space and complex space the L^2-norm can be expressed in a compact way by using inner products.

$$
\begin{aligned}
< x_1, x_2 > &= x_2^T \cdot x_1 = x_{2,1}x_{1,1} + x_{2,2}x_{1,2} + \dots + x_{2,n}x_{1,n} \\
< z_1, z_2 > &= z_2^* \cdot z_1 = \bar{z}_{2,1}z_{1,1} + \bar{z}_{2,2}z_{1,2} + \dots + \bar{z}_{2,n}z_{1,n}
\end{aligned} \tag{2.19}
$$

are the standard inner products for Euclidean space and complex space, respectively. x^T denotes the transpose of x. $\bar{z}$ and z^* denote the conjugation operation and conjugate transpose of z, respectively. So, the L^2-norms on these spaces are defined as follows

$$
||x||_2 = \sqrt{< x, x >_{\mathbf{R}^n}} \quad and \quad ||z||_2 = \sqrt{< z, z >_{\mathbf{C}^n}} \tag{2.20}
$$

L^2-norm can be generalized as L^p-norm as

$$
||x||_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \tag{2.21}
$$

where $1 \leq p < \infty$. As $p \to \infty$, L^p-norm approaches to L^∞-norm or so called maximum norm

$$
||x||_\infty = \max_{i=1,2,\dots,n} |x_i|. \tag{2.22}
$$

2.6.2 Matrix norms

Matrix norm is a vector norm in a vector space whose domain is the vector space of matrices. The matrix norm is a function $|| \cdot || : F^{m \times n} \to \mathbf{R}$ satisfying the following properties $\forall \alpha \in F,\ A, B \in F^{m \times n}$:

- $||\alpha A|| = |\alpha| ||A||$ (absolute homogeneity),
- $||A + B|| \leq ||A|| + ||B||$ (subadditivity or triangle inequality)
- $||A|| > 0$ and $||A|| = 0$ only if $A = 0$ (positive definite property).

Additionally, if $m = n$ some matrix norms satisfy

- $||AB|| \leq ||A|| ||B||$ (submultiplicativity).

In some literature submultiplicativity is sometimes extended to non-square matrices using different norms.

There are plenty of matrix norm definitions in the literature. Here, the matrix norm definitions used in this study are defined as follows.

The matrix norm induced by vector norm is defined as

$$\begin{aligned} \|A\| &= \sup\{\|Ax\| \ : \ x \in F^n \ with \ \|x\| = 1\} \\ &= \sup\left\{\frac{\|Ax\|}{\|x\|} \ : \ x \in F^n \ with \ \|x\| \neq 0\right\} \end{aligned} \tag{2.23}$$

If the p-norm for vectors ($1 \leq p \leq \infty$) is used for both spaces F^n and F^m, then the corresponding induced operator norm will be

$$\|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}. \tag{2.24}$$

If the vector norms of F^n and F^m are different, the matrix norm is defined as

$$\|A\|_{p,q} = \sup_{x \neq 0} \frac{\|Ax\|_q}{\|x\|_p}. \tag{2.25}$$

where $\|\cdot\|_p$ is defined on F^n whereas $\|\cdot\|_q$ is defined on F^m. The matrix norm $\|A\|_{\alpha,\beta}$ is called a subordinate norm. Subordinate norms are consistent with the norms that induce them.

$$\|Ax\|_q \leq \|A\|_{p,q}\|x\|_p. \tag{2.26}$$

Any induced operator norm is a submultiplicative matrix norm; this follows from

$$\|ABx\| \leq \|A\|\|Bx\| \leq \|A\|\|B\||x| \tag{2.27}$$

and $\sup_{\|x\|=1} \|ABx\| = \|AB\|$. Moreover, any induced norm satisfies the inequality

$$\|A^r\|^{1/r} \geq \rho(A) \tag{2.28}$$

where $\rho(A)$ is the spectral radius of A, i.e. largest absolute value of its eigenvalues.

The induced matrix norms can be expressed as

$$\begin{aligned} \|A\|_1 &= \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}|, \\ \|A\|_2 &= \sigma_{max}(A) = \sqrt{\lambda_{max}(A^*A)} \\ \|A\|_\infty &= \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}|. \end{aligned} \tag{2.29}$$

Here A^* denotes the conjugate transpose of A, $\sigma_{max}(A)$ represents the largest singular value of matrix A. Frobenius norm can be defined in various ways:

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} = \sqrt{trace(A^*A)} = \sqrt{\sum_{i=1}^{\min\{m,n\}} \sigma_i^2(A)} \tag{2.30}$$

where $\sigma_i(A)$ are the singular values of A and the trace function returns the sum of diagonal entries of a square matrix. The following inequality holds for every $A \in F^{m\times n}$

$$\|A\|_2 \leq \|A\|_F \tag{2.31}$$

The p-norms for $p = 1,2,...,\infty$ can be expressed as

$$\|A\|_p = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^p \right)^{1/p} \tag{2.32}$$

for $A \in F^{m\times n}$.

2.7 Other Lemmas

In this section, some lemmas which will be used in this dissertation are presented.

Lemma 1. *(Schur complement) Given constant matrices S_{11}, S_{12}, S_{22} with appropriate dimensions satisfying $S_{11} = S_{11}^T$ and $S_{22} = S_{22}^T$ and $S_{22} < 0$, the LMI $S = \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix} < 0$ is equivalent to $S_{11} + S_{12}S_{22}^{-1}S_{12}^T < 0$, [43].*

Lemma 2. *(Grönwall's lemma) If a differentiable function $\psi(t) > 0$ on the open interval $U = (a,b)$ (as well as $U = [a,b]$ or $U = [a,b)$) and*

$$\dot{u}(t) \leq \phi(t) + \psi(t)u(t)$$

then

$$u(t) \leq u(a)e^{\Psi(t)} + \int_a^t \phi(s)e^{\Psi(t)-\Psi(s)}ds$$

for $t < b$ where

$$\Psi(t) = \int_a^t \psi(s)ds,$$

[44].

Lemma 3. *(Jensen's inequality) For any symmetric positive definite matrix $M > 0$, scalars $a,b > 0$ with $b > a$ and an integrable vector function $x : [a,b] \to \mathbf{R}^n$, the following inequality holds, [27].*

$$\left(\int_a^b x(s)ds \right)^T M \left(\int_a^b x(s)ds \right) \leq (b-a) \left(\int_a^b x^T(s)Mx(s)ds \right)$$

3. FINITE-TIME STABILITY ANALYSIS FOR SWITCHED LINEAR SYSTEMS BY USING JORDAN DECOMPOSITION

In order to investigate the effects of the eigenvalues to FT stability let us consider the system (2.6). Let $\sigma_{i+1} := \sigma(t_i) \in \mathscr{I},\ i = 0,1,2,...,n$. Therefore, by definition σ_{i+1}^{th} subsystem is activated on the time interval $[t_i, t_{i+1})$. For $t \in [t_i, t_{i+1})$, let us define the number of activation of the σ_{i+1}^{th} subsystem and the number of activated subsystems on $[0,t]$ by $\eta_{\sigma_{i+1}}(t)$ and $N(t)$, respectively. For a given initial condition $x(0) = x_0$, the solution of (2.6) in $t \in [t_n, T_f]$ can be written as

$$x(t) = e^{A_{\sigma_{n+1}}(t-t_n)} e^{A_{\sigma_n}(t_n - t_{n-1})} .....e^{A_{\sigma_1} t_1} x_0. \tag{3.1}$$

In the following theorems, FT stability of switched linear systems is analyzed by using Jordan decomposition of the subsystem matrices and the vector and the matrix norms. In order to make that analysis, the definition of FT stability is revised as follows.

Definition 9. *Consider a constant scalar $T_f > 0$ and a positive definite matrix R, the system (2.6) is said to be FT stable with respect to ($\delta, \varepsilon, T_f, R$) if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|R^{1/2}x(t)\| < \sqrt{\varepsilon}$, whenever $\|R^{1/2}\| \cdot \|x_0\| < \sqrt{\delta}$, $\forall t \in [0, T_f]$.*

In literature the system (2.6) is said to be FT stable with respect to $(\delta, \varepsilon, T_f, R)$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $x_0^T R x_0 < \delta \Rightarrow x(t)^T R x(t) < \varepsilon$. Since $x_0^T R x_0 = \|R^{1/2}x_0\|^2 \leq \|R^{1/2}\|^2 \cdot \|x_0\|^2$, the Definition 9 still satisfies the FT stability.

It is a well-known fact that for any $n \times n$ matrix A (even non-diagonalizable), there exists a nonsingular matrix V such that

$$A = VJV^{-1} \tag{3.2}$$

where J is the Jordan canonical form of A.

Before we go further, we will give a following lemma to be used to state the main results.

Lemma 4. *[16] Let $J = diag(J_1, ..., J_k)$ be a matrix in Jordan form where each J_i is a Jordan block of size n_i with eigenvalue λ_i. Then the following inequality holds for the* *spectral norm of J*

$$\|e^{Jt}\| \leq e^{\lambda^*(J)t}, \tag{3.3}$$

where

$$\lambda^*(J) = \max_{i=1,\dots,k}\left(Re\{\lambda_i\} + cos\left(\frac{\pi}{n_i+1}\right)\right).$$

3.1 Finite-Time Stability Analysis in Terms of the Eigenvalues of the Subsystems

One of the major goal of this chapter is to state a stability condition in terms of the eigenvalues of the switching subsystems. Before we state our results, consider the subintervals $[t_p, t_{p+1})$ where $p = 0, 1, 2, \dots, n$ formed by the switching sequence Σ. For any $t \in [t_p, t_{p+1})$, define the maximum of the spectral norm bounds, matrix condition number and the maximum norm of the matrix exponential function as $\overline{\lambda}(t_p) = \max\{\lambda^*(J_{\sigma_1}), \lambda^*(J_{\sigma_2}), \dots, \lambda^*(J_{\sigma_p})\}$, $\kappa_{\sigma_p} = \|V_{\sigma_p}\| \cdot \|V_{\sigma_p}^{-1}\|$ and $m(a,b) = \sup_{t\in[a,b)} \|e^{J_{\sigma(a)}(t-a)}\|$. Now, we are ready to present our first result.

Theorem 1. *Assume that $\lambda^*(J_{\sigma_i}) > 0$, for all $\sigma_i \in \mathscr{I}$. The system (2.6) is FT stable with respect to $(\varepsilon, \delta, T_f, R)$ if there exist some positive integers $q_1, q_2, \dots, q_m$ satisfying the following condition*

$$\ln\delta + 2\cdot\left(\sum_{i=1}^{m} q_i \ln\kappa_i + \lambda^*_{max} T_f\right) \leq \ln\varepsilon \tag{3.4}$$

where $\lambda^*_{max} := \max\{\lambda^*(J_1), \lambda^*(J_2), \dots, \lambda^*(J_m)\}$ and m is the number of subsystems to be activated in $[0, T_f]$.

Proof. Assume that $\lambda^*(J_{\sigma_i}) > 0$, for all $\sigma_i \in \mathscr{I}$. Let $\delta > 0$ such that $\|R^{1/2}x_0\| \leq \|R^{1/2}\| \cdot \|x_0\| \leq \sqrt{\delta}$ and consider the solution $x(t)$ given in (3.1). For any $t \in [0, t_1)$, the following inequality is obtained by using the properties of the norm.

$$\begin{aligned}
\|R^{1/2}x(t)\| &= \|R^{1/2}V_{\sigma_1}e^{J_{\sigma_1}t}V_{\sigma_1}^{-1}x_0\| \\
&\leq \|R^{1/2}\| \cdot \|V_{\sigma_1}\| \cdot \|e^{J_{\sigma_1}t}\| \cdot \|V_{\sigma_1}^{-1}\| \cdot \|x_0\| \\
&\leq \|R^{1/2}\| \cdot \|V_{\sigma_1}\| \cdot m(0,t_1) \cdot \|V_{\sigma_1}^{-1}\| \cdot \|x_0\| \\
&\leq \sqrt{\delta} \cdot \|V_{\sigma_1}\| \cdot m(0,t_1) \cdot \|V_{\sigma_1}^{-1}\| \\
&= \sqrt{\delta} \cdot \kappa_{\sigma_1} \cdot m(0,t_1).
\end{aligned} \tag{3.5}$$

Let $T_1 := t_1$ and consider the interval $[t_1,t_2)$. For any $t \in [t_1,t_2)$, the norm of $R^{1/2}x(t)$ is written as follows:

$$\begin{aligned}\|R^{1/2}x(t)\| =&\|R^{1/2}V_{\sigma_2}e^{J_{\sigma_2}(t-t_1)}V_{\sigma_2}^{-1}V_{\sigma_1}e^{J_{\sigma_1}t_1}V_{\sigma_1}^{-1}x_0\| \\ \leq&\|R^{1/2}\|\cdot\|V_{\sigma_2}\|\cdot\|e^{J_{\sigma_2}(t-t_1)}\|\cdot\|V_{\sigma_2}^{-1}\| \\ &\cdot\|V_{\sigma_1}\|\cdot\|e^{J_{\sigma_1}T_1}\|\cdot\|V_{\sigma_1}^{-1}\|\cdot\|x_0\| \\ \leq&\sqrt{\delta}\cdot\kappa_{\sigma_2}\cdot\kappa_{\sigma_1}\cdot m(t_1,t_2)\cdot\|e^{J_{\sigma_1}T_1}\| \\ \leq&\sqrt{\delta}\cdot\left(\prod_{i=1}^{N(t_1)}\kappa_{\sigma_i}^{\eta_{\sigma_i}(t_1)}\right)\cdot m(t_1,t_2)\cdot\|e^{J_{\sigma_1}T_1}\|\end{aligned} \tag{3.6}$$

Note that $\|e^{J_{\sigma_1}T_1}\| \leq e^{\lambda^*(J_{\sigma_1})T_1}$ and two subsystems σ_1 and σ_2 are activated in this interval. So, $N(t_1)=2$, $\eta_{\sigma_1}(t_1)=\eta_{\sigma_2}(t_1)=1$ and $\overline{\lambda}(t_1)=\lambda^*(J_{\sigma_1})$. Now, let $T_p := t_p - t_{p-1}$ and consider each interval $[t_p,t_{p+1})$, for $p=2,...,n$. In each interval, same subsystems could be activated several times. Thus, we have

$$\begin{aligned}\|R^{1/2}x(t)\| =&\|R^{1/2}V_{\sigma_{p+1}}e^{J_{\sigma_{p+1}}(t-t_p)}V_{\sigma_{p+1}}^{-1}V_{\sigma_p}e^{J_{\sigma_p}T_p}V_{\sigma_p}^{-1}\cdots V_{\sigma_1}e^{J_{\sigma_1}T_1}V_{\sigma_1}^{-1}x_0\| \\ \leq&\|R^{1/2}\|\cdot\|V_{\sigma_{p+1}}\|\cdot\|e^{J_{\sigma_{p+1}}(t-t_p)}\|\cdot\|V_{\sigma_{p+1}}^{-1}\|\cdot\|V_{\sigma_p}\|\cdot\|e^{J_{\sigma_p}T_p}\| \\ &\cdot\|V_{\sigma_p}^{-1}\|\cdots\|V_{\sigma_1}\|\cdot\|e^{J_{\sigma_1}T_1}\|\cdot\|V_{\sigma_1}^{-1}\|\cdot\|x_0\| \\ \leq&\sqrt{\delta}\cdot\left(\prod_{i=1}^{N(t_p)}\kappa_{\sigma_i}^{\eta_{\sigma_i}(t_p)}\right)\cdot m(t_p,t_{p+1})\cdot\left(\prod_{k=1}^{p}\|e^{J_{\sigma_k}T_k}\|\right).\end{aligned} \tag{3.7}$$

From Lemma 4, it is possible to write

$$\|e^{J_{\sigma_k}T_k}\| \leq e^{\lambda^*(J_{\sigma_k})\cdot T_k}. \tag{3.8}$$

and

$$\prod_{k=1}^{p}\|e^{J_{\sigma_k}T_k}\| \leq \prod_{k=1}^{p}e^{\lambda^*(J_{\sigma_k})\cdot T_k} = e^{\sum_{k=1}^{p}\lambda^*(J_{\sigma_k})\cdot T_k} \tag{3.9}$$

Then, from (3.7) and (3.9), we obtain

$$\|R^{1/2}x(t)\| \leq\sqrt{\delta}\cdot\left(\prod_{i=1}^{N(t_p)}\kappa_{\sigma_i}^{\eta_{\sigma_i}(t_p)}\right)\cdot m(t_p,t_{p+1})\cdot e^{\sum_{k=1}^{p}\lambda^*(J_{\sigma_k})\cdot T_k}. \tag{3.10}$$

Since $\sum_{k=1}^{p}T_k = t_p$ then, $e^{\sum_{k=1}^{p}\lambda^*(J_{\sigma_k})\cdot T_k} \leq e^{\overline{\lambda}(t_p)\cdot t_p}$ and

$$\|R^{1/2}x(t)\| \leq\sqrt{\delta}\cdot\left(\prod_{i=1}^{N(t_p)}\kappa_{\sigma_i}^{\eta_{\sigma_i}(t_p)}\right)\cdot m(t_p,t_{p+1})\cdot e^{\overline{\lambda}(t_p)\cdot t_p}. \tag{3.11}$$

Now, let us recall the inequality (3.5). Since $\lambda^*(J_{\sigma_1}) > 0$, then

$$
\begin{aligned}
m(0,t_1) = & \sup_{t\in[0,t_1)} \|e^{J_{\sigma_1} t}\| \\
\leq & e^{\lambda^*(J_{\sigma_1}) \sup_{t\in[0,t_1)} t} \\
= & e^{\lambda^*(J_{\sigma_1}) \lim_{t\to t_1^-} t} \\
\leq & e^{\lambda^*(J_{\sigma_1}) t_1}.
\end{aligned}
\tag{3.12}
$$

In $[0,t_1)$ only the subsystem σ_1 is activated and $\lambda^*(J_{\sigma_1}) = \overline{\lambda}(t_1)$. Then, by the inequalities (3.5) and (3.12), we have

$$
\|R^{1/2}x(t)\| \leq \sqrt{\delta} \cdot \kappa_{\sigma_1} \cdot e^{\overline{\lambda}(t_1)\cdot t_1}. \tag{3.13}
$$

Similar to (3.12), $m(t_p,t_{p+1}) \leq e^{\lambda^*(J_{\sigma_{p+1}})(t_{p+1}-t_p)}$, for all $p = 1,2,...,n$. By defining $\hat{\lambda}(t_p) := \max\{\lambda^*(J_{\sigma_{p+1}}), \overline{\lambda}(t_p)\}$, for all $p = 1,2,...,n$, we write

$$
\begin{aligned}
m(t_p,t_{p+1}) \cdot e^{\overline{\lambda}(t_p)\cdot t_p} & \leq e^{\lambda^*(J_{\sigma(t_{p+1})})(t_{p+1}-t_p)} \cdot e^{\overline{\lambda}(t_p)\cdot t_p} \\
& \leq e^{\hat{\lambda}(t_p)\cdot t_{p+1}}
\end{aligned}
\tag{3.14}
$$

Thus, by (3.11) and (3.14), we obtain

$$
\|R^{1/2}x(t)\| \leq \sqrt{\delta} \cdot \left(\prod_{i=1}^{N(t_p)} \kappa_{\sigma_i}^{\eta_{\sigma_i}(t_p)} \right) \cdot e^{\hat{\lambda}(t_p)\cdot t_{p+1}}. \tag{3.15}
$$

Since $\lambda^*(J_{\sigma_i}) > 0$ and $\kappa_{\sigma_i} \geq 1$, for all $\sigma_i \in \mathscr{I}$ and (3.13) is included in (3.15) for $p = 1,2,...,n$, the exponential in (3.15) is increasing. So, the upper bound of (3.15) is given for $p = n$. Thus, if

$$
\|R^{1/2}x(t)\| \leq \sqrt{\delta} \cdot \left(\prod_{i=1}^{N(t_n)} \kappa_{\sigma_i}^{\eta_{\sigma_i}(t_n)} \right) \cdot e^{\hat{\lambda}(t_n)\cdot T_f} \leq \sqrt{\varepsilon} \tag{3.16}
$$

then the system (2.6) is FT stable with respect to $(\varepsilon,\delta,T_f,R)$. Note that $\hat{\lambda}(t_n) = \max\{\lambda^*(J_{\sigma_1}),\lambda^*(J_{\sigma_2}),...,\lambda^*(J_{\sigma_{n+1}})\} = \max\{\lambda^*(J_1),\lambda^*(J_2),...,\lambda^*(J_m)\} = \lambda^*_{max}$ and $\{\kappa_{\sigma_1},...,\kappa_{\sigma_{n+1}}\} = \{\kappa_1,\kappa_2,...,\kappa_m\}$. Thus, define $q_i := \eta_{\sigma_i}$, for $i = 1,2,...,m$. By using these notations the inequality (3.16) is written as follows:

$$
\sqrt{\delta} \cdot \left(\prod_{i=1}^{m} \kappa_i^{q_i} \right) \cdot e^{\lambda^*_{max}\cdot T_f} \leq \sqrt{\varepsilon} \tag{3.17}
$$

The unknown variables in this inequality are the numbers of the activations q_i of the subsystems. By taking the natural logarithm of both sides of (3.17), we obtain the result given in (3.4). □

Theorem 2. *Assume that* $\lambda^*(J_{\sigma_i}) \leq 0$*, for all* $\sigma_i \in \mathscr{I}$*. The system* (2.6) *is FT stable with respect to* $(\varepsilon, \delta, T_f, R)$ *if there exist some positive integers* $q_1, q_2, ..., q_m$ *and a real number* t_n*,* $0 < t_n \leq T_f$ *satisfying the following conditions*

$$\varepsilon - \delta \kappa_{\sigma_i}^2 \geq 0, \; for\ all\ \sigma_i \in \mathscr{I} \tag{3.18a}$$

$$\ln \delta + 2 \cdot \left(\sum_{i=1}^{m} q_i \cdot \ln \kappa_i + \lambda_{max}^* \cdot t_n \right) \leq \ln \varepsilon. \tag{3.18b}$$

Here λ_{max}^* *and m are as defined in Theorem 1.*

Proof. Assume that $\lambda^*(J_{\sigma_i}) \leq 0$, for all $\sigma_i \in \mathscr{I}$ and let $\delta > 0$ such that $\|R^{1/2}x_0\| \leq \|R^{1/2}\| \cdot \|x_0\| \leq \sqrt{\delta}$. Consider the inequality (3.5). Since $\lambda^*(J_{\sigma_1}) \leq 0$ then for $t \in [0, t_1)$

$$\begin{aligned} m(0,t_1) = & \sup_{t \in [0,t_1)} \|e^{J_{\sigma_1} t}\| \\ \leq & e^{\lambda^*(J_{\sigma_1}) \inf_{t \in [0,t_1)} t} \\ \leq & 1. \end{aligned} \tag{3.19}$$

By the inequalities (3.5) and (3.19), we obtain

$$\|R^{1/2}x(t)\| \leq \sqrt{\delta} \cdot \kappa_{\sigma_1}. \tag{3.20}$$

If $\|R^{1/2}x(t)\| \leq \sqrt{\delta} \cdot \kappa_{\sigma_1} \leq \sqrt{\varepsilon}$, then the system (2.6) is FT stable with respect to $(\varepsilon, \delta, T_f, R)$ in $[0, t_1)$. Since σ_1 could be any number in $\mathscr{I}$ then, the inequality (3.18a) is obtained.

Now, consider the inequality in (3.11). For any $t \in [t_p, t_{p+1})$, the subsystem σ_{p+1} is activated. Since $\lambda^*(J_{\sigma_i}) \leq 0$, for all $\sigma_i \in \mathscr{I}$, we have $m(t_p, t_{p+1}) \leq 1$ similarly as (3.19). Then, (3.11) can be written as follows:

$$\|R^{1/2}x(t)\| \leq \sqrt{\delta} \cdot \left(\prod_{i=1}^{N(t_p)} \kappa_{\sigma_i}^{\eta_{\sigma_i}(t_p)} \right) \cdot e^{\overline{\lambda}(t_p) \cdot t_p} \tag{3.21}$$

Note that $e^{\overline{\lambda}(t_p)} \geq e^{\overline{\lambda}(t_{p-1})} \geq 1$, for $p = 2, ..., n$ and $\kappa_{\sigma_i} \geq 1$, for all $\sigma_i \in \mathscr{I}$. So, the exponential in (3.21) is increasing in terms of t_p and its upper bound is attained at $p = n$. In $[0, T_f]$ all the subsystems including σ_{n+1} had been activated. Thus, $\prod_{i=1}^{N(t_n)} \kappa_{\sigma_i}^{\eta_{\sigma_i}(t_n)} = \prod_{i=1}^{m} \kappa_i^{q_i}$ and $\overline{\lambda}(t_n) \leq \max\{\lambda^*(J_1), \lambda^*(J_2), ..., \lambda^*(J_m)\} = \lambda_{max}^*$. Consequently, for any $t \in [t_n, T_f]$ if

$$\|R^{1/2}x(t)\| \leq \sqrt{\delta} \cdot \left(\prod_{i=1}^{m} \kappa_i^{q_i} \right) \cdot e^{\lambda_{max}^* \cdot t_n} \leq \sqrt{\varepsilon} \tag{3.22}$$

then the system (2.6) is FT stable with respect to $(\varepsilon, \delta, T_f, R)$ when the condition (3.18b) is satisfied. □

Remark 1. *Theorem 1 and 2 depend on the solution of the integers* $q_1, q_2, ..., q_n$ *on* $[0, T_f]$ *and* $[0, t_n]$*, respectively. These "subsystem activation configurations" can be chosen from the feasible set defined as*

$$\mathscr{F}_I := \Big\{ (q_1, q_2, ..., q_m) \ \Big| \ q_i \in \mathbb{Z}^+ \ for\ all\ i = 1, 2, ..., m\ satisfying \\ \sum_{i=1}^{m} q_i \ln \kappa_i \leq \frac{\ln \varepsilon - \ln \delta}{2} - \lambda^*_{max} \cdot \ell(I) \Big\} \tag{3.23}$$

where $\ell(I)$ *is the length of the closed interval* I *and* λ^*_{max} *is as defined in Theorem 1 and 2 by using the appropriate closed interval.*

Remark 2. *Note that, Theorems 1 and 2 are conservative because they have the constraints* $\lambda^*(J_{\sigma(t_i)}) > 0$ *and* $\lambda^*(J_{\sigma(t_i)}) \leq 0$*, for all* $\sigma(t_i) \in \mathscr{I}$*, respectively. However, this conservativeness can be relaxed by seperating the subsystems having* $\lambda^*(J_i) \leq 0$ *and* $\lambda^*(J_i) > 0$ *as in the following Theorem.*

Let $\mathscr{I}^-$ and $\mathscr{I}^+$ be the subsets of $\mathscr{I}$, for which $\lambda^*(J_{\sigma_i}) \leq 0$ and $\lambda^*(J_{\sigma_i}) > 0$, respectively. Note that, $\mathscr{I} = \mathscr{I}^- \cup \mathscr{I}^+$. Let us also define

$$\lambda^-_{\min} := \min_{\sigma_i \in \mathscr{I}^-} \{|\lambda^*(J_{\sigma_i})|\}$$
$$\lambda^+_{\max} := \max_{\sigma_i \in \mathscr{I}^+} \{\lambda^*(J_{\sigma_i})\}$$

Let T^+ and T^- be the activation times of the systems belonging to the subsets $\mathscr{I}^+$ and $\mathscr{I}^-$, respectively and $T^+ + T^- \leq T_f$. Now, we have the following Theorem.

Theorem 3. *Assume that* $\mathscr{I}^- \neq \emptyset$*,* $\mathscr{I}^+ \neq \emptyset$*. The system* (2.6) *is FT stable with respect to* $(\varepsilon, \delta, T_f, R)$ *if there exist positive integers* $q_1, q_2, ..., q_m$ *satisfying the following condition:*

$$\ln \delta + 2 \cdot \left(\sum_{i=1}^{m} q_i \cdot \ln \kappa_i + \lambda^+_{\max} T^+ - \lambda^-_{\min} T^- \right) \leq \ln \varepsilon \tag{3.24}$$

Proof. By (3.12) and (3.19), it is clear that

$$m(t_p, t_{p+1}) \leq \begin{cases} 1 & , \ \text{if } \sigma_{p+1} \in \mathscr{I}^- \\ e^{\lambda^*(J_{\sigma_{p+1}}) \cdot t_{p+1}} & , \ \text{if } \sigma_{p+1} \in \mathscr{I}^+ \end{cases} \tag{3.25}$$

Consider the inequality in (3.11). Since $e^{\lambda^*(J_{\sigma_{p+1}}) \cdot t_{p+1}} > 1$, by (3.25) we have

$$m(t_p, t_{p+1}) \cdot e^{\overline{\lambda}(t_p) \cdot t_p} \leq e^{\lambda^+_{\max} T^+_{p+1} - \lambda^-_{\min} T^-_{p+1}} \tag{3.26}$$

where T^+_{p+1} and T^-_{p+1} are the activation times of the systems belonging to the subsets $\mathscr{I}^+$ and $\mathscr{I}^-$, respectively in the time interval $[0, t_{p+1})$. If (3.26) is substituted into (3.11) at $p = n$ and the result is bounded above by $\sqrt{\varepsilon}$ then, $\|R^{1/2}x(t)\| \leq \sqrt{\varepsilon}$ is obtained. Thus, the system (2.6) is FT stable with respect to $(\varepsilon, \delta, T_f, R)$ in $[0, T_f]$ under the condition in (3.24). □

3.2 Average Dwell-Time Condition for the Switching Systems

Consider the constrained set of all switching signals as follows:

$$\mathscr{S} = \mathscr{S}_{avg.\ dwell}[\tau_a, N_0] = \left\{ \sigma \in \Sigma \,\middle|\, N_\sigma(t) \leq N_0 + \frac{T_f - t}{\tau_a} \right\}. \tag{3.27}$$

The infimum τ_a for which the switched system is FT stable is called the *average dwell time* (ADT) and it is denoted by τ_a^*. Let $\kappa_{\max} := \max\{\kappa_{\sigma_1}, \kappa_{\sigma_2}, ..., \kappa_{\sigma_{n+1}}\}$. Now, we are ready to present the following.

Theorem 4. *The switched system* (2.6) *is FT stable with respect to* $(\varepsilon, \delta, T_f, R)$*, if the switching signal satisfies the ADT* τ_a^* *for the following cases:*

(i) Assume that $\mathscr{I}^- = \mathscr{I}$*. Then,*

$$\tau_a \geq \tau_a^* = \frac{2T_f \ln \kappa_{\max}}{\ln \varepsilon - \left(\ln \delta + 2N_0 \ln \kappa_{\max} - 2\lambda^-_{\min} t_n\right)}. \tag{3.28}$$

(ii) Assume that $\mathscr{I}^+ = \mathscr{I}$*. Then,*

$$\tau_a \geq \tau_a^* = \frac{2T_f \ln \kappa_{\max}}{\ln \varepsilon - \left(\ln \delta + 2N_0 \ln \kappa_{\max} + 2\lambda^+_{\max} T_f\right)}. \tag{3.29}$$

(iii) Assume that $\mathscr{I}^- \neq \emptyset$ *and* $\mathscr{I}^+ \neq \emptyset$*. Then,*

$$\tau_a \geq \tau_a^* = \frac{2T_f \ln \kappa_{\max}}{\ln \varepsilon - \left(\ln \delta + 2N_0 \ln \kappa_{\max} + 2(\lambda^+_{\max} T^+ - \lambda^-_{\min} T^-)\right)}. \tag{3.30}$$

Here T^+ *and* T^- *are the activation times of the systems belonging to the subsets* $\mathscr{I}^+$ *and* $\mathscr{I}^-$*, respectively.*

Proof. Consider the inequality (3.11) for $p = n$. Since $\kappa_{\max} > 1$, for any $t \in [t_n, T_f]$, we have

$$\|R^{1/2}x(t)\| \leq \sqrt{\delta} \cdot \left(\kappa_{max}^{N_\sigma(t)}\right) \cdot m(t_n, T_f) \cdot e^{\overline{\lambda}(t_n) \cdot t_n}. \tag{3.31}$$

Since $\kappa_{\max} > 1$, by the definition $N_\sigma(t)$ it is clear that $\kappa_{max}^{N_\sigma(t)} \leq \kappa_{max}^{N_0+\frac{T_f-t}{\tau_a}} \leq \kappa_{max}^{N_0+\frac{T_f}{\tau_a}}$. By considering (3.25) for $p = n$, we get

$$m(t_n, T_f) \leq \begin{cases} 1 & \text{if } \sigma_{n+1} \in \mathscr{I}^- \\ e^{\lambda^*(J_{\sigma_{n+1}})\cdot T_f} & \text{if } \sigma_{n+1} \in \mathscr{I}^+ \end{cases} \tag{3.32}$$

If $\sigma_{n+1} \in \mathscr{I}^-$ then, by (3.31) and (3.32) we obtain $m(t_n, T_f) \cdot e^{\overline{\lambda}(t_n)\cdot t_n} \leq e^{\overline{\lambda}(t_n)\cdot t_n}$. In order to obtain $\|R^{1/2}x(t)\| \leq \varepsilon$, we should have

$$\sqrt{\delta} \cdot \left(\kappa_{max}^{N_0+\frac{T_f}{\tau_a}} \right) \cdot e^{\overline{\lambda}(t_n)\cdot t_n} \leq \sqrt{\varepsilon}. \tag{3.33}$$

By taking the natural logarithm of both sides of (3.33) and rearranging the result we get

$$\tau_a \geq \frac{2T_f \ln \kappa_{\max}}{\ln \varepsilon - \left(\ln \delta + 2N_0 \ln \kappa_{\max} + 2\overline{\lambda}(t_n) \cdot t_n \right)}. \tag{3.34}$$

In the case that $\mathscr{I}^- = \mathscr{I}$, then $\overline{\lambda}(t_n) \cdot t_n \leq 0$. Since $\hat{\lambda}(t_n) = \max\{\overline{\lambda}(t_n), \lambda^*(J_{\sigma_{n+1}})\}$, it is clear that $\overline{\lambda}(t_n) \leq \hat{\lambda}(t_n) \leq 0$ and $\frac{1}{-\hat{\lambda}(t_n)} \geq \frac{1}{-\overline{\lambda}(t_n)}$. All subsystems are activated at this stage so that we have $-\hat{\lambda}(t_n) = \lambda^-_{\min}$. Thus, by substituting $\hat{\lambda}(t_n)$ with $-\lambda^-_{\min}$ in (3.34), we conclude that the switched system (2.6) is FT stable with respect to $(\varepsilon, \delta, T_f, R)$ if the condition (3.28) holds true.

In the other case that $\mathscr{I}^+ = \mathscr{I}$, then $\overline{\lambda}(t_n) \cdot t_n > 0$ and $\sigma_{n+1} \in \mathscr{I}^+$. Thus, $\hat{\lambda}(t_n) = \lambda^+_{\max}$. By using again the fact $T_f > t_n$, the switched system (2.6) is FT stable with respect to $(\varepsilon, \delta, T_f, R)$ if the condition (3.29) holds true.

Let us consider the last case that $\mathscr{I}^- \neq \emptyset$ and $\mathscr{I}^+ \neq \emptyset$. Since $e^{\lambda^*(J_{\sigma_{n+1}})\cdot T_f} \geq 1$ in (3.32) the following inequality can be written:

$$m(t_n, T_f) \cdot e^{\overline{\lambda}(t_n)\cdot t_n} \leq e^{\lambda^+_{\max} T^+ - \lambda^-_{\min} T^-} \tag{3.35}$$

Thus, the switched system (2.6) is FT stable with respect to $(\varepsilon, \delta, T_f, R)$ if the condition (3.30) holds true for the last case, which concludes the proof. □

In the existing literature, the ADT is obtained by the analysis of Lyapunov functionals. See the following theorem.

Theorem 5. *[45] For any $i \in \{1, 2, ..., m\}$, let $\tilde{Q}_i = R^{-1/2}$ and suppose there exist matrices $Q_i > 0$ and a constant $\alpha \geq 0$ such that*

$$A_i \tilde{Q}_i + \tilde{Q}_i A_i^T - 2\alpha \tilde{Q}_i < 0, \tag{3.36a}$$

$$\mu < \frac{\varepsilon}{\delta} e^{-2\alpha T_f} \tag{3.36b}$$

then, the system (2.6) *is FT stable with respect to* $(\varepsilon,\delta,T_f,R)$ *with ADT satisfying*

$$\tau_a > \tau_a^* = \frac{T_f \ln\mu}{\ln(\varepsilon/\delta) - \ln\mu - 2\alpha T_f}, \tag{3.36c}$$

where $\lambda_1 = \min_{\forall i\in\mathscr{I}}(\lambda_{min}(Q_i))$, $\lambda_2 = \max_{\forall i\in\mathscr{I}}(\lambda_{max}(Q_i))$, $\mu = \frac{\lambda_2}{\lambda_1}$.

The ADT constraints obtained in Theorem 4 are quite similar to the ADT constraints in Theorem 5 by formulation. However, these constraints differ in calculation. The comparison and differences are given in the following remark.

Remark 3. *There are two major differences in calculation of the ADT bounds in Theorem 4 and 5. Firstly,* λ_1 *and* λ_2 *(as well as* μ*) depend on the solutions of the LMIs* (3.36b) *in Theorem 5. Thus,* μ *has a freedom of choice. However,* κ_{max} *is directly calculated by the matrix condition numbers of the subsystems in Theorem 4. Secondly,* α *is a parameter which is dependent on the solution of the LMIs* (3.36b). *In other words,* α *can be freely chosen to shift the eigenvalues of the subsystem matrices* A_i *by* α *to make* $(A_i - \alpha I) < 0$. *On the other hand, the eigenvalues of the subsystem matrices are directly used to calculate* λ_{max}^+ *and* λ_{max}^- *in Theorem 4. The restriction of the freedom of choice gives an opportunity to have a better estimation for the ADT as it can be seen in Example 9 of the next section.*

3.3 Numerical Examples

Now, the following examples are presented to demonstrate the FT stabilizability of switching systems by using Theorem 1 and 4.

Example 5. *Consider the switched linear system* (2.6) *with following three subsystems*

$$A_1 = \begin{bmatrix} 0.06 & 0.13 & 0.04 \\ 0.06 & 0.02 & -0.06 \\ 0.04 & -0.13 & 0.06 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0.2 & -0.4 \\ 0.21 & 0.2 & 0.2 \\ -0.21 & 0.1 & 0.1 \end{bmatrix}, A_3 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.36 & -0.02 \\ 0 & 0 & 0.3 \end{bmatrix}.$$

Calculating the Jordan forms, we have

$$J_1 = V_1 A_1 V_1^{-1} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & -0.105 & 0 \\ 0 & 0 & 0.15 \end{bmatrix}, V_1 = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0.96 & -0.96 \\ 1 & 1 & 1 \end{bmatrix},$$

$$J_2 = V_2 A_2 V_2^{-1} = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.355 & 0 \\ 0 & 0 & -0.355 \end{bmatrix}, V_2 = \begin{bmatrix} 0 & -1.69 & 1.69 \\ 2 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$J_3 = V_3 A_3 V_3^{-1} = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.36 \end{bmatrix}, V_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0.33 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

$\lambda^*(J_i)$'s are calculated as 0.15*,* 0.355 *and* 0.36*, respectively. Note that all $\lambda^*(J_i)$'s are positive so that we can apply Theorem 1. By choosing $\delta = 1$, $\varepsilon = 100$, $T_f = 2$ and from* (3.4)*, we have*

$$0.3866 \cdot q_1 + 0.3466 \cdot q_2 + 0.3318 \cdot q_3 \leq 1.5826.$$

So, this allows us to determine the feasible set of the following subsystem activation configurations: $\mathscr{F}_{[0,2]} = \{(1,1,1),(2,1,1),(1,2,1),(1,1,2)\}$. Let us see this with a simulation experiment.

Take the initial state $x_0 = [-0.1 \;\; 5 \;\; 0.5]^T$ and $R = \begin{bmatrix} 4 & 0 & 3 \\ 0 & 0.01 & 0 \\ 3 & 0 & 4 \end{bmatrix}$ satisfying the initial condition $x_0^T R x_0 \leq 1$. Let us also consider the switching signal

$$\sigma(t) = \begin{cases} 2, & t \in [0, 0.6) \cup [1.6, 2) \\ 3, & t \in [0.6, 1.2) \\ 1, & t \in [1.2, 1.6) \end{cases}$$

which satisfies the subsystem activation configuration $(1,2,1) \in \mathscr{F}_{[0,2]}$. The simulation results in Figure 3.1 verify that the system is FT stable with respect to the chosen parameters.

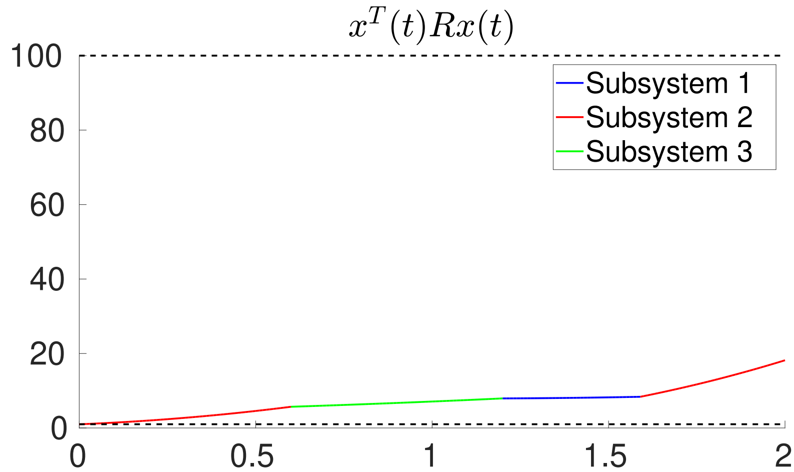

Figure 3.1 : Simulation of $x(t)^T R x(t)$ Under Non-Periodic Switching $\sigma(t)$.

Example 6. *Consider again the switched linear system given as the motivating example in Chapter 2 (Example 4 of Chapter 2). This motivating example shows us the existence of an ADT which makes the switched system FT stable with respect to given parameters. When periodic switching is concerned, the existence of such an ADT is equivalent to the existence of a critical period for the systems to ensure FT stability.*

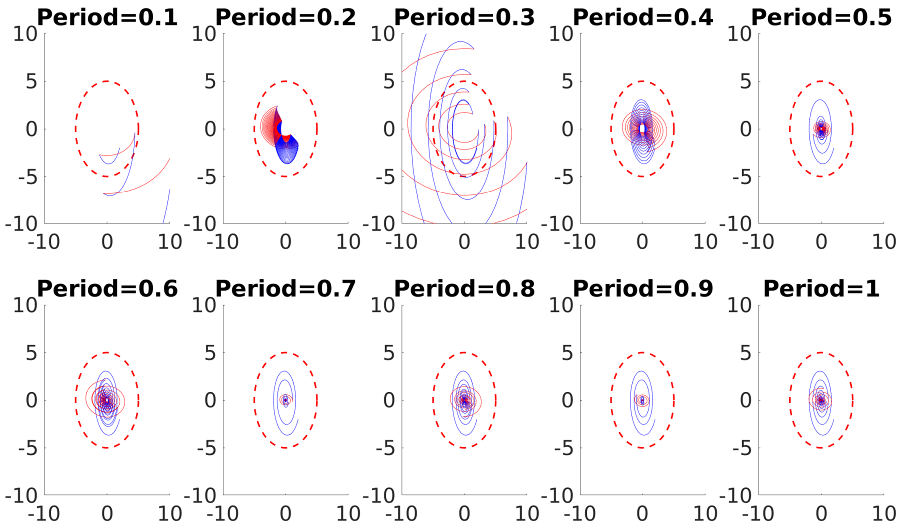

Figure 3.2 : Periodic Switching of A_1 and A_2 with Various Periods and $T_f = 10$.

In order to see this phenomenon, the simulations of the given switched system with various periods are depicted in Figure 3.2.

As it can be seen in Figure 3.2, there is a critical period between the periods 0.3 and 0.4 that the system satisfies $x^T(t)x(t) \leq 25$, $\forall t \in [0,10]$.

Now, let us apply Theorem 4. Calculating the Jordan forms, we have

$$
\begin{aligned}
J_1 = & V_1 A_1 V_1^{-1} = J_2 = V_2 A_2 V_2^{-1} = \begin{bmatrix} -1-17.2916j & 0 \\ 0 & -1-17.2916j \end{bmatrix}, \\
V_1 = & \begin{bmatrix} -0.0333+0.5764j & -0.0333-0.5764j \\ 1 & 1 \end{bmatrix}, \\
V_2 = & \begin{bmatrix} -0.1-1.7292j & -0.1+1.7292j \\ 1 & 1 \end{bmatrix}.
\end{aligned}
$$

Thus, $\lambda^*(J_1) = \lambda^*(J_2) = -1$ and $\lambda_{min}^- = 1$. Now, in order to apply Theorem 4, we should determine the last switching instant t_n. In order to represent the cases presented in Figure 3.2, we choose different t_n's, calculate different ADT bounds for each t_n and present the results in Table 3.1.

These results are consistent with the simulation results presented in Figure 3.2.

Example 7. *A switched linear system* (2.6) *with two subsystems is given as follows*

$$
A_1 = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}, \; A_2 = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}
$$

Table 3.1 : ADT Bounds for Different Switching Periods and Corresponding t_n's

Period	t_n	τ_a^*
0.1	9.9	0.5278
0.2	9.8	0.5329
0.3	9.9	0.5278
0.4	9.6	0.5434
0.5	9.5	0.5488
0.6	9.6	0.5434
0.7	9.8	0.5329
0.8	9.6	0.5434
0.9	9.9	0.5278
1	9	0.5775

where A_1 is unstable while A_2 is Hurwitz stable and both of them are diagonalizable matrices.

$$J_1 = V_1^{-1}A_1V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \; V_1 = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \lambda_{\max}^{+} = 4$$

$$J_2 = V_2^{-1}A_2V_2 = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, \; V_2 = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \quad \Rightarrow \lambda_{\min}^{-} = 1$$

Choose

$$\delta = 500, \; R = I, \; T^{+} = 2.4, \; T^{-} = 21.6, \; T_f = 24, N_0 = 0, \varepsilon = 900$$

the initial state $x_0 = [10 \quad -20]^T$ satisfying the initial condition $x_0^T x_0 \leq 500$. By applying (3.30) *of Theorem 4, the ADT is found as $\tau_a^* = 0.5560$ which demonstrates the validity of the proposed theorem.*

Note that, in [46] the ADT is obtained as $\tau_a^* = 2.4$. The major reason for such a difference in ADT is that exponential stability is analyzed in [46]. However, in our example FT stability is analyzed which is much more relaxed stability notion than exponential stability that allows more frequent switching. The simulation results in Figure 3.3 verify that the system is FT stable with respect to the chosen parameters and with a periodic activation of A_1 and A_2 over time periods 0.12 *and* 1.08 *(i.e. $\tau_a = 0.6$), respectively.*

Let us give another example with three subsystems.

Example 8. *A switched linear system* (2.6) *with three subsystems is given as follows*

$$A_1 = \begin{bmatrix} -1.35 & 0 & 0 \\ -0.5 & -1.35 & 0 \\ -0.5 & 0 & -1.35 \end{bmatrix}, A_2 = \begin{bmatrix} 0.35 & 0 & 0 \\ 0 & 0.35 & 0 \\ 1 & 1 & 0.35 \end{bmatrix}, A_3 = \begin{bmatrix} -0.85 & 1 & -1 \\ 0.67 & -0.85 & 0 \\ 0.67 & 0 & -0.85 \end{bmatrix}$$

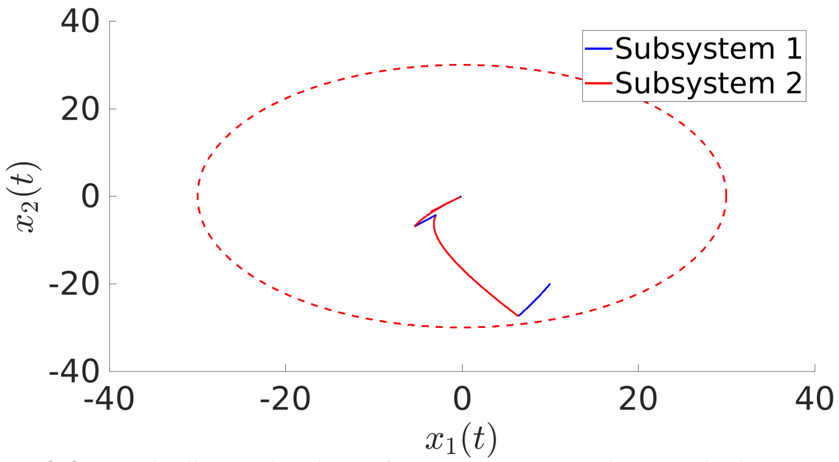

Figure 3.3 : Periodic Activation of A_1 and A_2 over time periods 0.12 and 1.08, respectively.

where A_1 and A_3 are Hurwitz stable while A_2 is unstable matrices. Calculating the Jordan forms, we have

$$J_1 = V_1 A_1 V_1^{-1} = \begin{bmatrix} -1.35 & 1 & 0 \\ 0 & -1.35 & 0 \\ 0 & 0 & -1.35 \end{bmatrix}, \; V_1 = \begin{bmatrix} 0 & -1 & 0 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0 & -0.5 \end{bmatrix},$$

$$J_2 = V_2 A_2 V_2^{-1} = \begin{bmatrix} 0.35 & 1 & 0 \\ 0 & 0.35 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}, \; V_2 = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0 & 0.5 & -0.5 \\ 1 & 0 & 0 \end{bmatrix},$$

$$J_3 = V_3 A_3 V_3^{-1} = \begin{bmatrix} -0.85 & 1 & 0 \\ 0 & -0.85 & 1 \\ 0 & 0 & -0.85 \end{bmatrix}, \; V_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0.67 & 0 & 0.33 \\ 0.67 & 0 & -0.67 \end{bmatrix}.$$

$\lambda^*(J_i)$'s are calculated as -0.85, 0.85 and -0.14, respectively. According to $\lambda^*(J_i)$'s, we get $\lambda^+_{max} = \lambda^-_{min} = 0.85$. Choose

$$\delta = 16, \; R = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 6 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \; T^+ = 7, \; T^- = 9, \; T_f = 16, N_0 = 0, \varepsilon = 25$$

the initial state $x_0 = \begin{bmatrix} -1 & 1 & 2 \end{bmatrix}^T$ satisfying the initial condition $x_0^T R x_0 \leq 16$. By applying (3.30) of Theorem 4, the ADT is found as $\tau_a^* = 1.7157$.

To verify this result, let us consider the switching signal

$$\sigma(t) = \begin{cases} 1, & t \in [0,2) \cup [12,14) \\ 2, & t \in [2,5) \cup [7,9) \cup [10,12) \\ 3, & t \in [5,7) \cup [9,10) \cup [14,16) \end{cases}$$

The simulation results in Figure 3.4 verify that the system is FT stable with respect to the chosen parameters and with a switching signal satisfying the calculated ADT bound.

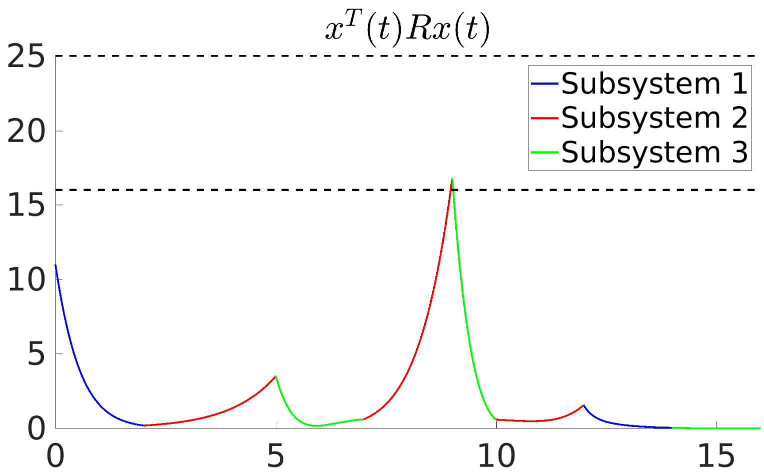

Figure 3.4 : Simulation of $x(t)^T Rx(t)$ Under Non-Periodic Switching $\sigma(t)$.

Example 9. *Let us consider the switched system* (2.6) *with the same subsystem matrices in [45] as*

$$A_1 = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}.$$

By Jordan decomposition, we have

$$J_1 = V_1^{-1} A_1 V_1 = J_2 = V_2^{-1} A_2 V_2 = \begin{bmatrix} -1.4142j & 0 \\ 0 & 1.4142j \end{bmatrix},$$

$$V_1 = \begin{bmatrix} -0.7071j & 0.7071j \\ 1 & 1 \end{bmatrix},\ V_2 = \begin{bmatrix} -1.4142j & 1.4142j \\ 1 & 1 \end{bmatrix}.$$

Thus, we have $\lambda_{min}^{-} = 0$. *Note that, the term* $2\lambda_{\min}^{-} t_n$ *vanishes in* (3.28), *since* $\lambda_{min}^{-} = 0$. *Therefore, there is no need to determine* t_n *to apply Theorem 4 for this example. For given* $\delta = 1$, $\varepsilon = 20$, $T_f = 10$, $R = I$, *let us apply Theorem 4, we have the ADT bound* $\tau_a^* = 2.3138$ *which is a better estimation than* $\tau_a^* = 3.1539$ *found in [45]. On the other hand, we obtain the same ADT bound* $\tau_a^* = 3.1539$ *as in [45] by taking* $\varepsilon = 9.005$, *which is a better estimation for* ε *as it is depicted in Figure 3.5.*

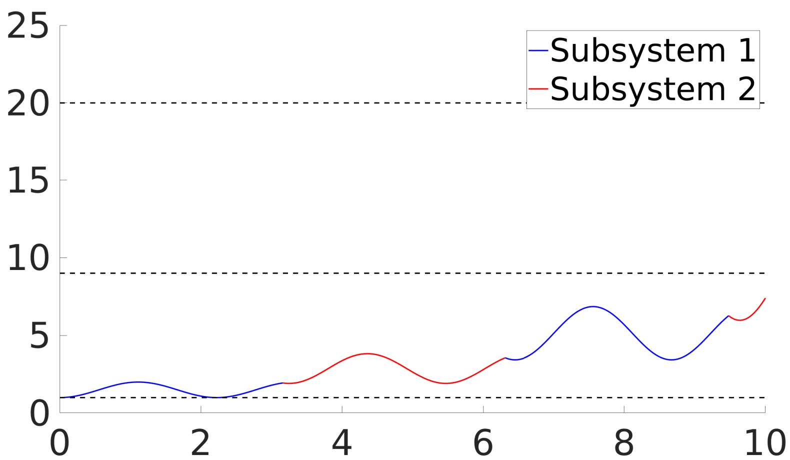

Figure 3.5 : Simulation of $x(t)^T Rx(t)$ Under Periodic Switching with Period 3.16.

4. STATE FEEDBACK DESIGN FOR FINITE-TIME BOUNDEDNESS

In this part, finite-time control of switched linear systems with interval time-delay is considered. State feedback is applied in order to ensure finite-time boundedness of the system. Sufficient conditions and average dwell-time bounds are obtained. Because of non-convex terms in the average dwell-time constraint, a technique which converts the nonlinear terms into linear matrix inequality conditions is expressed in terms of the cone-complementarity linearization method. Finally, numerical examples are given for the effectiveness and validity of the proposed solutions.

4.1 Problem Statement

Consider a switched linear system with an interval time-varying delay in the state vector, where

$$\dot{x}(t) = A_{\sigma(t)}x(t) + A_{d\sigma(t)}x(t-h(t)) + B_{\sigma(t)}u(t) + B_{w\sigma(t)}w(t), \tag{4.1}$$

with the initial conditon function

$$x(t) = \phi(t),\ t \in [-h_2, 0]. \tag{4.2}$$

Here $x(t) \in \mathbf{R}^n$ is the state vector and $u(t) \in \mathbf{R}^m$ the control input, respectively. $A_{\sigma(t)}$, $A_{d\sigma(t)}$, $B_{\sigma(t)}$ and $B_{w\sigma(t)}$ are real constant matrices of appropriate dimensions, $\phi \in \mathscr{C}([-h_2, 0], \mathbf{R}^n)$ is the initial function and h is the delay function satisfying

$$0 \leq h_1 \leq h(t) \leq h_2, \quad \dot{h}(t) \leq h_d < \infty. \tag{4.3}$$

Unless otherwise stated, the expression “Switched Systems with Stable Subsystems" means that $A_1, A_2, ..., A_N$ are all Hurwitz stable. $w(t)$ is the exogenous disturbance satisfying

$$\int_0^\infty w^T(t)w(t)dt < d,\ d \geq 0 \tag{4.4}$$

Consider the control law

$$u(t) = -K_{\sigma(t)}x(t). \tag{4.5}$$

The closed-loop system is given as follows

$$\dot{x}(t) = A_{K\sigma(t)}x(t) + A_{d\sigma(t)}x(t-h(t)) + B_{w\sigma(t)}w(t), \tag{4.6}$$

where $A_{K\sigma(t)} = A_{\sigma(t)} - B_{\sigma(t)}K_{\sigma(t)}$.

4.2 FT Boundedness Analysis

In this section, we suppose that A_1, A_2,..., A_r, $(1 \leq r < N)$ in system (4.1) are Hurwitz stable and the remaining matrices are unstable. Let us define

$$\psi_i = \begin{cases} -\alpha_i & i \in \mathscr{I}_{st} \\ \alpha_i & i \in \mathscr{I}_{un} \end{cases}$$

where $\mathscr{I}_{st}$ and $\mathscr{I}_{un}$ are the index set of all Hurwitz stable and unstable subsystems, respectively. Note that $\mathscr{I} = \mathscr{I}_{st} \cup \mathscr{I}_{un}$. For a given switching sequence Σ, the total activation times of stable and unstable subsystems are defined as T^- and T^+, respectively in a finite interval $[0, T_f]$. Thus, $T_f = T^+ + T^-$.

Theorem 6. *Consider the switched system* (4.1) *with r Hurwitz stable and $N-r$ unstable subsystems. The system* (4.1) *is FT bounded with respect to* $(\delta, \varepsilon, T_f, d, R)$, *for given constants $\alpha_i \geq 0$, $\mu \geq 1$, $T^+ > 0$ and $T^- > 0$ such that $T_f = T^+ + T^-$, if there exist a set of symmetric matrices for every i^{th} system $P_i > 0$, $Q_{1i} > 0$, $Q_{2i} > 0$, $S_{1i} > 0$, $S_{2i} > 0$, $T_i > 0$, $W_i > 0$, Y_i, M_{1i}, M_{2i}, N_{1i}, N_{2i} satisfying*

$$\Upsilon_i = \begin{bmatrix} \Omega_i & -M_i & -N_i & Z_i \\ * & -e^{2\psi_i h_2}S_{2i} & 0 & 0 \\ * & * & -e^{2\psi_i h_2}S_{2i} & 0 \\ * & * & * & -I \end{bmatrix} < 0 \tag{4.7a}$$

$$e^{2\alpha^+_{max}T^+}\eta'_+ \leq \lambda_1 e^{2\alpha^-_{min}T^-}\varepsilon \tag{4.7b}$$

$$P_j \leq \mu P_i,\; Q_{kj} \leq \mu Q_{ki}, S_{kj} \leq \mu S_{ki}, T_j \leq \mu T_i, \tag{4.7c}$$

for $i, j \in \mathscr{I}$ and $k = 1, 2$, where

$$\Omega_i = \begin{bmatrix} \Omega_{11,i} & \Omega_{12,i} & \Omega_{13,i} & -N_{1i} & B_{wi} & \Omega_{16,i} \\ * & \Omega_{22,i} & M_{2i} & -N_{2i} & 0 & P_iA_{di}^T \\ * & * & \Omega_{33,i} & 0 & 0 & 0 \\ * & * & * & \Omega_{44,i} & 0 & 0 \\ * & * & * & * & -W_i & B_{wi}^T \\ * & * & * & * & * & \Omega_{66,i} \end{bmatrix} \tag{4.8}$$

with entries

$$\Omega_{11,i} = A_iP_i + P_iA_i^T - B_iY_i - Y_i^TB_i^T + Q_{1i} + Q_{2i} - e^{2\psi_i h_1}S_{1i} - 2\psi_iP_i + T_i,$$

$$\Omega_{12,i} = A_{di}P_i - M_{1i} + N_{1i},\ \Omega_{13,i} = e^{2\psi_i h_1}S_{1i} + M_{1i},\ \Omega_{16,i} = P_iA_i^T - Y_i^TB_i^T,$$

$$\Omega_{22,i} = N_{2i} + N_{2i}^T - M_{2i} - M_{2i}^T - (1-h_d)e^{2\psi_i h_2}T_i$$

$$\Omega_{33,i} = -e^{2\psi_i h_1}(Q_{1i} + S_{1i}),\ \Omega_{44,i} = -e^{2\psi_i h_2}Q_{2i},\ \Omega_{66,i} = h_1^2S_{1i} + h_{12}^2S_{2i} - 2P_i.$$

Then the ADT of the switching signal satisfies

$$\tau_a > \tau_a^* = \frac{T_f \ln\mu}{\ln(\lambda_1\varepsilon) - \ln\eta'_+ - 2\alpha_{max}^+T^+ + 2\alpha_{min}^-T^- - N_0\ln\mu} \tag{4.9}$$

where $\alpha_{max}^+ = \max_{i\in\mathscr{I}_{un}}\{\alpha_i\}$, $\alpha_{min}^- = \min_{i\in\mathscr{I}_{st}}\{\alpha_i\}$ *and*

$$\begin{aligned}\eta'_+ =& \lambda_2\delta + \lambda'_3h_1e^{2\alpha_{max}^+h_1}\delta + \lambda'_4h_2e^{2\alpha_{max}^+h_2}\delta + \lambda'_5h_1^3e^{2\alpha_{max}^+h_1}\delta' \\ &+ \lambda'_6h_{12}^2(h_1e^{2\alpha_{max}^+h_1} + h_{12}e^{2\alpha_{max}^+h_2})\delta' + \lambda'_7h_2e^{2\alpha_{max}^+h_2}\delta + \lambda_8d.\end{aligned} \tag{4.10}$$

with matrix transformations

$$\begin{aligned}\hat{Q}_{1i} =& R^{1/2}Q_{1i}R^{1/2},\ \hat{Q}_{2i} = R^{1/2}Q_{2i}R^{1/2},\\ \hat{S}_{1i} =& R^{1/2}S_{1i}R^{1/2},\ \hat{S}_{2i} = R^{1/2}S_{2i}R^{1/2},\\ Q_{1i} =& P_i\overline{Q}_{1i}P_i,\ Q_{2i} = P_i\overline{Q}_{2i}P_i,\ S_{1i} = P_i\overline{S}_{1i}P_i,\ S_{2i} = P_i\overline{S}_{2i}P_i,\\ \hat{T}_i =& R^{1/2}T_iR^{1/2},\ T_i = P_i\overline{T}_iP_i,\\ M_{1i} =& P_i\overline{M}_{1i}P_i,\ M_{2i} = P_i\overline{M}_{2i}P_i,\ N_{1i} = P_i\overline{N}_{1i}P_i,\ N_{2i} = P_i\overline{N}_{2i}P_i\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\lambda_1 =& \inf_{i\in\mathscr{I}}\{\lambda_{min}(\tilde{P}_i^{-1})\},\ \lambda_2 = \sup_{i\in\mathscr{I}}\{\lambda_{max}(\tilde{P}_i^{-1})\},\\ \lambda'_3 =& \sup_{i\in\mathscr{I}}\{\lambda_{max}(\tilde{P}_i^{-1}\hat{Q}_{1i}\tilde{P}_i^{-1})\},\ \lambda'_4 = \sup_{i\in\mathscr{I}}\{\lambda_{max}(\tilde{P}_i^{-1}\hat{Q}_{2i}\tilde{P}_i^{-1})\},\\ \lambda'_5 =& \sup_{i\in\mathscr{I}}\{\lambda_{max}(\tilde{P}_i^{-1}\hat{S}_{1i}\tilde{P}_i^{-1})\},\ \lambda'_6 = \sup_{i\in\mathscr{I}}\{\lambda_{max}(\tilde{P}_i^{-1}\hat{S}_{2i}\tilde{P}_i^{-1})\},\\ \lambda'_7 =& \sup_{i\in\mathscr{I}}\{\lambda_{max}(\tilde{P}_i^{-1}\hat{T}_i\tilde{P}_i^{-1})\},\ \lambda_8 = \sup_{i\in\mathscr{I}}\{\lambda_{max}(W_i)\},\\ \delta' =& \sup_{s\in[-h_2,0]}\left\{\dot{x}^T(s)R\dot{x}(s)\right\},\ h_{12} = h_2 - h_1,\\ Z_i =& [0\ 0\ 0\ 0\ C_iP_i\ 0\ 0]^T\ M_i = [M_{1i}\ M_{2i}\ 0\ 0\ 0\ 0\ 0]^T,\ N_i = [N_{1i}\ N_{2i}\ 0\ 0\ 0\ 0\ 0]^T.\end{aligned}$$

The gain matrices K_i *of controller are perceived as*

$$K_i = Y_iP_i^{-1}. \tag{4.12}$$

Proof. Consider the following Lyapunov-Krasovskii candidate functional as

$$V_i(x(t)) = \sum_{j=1}^{6} V_{ji}(x(t)) \tag{4.13}$$

where

$$\begin{aligned}
V_{1i}(x(t)) =& x^T(t)P_i^{-1}x(t) \\
V_{2i}(x(t)) =& \int_{t-h_1}^{t} e^{2\psi_i(t-s)}x^T(s)\overline{Q}_{1i}x(s)ds \\
V_{3i}(x(t)) =& \int_{t-h_2}^{t} e^{2\psi_i(t-s)}x^T(s)\overline{Q}_{2i}x(s)ds \\
V_{4i}(x(t)) =& \int_{-h_1}^{0}\int_{t+\theta}^{t} h_1 e^{2\psi_i(t-s)}\dot{x}^T(s)\overline{S}_{1i}\dot{x}(s)dsd\theta \\
V_{5i}(x(t)) =& \int_{-h_2}^{-h_1}\int_{t+\theta}^{t} h_{12} e^{2\psi_i(t-s)}\dot{x}^T(s)\overline{S}_{2i}\dot{x}(s)dsd\theta \\
V_{6i}(x(t)) =& \int_{t-h(t)}^{t} e^{2\psi_i(t-s)}x^T(s)\overline{T}_i x(s)ds
\end{aligned} \tag{4.14}$$

The derivatives are obtained as follows

$$\begin{aligned}
\dot{V}_{1i}(x(t)) =& x^T(t)[P_i^{-1}A_{Ki} + A_{Ki}^T P_i^{-1}]x(t) \\
&+ 2x^T(t)P_i^{-1}A_{di}x(t-h(t)) + 2x^T(t)P_i^{-1}B_{wi}w(t) \\
\dot{V}_{2i}(x(t)) =& 2\psi_i V_{2i} + x^T(t)\overline{Q}_{1i}x(t) - e^{2\psi_i h_1}x^T(t-h_1)\overline{Q}_{1i}x(t-h_1) \\
\dot{V}_{3i}(x(t)) =& 2\psi_i V_{3i} + x^T(t)\overline{Q}_{2i}x(t) - e^{2\psi_i h_2}x^T(t-h_2)\overline{Q}_{2i}x(t-h_2) \\
\dot{V}_{4i}(x(t)) =& 2\psi_i V_{4i} + h_1^2\dot{x}^T(t)\overline{S}_{1i}\dot{x}(t) - e^{2\psi_i h_1}\int_{t-h_1}^{t} h_1\dot{x}^T(s)\overline{S}_{1i}\dot{x}(s)ds \\
\dot{V}_{5i}(x(t)) \leq& 2\psi_i V_{5i} + h_{12}^2\dot{x}^T(t)\overline{S}_{2i}\dot{x}(t) - e^{2\psi_i h_2}\int_{t-h_2}^{t-h_1} h_{12}\dot{x}^T(s)\overline{S}_{2i}\dot{x}(s)ds \\
\dot{V}_{6i}(x(t)) \leq& 2\psi_i V_{6i} + x^T(t)\overline{T}_i x(t) - (1-h_d)e^{2\psi_i h_2}x^T(t-h(t))\overline{T}_i x(t-h(t))
\end{aligned} \tag{4.15}$$

By Jensen's Inequality, $\dot{V}_{4i}(x(t))$ can be written as

$$\begin{aligned}
\dot{V}_{4i}(x(t)) \leq& 2\psi_i V_{4i}(x(t)) + h_1^2\dot{x}^T(t)\overline{S}_{1i}\dot{x}(t) - e^{2\psi_i h_1}x^T(t)\overline{S}_{1i}x(t) \\
&+ 2e^{2\psi_i h_1}x^T(t)\overline{S}_{1i}x(t-h_1) - e^{2\psi_i h_1}x^T(t-h_1)\overline{S}_{1i}x(t-h_1)
\end{aligned} \tag{4.16}$$

From (4.3), it is clear that $-(h_2-h_1) \leq -(h_2-h(t))$ and $-(h_2-h_1) \leq -(h(t)-h_1)$. Thus

$$\begin{aligned}
-h_{12}\int_{t-h_2}^{t-h_1}\dot{x}^T(s)\overline{S}_{2i}\dot{x}(s)ds \leq& -(h_2-h(t))\int_{t-h_2}^{t-h(t)}\dot{x}^T(s)\overline{S}_{2i}\dot{x}(s)ds \\
&-(h(t)-h_1)\int_{t-h(t)}^{t-h_1}\dot{x}^T(s)\overline{S}_{2i}\dot{x}(s)ds
\end{aligned} \tag{4.17}$$

Let $\int_{t-h(t)}^{t-h_1}\dot{x}(s)ds =: i_{h_1}(t)$ and $\int_{t-h_2}^{t-h(t)}\dot{x}(s)ds =: i_{h_2}(t)$. Then, by Jensen's Inequality, (4.17) is written as follows

$$-h_{12}\int_{t-h_2}^{t-h_1}\dot{x}^T(s)\overline{S}_{2i}\dot{x}(s)ds \leq -i_{h_2}^T(t)\overline{S}_{2i}i_{h_2}(t) - i_{h_1}^T(t)\overline{S}_{2i}i_{h_1}(t) \tag{4.18}$$

Now, define

$$\xi(t) = \left[x^T(t)\; x^T(t-h(t))\; x^T(t-h_1)\; x^T(t-h_2)\;\; w^T(t)\; \dot{x}^T(t)\; i_{h_1}^T(t)\; i_{h_2}^T(t)\right]^T. \quad (4.19)$$

By Leibniz's formula, we have

$$\begin{aligned} 2\xi^T(t)\mathscr{M}_i\Big[x(t-h_1)-x(t-h(t))-i_{h_1}(t)\Big] &= 0 \\ 2\xi^T(t)\mathscr{N}_i\Big[x(t-h(t))-x(t-h_2)-i_{h_2}(t)\Big] &= 0 \end{aligned} \quad (4.20)$$

Also from (4.6), it can be written

$$2\dot{x}^T(t)P_i^{-1}\Big[A_{Ki}x(t)+A_{di}x(t-h(t))+B_{wi}w(t)-\dot{x}(t)\Big] = 0 \quad (4.21)$$

On the other hand, for a positive definite matrix W_i the following holds

$$\left[w^T(t)W_iw(t)-w^T(t)W_iw(t)\right] = 0 \quad (4.22)$$

Then, by the equations (4.13)-(4.22), we obtain

$$\dot{V}_i(x(t))-2\psi_iV_i(x(t)) \leq \xi^T(t)\Sigma_i\xi(t)+w^T(t)W_iw(t). \quad (4.23)$$

Here

$$\Sigma_i = \begin{bmatrix} \Xi_i & -\mathscr{M}_i & -\mathscr{N}_i \\ * & -e^{2\psi_ih_2}\overline{S}_{2i} & 0 \\ * & * & -e^{2\psi_ih_2}\overline{S}_{2i} \end{bmatrix} \quad (4.24)$$

for $i,j\in\mathscr{I}$ and $k=1,2$, where

$$\Xi_i = \begin{bmatrix} \Xi_{11,i} & \Xi_{12,i} & \Xi_{13,i} & -\overline{N}_{1i} & P_i^{-1}B_{wi} & A_{Ki}^TP_i^{-1} \\ * & \Xi_{22,i} & \overline{M}_{2i} & -\overline{N}_{2i} & 0 & A_{di}^TP_i^{-1} \\ * & * & \Xi_{33,i} & 0 & 0 & 0 \\ * & * & * & \Xi_{44,i} & 0 & 0 \\ * & * & * & * & -W_i & B_{wi}^TP_i^{-1} \\ * & * & * & * & * & \Xi_{66,i} \end{bmatrix} \quad (4.25)$$

with entries

$$\begin{aligned} \Xi_{11,i} =& P_i^{-1}A_{Ki}+A_{Ki}^TP_i^{-1}+\overline{Q}_{1i}+\overline{Q}_{2i}-e^{2\psi_ih_1}\overline{S}_{1i}-2\psi_iP_i^{-1}+\overline{T}_i, \\ \Xi_{12,i} =& P_i^{-1}A_{di}-\overline{M}_{1i}+\overline{N}_{1i},\ \Xi_{13,i}=e^{2\psi_ih_1}\overline{S}_{1i}+\overline{M}_{1i}, \\ \Xi_{22,i} =& \overline{N}_{2i}+\overline{N}_{2i}^T-\overline{M}_{2i}-\overline{M}_{2i}^T-(1-h_d)e^{2\psi_ih_2}\overline{T}_i, \\ \Xi_{33,i} =& -e^{2\psi_ih_1}(\overline{Q}_{1i}+\overline{S}_{1i}),\ \Xi_{44,i}=-e^{2\psi_ih_2}\overline{Q}_{2i},\ \Xi_{66,i}=h_1^2\overline{S}_{1i}+h_{12}^2\overline{S}_{2i}-2P_i^{-1} \\ \mathscr{M}_i =& [\overline{M}_{1i}^T\ \overline{M}_{2i}^T\ 0\ 0\ 0\ 0\ 0]^T,\ \mathscr{N}_i=[\overline{N}_{1i}^T\ \overline{N}_{2i}^T\ 0\ 0\ 0\ 0\ 0]^T \end{aligned}$$

By pre- and post-multiplying both sides of the Inequalities in (4.24) with (4.25) by $\mathscr{D}_i = diag\{P_i, P_i, P_i, P_i, I, P_i, P_i, P_i\}$, Υ_i of (4.7a) are obtained. From (4.7a)

$$\dot{V}_i(x(t)) - 2\psi_i V_i(x(t)) \leq w^T(t) W_i w(t) \tag{4.26}$$

is obtained.

On the other hand, by applying Grönwall's Lemma on $t \in [t_k, t_{k+1})$ we have

$$V_{\sigma(t)}(x(t)) \leq e^{2\psi_{\sigma(t_k)}(t-t_k)} V_{\sigma(t_k)}(x(t_k)) + \int_{t_k}^{t} e^{2\psi_{\sigma(t_k)}(t-s)} w^T(s) W_{\sigma(t_k)} w(s) ds. \tag{4.27}$$

Consider (4.7c) and assume $\sigma(t_k) = i$ and $\sigma(t_k^-) = j$, we have

$$V_{\sigma(t_k)}(x(t_k)) \leq \mu V_{\sigma(t_k^-)}(x(t_k^-)) \tag{4.28}$$

If Grönwall's Lemma and (4.28) is applied to (4.26) until $[0, t_1)$ iteratively, we get

$$\begin{aligned} V_{\sigma(t)}(x(t)) \leq & e^{2\psi_{\sigma(t_k)}(t-t_k)+\ldots+2\psi_{\sigma(0)}(t_1-0)} \mu^N V_{\sigma(0)}(x(0)) \\ & + \mu^N \int_0^{t_1} e^{2\psi_{\sigma(t_k)}(t-t_k)+\ldots+2\psi_{\sigma(0)}(t_1-s)} w^T(s) W_{\sigma(0)} w(s) ds \\ & + \ldots \\ & + \int_{t_k}^{t} e^{2\psi_{\sigma(t_k)}(t-s)} w^T(s) W_{\sigma(t_k)} w(s) ds \end{aligned} \tag{4.29}$$

By considering the activation times T^- and T^+ for stable and unstable subsystems, respectively, the inequality (4.29) can be written as follows:

$$V_{\sigma(t)}(x(t)) \leq e^{2\alpha_{max}^+ T^+ - 2\alpha_{min}^- T^-} \mu^N \big(V_{\sigma(0)}(x(0)) + \lambda_8 d\big). \tag{4.30}$$

where N denotes the switching number of $\sigma(t)$ over $(0, T_f)$. Moreover,

$$\begin{aligned} V_{\sigma(t)}(x(0)) = & x^T(0) P_{\sigma(0)}^{-1} x(0) \\ & + \int_{-h_1}^{0} e^{-2\psi_{\sigma(0)} s} x^T(s) \overline{Q}_{1\sigma(0)} x(s) ds \\ & + \int_{-h_2}^{0} e^{-2\psi_{\sigma(0)} s} x^T(s) \overline{Q}_{2\sigma(0)} x(s) ds \\ & + \int_{-h_1}^{0} \int_{\theta}^{0} h_1 e^{-2\psi_{\sigma(0)} s} \dot{x}^T(s) \overline{S}_{1\sigma(0)} \dot{x}(s) ds d\theta \\ & + \int_{-h_2}^{-h_1} \int_{\theta}^{0} h_{12} e^{-2\psi_{\sigma(0)} s} \dot{x}^T(s) \overline{S}_{2\sigma(0)} \dot{x}(s) ds d\theta \\ & + \int_{-h(0)}^{0} e^{-2\psi_{\sigma(0)} s} x^T(s) \overline{T}_{\sigma(0)} x(s) ds. \end{aligned} \tag{4.31}$$

When the orders of the double integrals are changed and the matrices in (4.11) are substituted, we have

$$\begin{aligned}
V_{\sigma(0)}(x(0)) =& x^T(0)P_{\sigma(0)}^{-1}x(0) \\
&+\int_{-h_1}^{0} e^{-2\psi_{\sigma(0)}s}x^T(s)P_{\sigma(0)}^{-1}Q_{1\sigma(0)}P_{\sigma(0)}^{-1}x(s)ds \\
&+\int_{-h_2}^{0} e^{-2\psi_{\sigma(0)}s}x^T(s)P_{\sigma(0)}^{-1}Q_{2\sigma(0)}P_{\sigma(0)}^{-1}x(s)ds \\
&+\int_{-h_1}^{0}\int_{-h_1}^{s} h_1 e^{-2\psi_{\sigma(0)}s}\dot{x}^T(s)P_{\sigma(0)}^{-1}S_{1\sigma(0)}P_{\sigma(0)}^{-1}\dot{x}(s)d\theta ds \\
&+\int_{-h_2}^{-h_1}\int_{-h_2}^{s} h_{12} e^{-2\psi_{\sigma(0)}s}\dot{x}^T(s)P_{\sigma(0)}^{-1}S_{2\sigma(0)}P_{\sigma(0)}^{-1}\dot{x}(s)d\theta ds \\
&+\int_{-h_1}^{0}\int_{-h_2}^{-h_1} h_{12} e^{-2\psi_{\sigma(0)}s}\dot{x}^T(s)P_{\sigma(0)}^{-1}S_{2\sigma(0)}P_{\sigma(0)}^{-1}\dot{x}(s)d\theta ds \\
&+\int_{-h(0)}^{0} e^{-2\psi_{\sigma(0)}s}x^T(s)P_{\sigma(0)}^{-1}T_{\sigma(0)}P_{\sigma(0)}^{-1}x(s)ds.
\end{aligned}$$

From (4.11), each matrix can be bounded as

$$P_{\sigma(0)}^{-1}Q_{1\sigma(0)}P_{\sigma(0)}^{-1} = R^{1/2}\tilde{P}_{\sigma(0)}^{-1}\hat{Q}_{1\sigma(0)}\tilde{P}_{\sigma(0)}^{-1}R^{1/2} \leq \lambda_{max}(\tilde{P}_{\sigma(0)}^{-1}\hat{Q}_{1\sigma(0)}\tilde{P}_{\sigma(0)}^{-1})R \leq \lambda_3' R.$$

Also, note that

$$\begin{aligned}
\sup_{s\in[-h(0),0]}\{e^{-2\psi_{\sigma(0)}s}\} \leq \sup_{s\in[-h_1,0]}\{e^{-2\psi_{\sigma(0)}s}\} &= e^{2\alpha_{max}^{+}h_1}, \\
\sup_{s\in[-h_2,0]}\{e^{-2\psi_{\sigma(0)}s}\} &= e^{2\alpha_{max}^{+}h_2}
\end{aligned} \tag{4.32}$$

Here, an upper bound for $V_{\sigma(0)}(0)$ can be written as follows

$$\begin{aligned}
V_{\sigma(0)}(x(0)) \leq& \lambda_2\delta + \lambda_3' h_1 e^{2\alpha_{max}^{+}h_1}\delta + \lambda_4' h_2 e^{2\alpha_{max}^{+}h_2}\delta + \lambda_5' h_1^3 e^{2\alpha_{max}^{+}h_1}\delta' \\
&+ \lambda_6' h_{12}^2(h_1 e^{2\alpha_{max}^{+}h_1} + h_{12}e^{2\alpha_{max}^{+}h_2})\delta' + \lambda_7' h_2 e^{2\alpha_{max}^{+}h_2}\delta.
\end{aligned} \tag{4.33}$$

Since,

$$\begin{aligned}
V_{\sigma(t)}(x(t)) \geq x^T(t)P_i^{-1}x(t) &= x^T(t)R^{1/2}\tilde{P}_i^{-1}R^{1/2}x(t) \\
&\geq \inf_{i\in I}\left(\lambda_{min}(\tilde{P}_i^{-1})\right)x^T(t)Rx(t) \\
&= \lambda_1 x^T(t)Rx(t).
\end{aligned} \tag{4.34}$$

By the equations (4.30), (4.33) and (4.34) the inequality $x^T(t)Rx(t) < \varepsilon$ is obtained, which tells that the switched system (4.1) is FT bounded. Then, for $\mu = 1$ the inequality in (4.7b) and for $\mu > 1$ the ADT bound in (4.9) are calculated. □

Remark 4. *Note that, Theorem 6 has* $A_iP_i + P_iA_i^T - 2\psi_iP_i - B_iY_i - Y_i^TB_i^T$ *in* (4.7a) *with* (4.8). *By adjusting* $(A_i - \psi_iI)$ *by* ψ_i *each pair of* (A_i, B_i) *do not have to* *be controllable on each mode. For example, consider the simple system* $\dot{x}(t) = Ax(t) + bu(t)$, *take* $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *and use the state feedback controller* $u(t) = kx(t) = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T$. *When the state feedback controller is applied the* $A + bk = \begin{bmatrix} -1 + k_1 & k_2 \\ 0 & 1 \end{bmatrix}$, *the unstable mode remains unstable. However* ψ *enables us to stabilize* $A - \psi I + bk = \begin{bmatrix} -1 - \psi + k_1 & k_2 \\ 0 & 1 - \psi \end{bmatrix}$.

Remark 5. *Note also that the condition* (4.7b) *contains the constants* λ_1, λ_2, λ_3', λ_4', λ_5',λ_6', λ_7' *and* λ_8. *The existance of these constants depends on the solutions of the following inequalities*

$$\begin{aligned} &\lambda_1 I < \tilde{P}_i^{-1} < \lambda_2 I \\ &0 < \tilde{P}_i^{-1} \hat{Q}_{1i} \tilde{P}_i^{-1} < \lambda_3' I,\ 0 < \tilde{P}_i^{-1} \hat{Q}_{2i} \tilde{P}_i^{-1} < \lambda_4' I \\ &0 < \tilde{P}_i^{-1} \hat{S}_{1i} \tilde{P}_i^{-1} < \lambda_5' I,\ 0 < \tilde{P}_i^{-1} \hat{S}_{2i} \tilde{P}_i^{-1} < \lambda_6' I, \\ &0 < \tilde{P}_i^{-1} \hat{T}_i \tilde{P}_i^{-1} < \lambda_7' I,\ \ 0 < W_i < \lambda_8 I. \end{aligned} \tag{4.35}$$

For more details see [47].

To solve the inequalities in (4.35), *it is necessary to put them into LMIs form. Thus, consider* $0 < \tilde{P}_i^{-1} \hat{Q}_{1i} \tilde{P}_i^{-1} < \lambda_3' I$, *write it as* $-\lambda_3' I + \tilde{P}_i^{-1} \hat{Q}_{1i} \tilde{P}_i^{-1} < 0$ *and use Schur Complement*

$$\begin{bmatrix} -\lambda_3' I & \tilde{P}_i^{-1} \\ * & -\hat{Q}_{1i}^{-1} \end{bmatrix} \leq 0 \iff \begin{bmatrix} -\lambda_3' I & J_i \\ * & -E_{1i} \end{bmatrix} \leq 0 \tag{4.36}$$

where $J_i := \tilde{P}_i^{-1}$ *and* $E_{1i} := \hat{Q}_{1i}^{-1}$ *(or equivalently* $J_i \tilde{P}_i = I$ *and* $E_{1i} \hat{Q}_{1i} = I$*). By applying same procedure to the other nonlinear inequalities from* (4.35) *and defining the matrices* E_{2i}, F_{1i}, F_{2i} *and* G_i *for the matrix inverse approximates of* $\hat{Q}_{2i}$, $\hat{S}_{1i}$, $\hat{S}_{2i}$ *and* $\hat{T}_i$, *the following inequalities can be stated in terms of cone-complementarity algorithm given in [48].*

$$\begin{aligned} &\lambda_1 I < J_i < \lambda_2 I,\ 0 \leq \begin{bmatrix} \tilde{P}_i & I \\ * & J_i \end{bmatrix}, \\ &\begin{bmatrix} -\lambda_3' I & J_i \\ * & -E_{1i} \end{bmatrix} \leq 0, 0 \leq \begin{bmatrix} \hat{Q}_{1i} & I \\ * & E_{1i} \end{bmatrix}, \begin{bmatrix} -\lambda_4' I & J_i \\ * & -E_{2i} \end{bmatrix} \leq 0, 0 \leq \begin{bmatrix} \hat{Q}_{2i} & I \\ * & E_{2i} \end{bmatrix}, \\ &\begin{bmatrix} -\lambda_5' I & J_i \\ * & -F_{1i} \end{bmatrix} \leq 0, 0 \leq \begin{bmatrix} \hat{S}_{1i} & I \\ * & F_{1i} \end{bmatrix}, \begin{bmatrix} -\lambda_6' I & J_i \\ * & -F_{2i} \end{bmatrix} \leq 0, 0 \leq \begin{bmatrix} \hat{S}_{2i} & I \\ * & F_{2i} \end{bmatrix}, \\ &\begin{bmatrix} -\lambda_7' I & J_i \\ * & -G_i \end{bmatrix} \leq 0, 0 \leq \begin{bmatrix} \hat{T}_i & I \\ * & G_i \end{bmatrix}, 0 < W_i < \lambda_8 I, \end{aligned} \tag{4.37}$$

Algorithm 1. *This algorithm is derived for Theorem 6.*

- ***Step 1:*** *Find a feasible set*

$$(P_i^0, Q_{1i}^0, Q_{2i}^0, S_{1i}^0, S_{2i}^0, T_i^0, J_i^0, E_{1i}^0, E_{2i}^0, F_{1i}^0, F_{2i}^0, G_i^0, W_i^0, T_i^0, M_{1i}^0, M_{2i}^0, N_{1i}^0, N_{2i}^0)$$

satisfying the inequalities in (4.7a), (4.7b), (4.7c) *and* (4.37). *Set* $k = 0$.

- ***Step 2:*** *Solve the following LMI problem for the variables*

$$(P_i, Q_{1i}, Q_{2i}, S_{1i}, S_{2i}, T_i, J_i, E_{1i}, E_{2i}, F_{1i}, F_{2i}, G_i, W_i, T_i, M_{1i}, M_{2i}, N_{1i}, N_{2i})$$

according to the following minimization problem

$$\begin{aligned} \textit{minimize } tr\Bigg(\sum_{i\in\mathscr{I}} & J_i^k \tilde{P}_i + J_i \tilde{P}_i^k + E_{1i}^k \hat{Q}_{1i} + E_{1i}\hat{Q}_{1i}^k + E_{2i}^k \hat{Q}_{2i} + E_{2i}\hat{Q}_{2i}^k \\ & + F_{1i}^k \hat{S}_{1i} + F_{1i}\hat{S}_{1i}^k + F_{2i}^k \hat{S}_{2i} + F_{2i}\hat{S}_{2i}^k + G_i^k \hat{T}_i + G_i \hat{T}_i^k \Bigg) \\ \textit{subject to } & (4.7a),\ (4.7b),\ (4.7c) \textit{ and } (4.37) \end{aligned}$$

- ***Step 3:*** *If a stopping criteria is satisfied, then exit. Otherwise, set*

$$\tilde{P}_i^k = \tilde{P}_i, \hat{Q}_{1i}^k = \hat{Q}_{1i}, \hat{Q}_{2i}^k = \hat{Q}_{2i}, \hat{S}_{1i}^k = \hat{S}_{1i}, \hat{S}_{2i}^k = \hat{S}_{2i},$$

$$\hat{T}_i^k = \hat{T}_i, J_i^k = J_i, E_{1i}^k = E_{1i}, E_{2i}^k = E_{2i}, F_{1i}^k = F_{1i}, F_{2i}^k = F_{2i}, G_i^k = G_i$$

and set $k = k+1$ *and go to Step 2.*

4.3 Numerical Example

A numerical example is presented in order to show the effect of the Algorithm 1.

Example 10. *Consider the switched system with time delay* (4.1) *with two subsystems*

$$A_1 = \begin{bmatrix} 0.4 & 0 \\ 0 & -0.34 \end{bmatrix}, A_2 = \begin{bmatrix} -1.6 & 0 \\ 0 & -0.14 \end{bmatrix},$$
$$A_{d1} = \begin{bmatrix} -0.06 & 0 \\ 0.06 & -0.03 \end{bmatrix}, A_{d2} = \begin{bmatrix} -0.03 & 0 \\ -0.69 & -0.12 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0.4 \\ 0.1 \end{bmatrix}, B_2 = \begin{bmatrix} 0.3 \\ 0.15 \end{bmatrix}, B_{w1} = \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix}, B_{w2} = \begin{bmatrix} 0.15 \\ 0.3 \end{bmatrix}.$$

Note that, A_1 *is Hurwitz unstable and* A_2 *is Hurwitz stable. The activation times of the unstable and unstable subsystems are chosen as* $T^+ = 0.6$ *and* $T^- = 1.4$,, *respectively. The constants*

$$\psi_1 = 0.5,\ \psi_2 = -0.05,\ h_1 = 0,\ h_2 = 0.1,\ h_d = 0.01,$$

$$R = I,\ \delta = 4,\ \delta' = 4,\ \varepsilon = 25,\ \mu = 1.01,\ d = 0.01,$$

$$T_f = 2,\ N_0 = 0.$$

are chosen and by Algorithm 1, we get a feasible solution with controller gains

$$K_1 = \begin{bmatrix} 1850.6 & 388.3 \end{bmatrix},\ K_2 = \begin{bmatrix} -662.5 & 1760.7 \end{bmatrix}$$

with the ADT $\tau_a^* = 0.2180$.

5. OBSERVER-BASED CONTROL FOR FINITE-TIME BOUNDEDNESS

In this part, interval time-delay switched systems having completely unstable and mixed stable matrices of the state vector are considered. Observer-based controller is designed for finite-time boundedness of these systems. New sufficient conditions on the existence of desired observer are developed and new average dwell-time bounds are introduced seperately in case of unstable and mixed stable subsystems. An algorithm is presented for the calculation of unknown constants in the average dwell-time bounds which depend on nonlinear matrices in terms of cone complementarity linearization method. Finally, numerical examples are given for the effectiveness and validity of the proposed solutions.

5.1 Problem Statement

Consider a switched linear system with an interval time-varying delay in the state vector, where

$$\begin{aligned} \dot{x}(t) =& A_{\sigma(t)}x(t) + A_{d\sigma(t)}x(t-h(t)) + B_{\sigma(t)}u(t) + B_{w\sigma(t)}w(t), \\ y(t) =& C_{\sigma(t)}x(t) \end{aligned} \tag{5.1}$$

with the initial conditon function

$$x(t) = \phi(t),\ t \in [-h_2, 0]. \tag{5.2}$$

Here $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^m$ the control input and $y(t) \in \mathbf{R}^q$ the measurement output. $A_{\sigma(t)}$, $A_{d\sigma(t)}$, $B_{\sigma(t)}$, $B_{w\sigma(t)}$ and $C_{\sigma(t)}$ are real constant matrices of appropriate dimensions, $\phi \in \mathscr{C}([-h_2, 0], \mathbf{R})$ is the initial function, h is the delay function satisfying (4.3) and $w(t)$ is the exogenous disturbance satisfying (4.4). Consider the observer based feedback controller

$$\begin{aligned} \dot{\hat{x}}(t) &= A_{\sigma(t)}\hat{x}(t) + B_{\sigma(t)}u(t) + L_{\sigma(t)}(y(t) - \hat{y}(t)), \\ \hat{y}(t) &= C_{\sigma(t)}\hat{x}(t), \\ \hat{x}(t) &= 0,\ \forall t \in [-h_2, 0], \end{aligned} \tag{5.3}$$

and the control law

$$u(t) = -K_{\sigma(t)}\hat{x}(t). \tag{5.4}$$

Here, $K_{\sigma(t)}$ and $L_{\sigma(t)}$ are controller and observer gains, respectively. Define an error vector $e(t) = x(t) - \hat{x}(t)$. The closed-loop system will be

$$\begin{aligned} \dot{x}(t) =& A_{K\sigma(t)}x(t) + A_{d\sigma(t)}x(t-h(t)) + B_{\sigma(t)}K_{\sigma(t)}e(t) + B_{w\sigma(t)}w(t), \\ \dot{e}(t) =& A_{L\sigma(t)}e(t) + A_{d\sigma(t)}x(t-h(t)) + B_{w\sigma(t)}w(t) \end{aligned} \tag{5.5}$$

where

$$A_{K\sigma(t)} = A_{\sigma(t)} - B_{\sigma(t)}K_{\sigma(t)},$$
$$A_{L\sigma(t)} = A_{\sigma(t)} - L_{\sigma(t)}C_{\sigma(t)}.$$

5.2 FT Boundedness Analysis

In this section, first FT boundedness of the closed-loop switched interval time-delay system is analysed. ADT is calculated for switched system with completely unstable and also mixed stable subsystems. Full-order observer is designed that to guarantee the FT boundedness of these systems.

Lemma 5. *The closed-loop switched system* (5.5) *is FT bounded with respect to* $(\delta, \varepsilon, T_f, d, R)$, *if there exist a set of symmetric matrices for every* i^{th} *subsystem* $P_i^{-1} > 0$, $\overline{Q}_{1i} > 0$, $\overline{Q}_{2i} > 0$, $\overline{S}_{1i} > 0$, $\overline{S}_{2i} > 0$, $\overline{T}_i > 0$, $W_i > 0$, $\overline{M}_{1i}$, $\overline{M}_{2i}$, $\overline{N}_{1i}$, $\overline{N}_{2i}$ *and scalars* $\alpha_i \geq 0$ *and* $\mu \geq 1$ *satisfying*

$$\Sigma_i = \begin{bmatrix} \Xi_i & -\mathscr{M}_i & -\mathscr{N}_i \\ * & -e^{2\alpha_i h_2}\overline{S}_{2i} & 0 \\ * & * & -e^{2\alpha_i h_2}\overline{S}_{2i} \end{bmatrix} < 0, \tag{5.6a}$$

$$\mu^N e^{2\alpha_{max}T_f}\eta < \lambda_1 \varepsilon \tag{5.6b}$$

$$P_i^{-1} \leq \mu P_j^{-1},\ \overline{Q}_{ki} \leq \mu \overline{Q}_{kj},\ \overline{S}_{ki} \leq \mu \overline{S}_{kj},\ \overline{T}_i \leq \mu \overline{T}_j, \tag{5.6c}$$

for $i, j \in \mathscr{I}$ *and* $k = 1,2$, *where*

$$\Xi_i = \begin{bmatrix} \Xi_{11,i} & \Xi_{12,i} & \Xi_{13,i} & -\overline{N}_{1i} & \Xi_{15,i} & P_i^{-1}B_{wi} & A_{Ki}^T P_i^{-1} \\ * & \Xi_{22,i} & \overline{M}_{2i} & -\overline{N}_{2i} & A_{di}^T P_i^{-1} & 0 & A_{di}^T P_i^{-1} \\ * & * & \Xi_{33,i} & 0 & 0 & 0 & 0 \\ * & * & * & \Xi_{44,i} & 0 & 0 & 0 \\ * & * & * & * & \Xi_{55,i} & P_i^{-1}B_{wi} & \Xi_{57,i} \\ * & * & * & * & * & -W_i & B_{wi}^T P_i^{-1} \\ * & * & * & * & * & * & \Xi_{77,i} \end{bmatrix} \tag{5.7}$$

with entries

$$\Xi_{11,i} = P_i^{-1}A_{Ki} + A_{Ki}^T P_i^{-1} + \overline{Q}_{1i} + \overline{Q}_{2i} - e^{2\alpha_i h_1}\overline{S}_{1i} - 2\alpha_i P_i^{-1} + \overline{T}_i,$$
$$\Xi_{12,i} = P_i^{-1}A_{di} - \overline{M}_{1i} + \overline{N}_{1i}, \quad \Xi_{13,i} = e^{2\alpha_i h_1}\overline{S}_{1i} + \overline{M}_{1i}, \quad \Xi_{15,i} = P_i^{-1}B_iK_i,$$

$$\Xi_{22,i} = \overline{N}_{2i} + \overline{N}_{2i}^T - \overline{M}_{2i} - \overline{M}_{2i}^T - (1-h_d)e^{2\alpha_i h_2}\overline{T}_i,$$

$$\Xi_{33,i} = -e^{2\alpha_i h_1}(\overline{Q}_{1i} + \overline{S}_{1i}), \quad \Xi_{44,i} = -e^{2\alpha_i h_2}\overline{Q}_{2i},$$

$$\Xi_{55,i} = P_i^{-1}A_{Li} + A_{Li}^T P_i^{-1} - 2\alpha_i P_i^{-1}, \; \Xi_{57,i} = K_i^T B_i^T P_i^{-1},$$

$$\Xi_{77,i} = h_1^2\overline{S}_{1i} + h_{12}^2\overline{S}_{2i} - 2P_i^{-1}, \quad \mathscr{M}_i = [\overline{M}_{1i}^T \; \overline{M}_{2i}^T \; 0\;0\;0\;0\;0]^T,$$

$$\mathscr{N}_i = [\overline{N}_{1i}^T \; \overline{N}_{2i}^T \; 0\;0\;0\;0\;0]^T, \quad h_{12} = h_2 - h_1, \quad \alpha_{max} = \max\{\alpha_i, i \in \mathscr{I}\}$$

and

$$\eta = 2\lambda_2\delta + \lambda_3 h_1 e^{2\alpha_{max}h_1}\delta + \lambda_4 h_2 e^{2\alpha_{max}h_2}\delta + \lambda_5 h_1^3 e^{2\alpha_{max}h_1}\delta'$$
$$+ \lambda_6 h_{12}^2(h_1 e^{2\alpha_{max}h_1} + h_{12}e^{2\alpha_{max}h_2})\delta' + \lambda_7 h_2 e^{2\alpha_{max}h_2}\delta + \lambda_8 d.$$

with the matrix transformations

$$P_i^{-1} = R^{1/2}\tilde{P}_i^{-1}R^{1/2}, \; \overline{Q}_{1i} = R^{1/2}\tilde{Q}_{1i}R^{1/2}, \; \overline{Q}_{2i} = R^{1/2}\tilde{Q}_{2i}R^{1/2},$$
$$\overline{S}_{1i} = R^{1/2}\tilde{S}_{1i}R^{1/2}, \; \overline{S}_{2i} = R^{1/2}\tilde{S}_{2i}R^{1/2}, \; \overline{T}_i = R^{1/2}\tilde{T}_i R^{1/2},$$

while

$$\lambda_1 = \inf_{i\in\mathscr{I}}\{\lambda_{min}(\tilde{P}_i^{-1})\}, \; \lambda_2 = \sup_{i\in\mathscr{I}}\{\lambda_{max}(\tilde{P}_i^{-1})\}, \; \lambda_3 = \sup_{i\in\mathscr{I}}\{\lambda_{max}(\tilde{Q}_{1i})\},$$
$$\lambda_4 = \sup_{i\in\mathscr{I}}\{\lambda_{max}(\tilde{Q}_{2i})\}, \; \lambda_5 = \sup_{i\in\mathscr{I}}\{\lambda_{max}(\tilde{S}_{1i})\}, \; \lambda_6 = \sup_{i\in\mathscr{I}}\{\lambda_{max}(\tilde{S}_{2i})\},$$
$$\lambda_7 = \sup_{i\in\mathscr{I}}\{\lambda_{max}(\tilde{T}_i)\}, \; \lambda_8 = \sup_{i\in\mathscr{I}}\{\lambda_{max}(W_i)\}, \; \delta' = \sup_{s\in[-h_2,0]}\{\dot{x}^T(s)R\dot{x}(s)\}$$

Then the average dwell-time of the switching signal satisfies

$$\tau_a > \tau_a^* = \frac{T_f \ln\mu}{\ln(\lambda_1\varepsilon) - \ln\eta - 2\alpha_{max}T_f - N_0\ln\mu} \tag{5.8}$$

Proof. Consider the Lyapunov-Krasovskii candidate functional as

$$V_i(x(t)) = \sum_{j=1}^{6} V_{ji}(x(t)) \tag{5.9}$$

where

$$\begin{aligned}
V_{1i}(x(t)) &= x^T(t)P_i^{-1}x(t) + e^T(t)P_i^{-1}e(t)\\
V_{2i}(x(t)) &= \int_{t-h_1}^{t} e^{2\alpha_i(t-s)}x^T(s)\overline{Q}_{1i}x(s)ds\\
V_{3i}(x(t)) &= \int_{t-h_2}^{t} e^{2\alpha_i(t-s)}x^T(s)\overline{Q}_{2i}x(s)ds\\
V_{4i}(x(t)) &= \int_{-h_1}^{0}\int_{t+\theta}^{t} h_1 e^{2\alpha_i(t-s)}\dot{x}^T(s)\overline{S}_{1i}\dot{x}(s)dsd\theta\\
V_{5i}(x(t)) &= \int_{-h_2}^{-h_1}\int_{t+\theta}^{t} h_{12}e^{2\alpha_i(t-s)}\dot{x}^T(s)\overline{S}_{2i}\dot{x}(s)dsd\theta\\
V_{6i}(x(t)) &= \int_{t-h(t)}^{t} e^{2\alpha_i(t-s)}x^T(s)\overline{T}_i x(s)ds
\end{aligned} \tag{5.10}$$

The derivatives are obtained as follows

$$\begin{aligned}
\dot{V}_{1i}(x(t)) =& x^T(t)[P_i^{-1}A_{Ki}+A_{Ki}^TP_i^{-1}]x(t)+2x^T(t)P_i^{-1}A_{di}x(t-h(t))\\
&+2x^T(t)P_i^{-1}B_iK_ie(t)+2x^T(t)P_i^{-1}B_{wi}w(t)\\
&+e^T(t)[P_i^{-1}A_{Li}+A_{Li}^TP_i^{-1}]e(t)+2e^T(t)P_i^{-1}A_{di}x(t-h(t))\\
&+2e^T(t)P_i^{-1}B_{wi}w(t)\\
\dot{V}_{2i}(x(t)) =& 2\alpha_iV_{2i}+x^T(t)\overline{Q}_{1i}x(t)-e^{2\alpha_ih_1}x^T(t-h_1)\overline{Q}_{1i}x(t-h_1)\\
\dot{V}_{3i}(x(t)) =& 2\alpha_iV_{3i}+x^T(t)\overline{Q}_{2i}x(t)-e^{2\alpha_ih_2}x^T(t-h_2)\overline{Q}_{2i}x(t-h_2)\\
\dot{V}_{4i}(x(t)) =& 2\alpha_iV_{4i}+h_1^2\dot{x}^T(t)\overline{S}_{1i}\dot{x}(t)-e^{2\alpha_ih_1}\int_{t-h_1}^{t}h_1\dot{x}^T(s)\overline{S}_{1i}\dot{x}(s)ds\\
\dot{V}_{5i}(x(t)) \leq& 2\alpha_iV_{5i}+h_{12}^2\dot{x}^T(t)\overline{S}_{2i}\dot{x}(t)-e^{2\alpha_ih_2}\int_{t-h_2}^{t-h_1}h_{12}\dot{x}^T(s)\overline{S}_{2i}\dot{x}(s)ds\\
\dot{V}_{6i}(x(t)) \leq& 2\alpha_iV_{6i}+x^T(t)\overline{T}_ix(t)-(1-h_d)e^{2\alpha_ih_2}x^T(t-h(t))\overline{T}_ix(t-h(t))
\end{aligned} \tag{5.11}$$

By Jensen's inequality, $\dot{V}_{4i}(x(t))$ can be written as

$$\begin{aligned}
\dot{V}_{4i}(x(t)) \leq& 2\alpha_iV_{4i}(x(t))+h_1^2\dot{x}^T(t)\overline{S}_{1i}\dot{x}(t)-e^{2\alpha_ih_1}x^T(t)\overline{S}_{1i}x(t)\\
&+2e^{2\alpha_ih_1}x^T(t)\overline{S}_{1i}x(t-h_1)-e^{2\alpha_ih_1}x^T(t-h_1)\overline{S}_{1i}x(t-h_1)
\end{aligned} \tag{5.12}$$

From (4.3), it is clear that $-(h_2-h_1)\leq-(h_2-h(t))$ and $-(h_2-h_1)\leq-(h(t)-h_1)$. Thus

$$\begin{aligned}
-h_{12}\int_{t-h_2}^{t-h_1}\dot{x}^T(s)\overline{S}_{2i}\dot{x}(s)ds \leq& -(h_2-h(t))\int_{t-h_2}^{t-h(t)}\dot{x}^T(s)\overline{S}_{2i}\dot{x}(s)ds\\
&-(h(t)-h_1)\int_{t-h(t)}^{t-h_1}\dot{x}^T(s)\overline{S}_{2i}\dot{x}(s)ds
\end{aligned} \tag{5.13}$$

Let $\int_{t-h(t)}^{t-h_1}\dot{x}(s)ds =: i_{h_1}(t)$ and $\int_{t-h_2}^{t-h(t)}\dot{x}(s)ds =: i_{h_2}(t)$. Then, by Jensen's inequality, (5.13) is written as follows

$$-h_{12}\int_{t-h_2}^{t-h_1}\dot{x}^T(s)\overline{S}_{2i}\dot{x}(s)ds \leq -i_{h_2}^T(t)\overline{S}_{2i}i_{h_2}(t)-i_{h_1}^T(t)\overline{S}_{2i}i_{h_1}(t) \tag{5.14}$$

Now, define

$$\begin{aligned}
\xi(t) = \big[&x^T(t)\; x^T(t-h(t))\; x^T(t-h_1)\; x^T(t-h_2)\\
&e^T(t)\; w^T(t)\; \dot{x}^T(t)\; i_{h_1}^T(t)\; i_{h_2}^T(t)\big]^T.
\end{aligned} \tag{5.15}$$

It is clear that, by Leibniz's formula $i_{h_1}(t)=x(t-h_1)-x(t-h(t))$ and $i_{h_2}(t)=x(t-h(t))-x(t-h_2)$. In order to fill the upper triangular elements of (5.6a), we have

$$\begin{aligned}
2\xi^T(t)\mathscr{M}_i\Big[x(t-h_1)-x(t-h(t))-i_{h_1}(t)\Big]&=0\\
2\xi^T(t)\mathscr{N}_i\Big[x(t-h(t))-x(t-h_2)-i_{h_2}(t)\Big]&=0
\end{aligned} \tag{5.16}$$

Also from (5.5), it can be written in similar consideration

$$2\dot{x}^T(t)P_i^{-1}\Big[A_{Ki}x(t)+A_{di}x(t-h(t))+B_iK_ie(t)+B_{wi}w(t)-\dot{x}(t)\Big]=0 \qquad (5.17)$$

In order to fill $\Xi_{66,i}$ element of (5.7), the quadratic form of the positive definite matrix W_i can be added and subtracted to (5.7) as follows

$$\left[w^T(t)W_iw(t)-w^T(t)W_iw(t)\right]=0 \qquad (5.18)$$

Then, by the equations (5.9)-(5.18), we obtain

$$\dot{V}_i(x(t))-2\alpha_iV_i(x(t))\leq\xi^T(t)\Sigma_i\xi(t)+w^T(t)W_iw(t) \qquad (5.19)$$

From (5.6a)

$$\dot{V}_i(x(t))-2\alpha_iV_i(x(t))\leq w^T(t)W_iw(t) \qquad (5.20)$$

is obtained.

On the other hand, by applying Grönwall's lemma on $t\in[t_k,t_{k+1})$, we get

$$\begin{aligned}V_{\sigma(t)}(x(t))\leq&e^{2\alpha_{\sigma(t_k)}(t-t_k)}V_{\sigma(t_k)}(x(t_k))\\&+\int_{t_k}^{t}e^{2\alpha_{\sigma(t_k)}(t-s)}w^T(s)W_{\sigma(t_k)}w(s)ds.\end{aligned} \qquad (5.21)$$

Consider (5.6c) and assume $\sigma(t_k)=i$ and $\sigma(t_k^-)=j$, we have

$$V_{\sigma(t_k)}(x(t_k))\leq\mu V_{\sigma(t_k^-)}(x(t_k^-)) \qquad (5.22)$$

So by (5.21) and (5.22),

$$V_{\sigma(t)}(x(t))\leq e^{2\alpha_{\sigma(t_k)}(t-t_k)}\mu V_{\sigma(t_k^-)}(x(t_k^-))+\int_{t_k}^{t}e^{2\alpha_{\sigma(t_k)}(t-s)}w^T(s)W_{\sigma(t_k)}w(s)ds.$$

is obtained. When we apply Grönwall's lemma to (5.20) on $t_k\in[t_{k-1},t)$, we have

$$\begin{aligned}V_{\sigma(t_k^-)}(x(t_k^-))\leq V_{\sigma(t_k)}(t_k)\leq&e^{2\alpha_{\sigma(t_{k-1})}(t_k-t_{k-1})}V_{\sigma(t_{k-1})}(x(t_{k-1}))\\&+\int_{t_{k-1}}^{t_k}e^{2\alpha_{\sigma(t_{k-1})}(t_k-s)}w^T(s)W_{\sigma(t_{k-1})}w(s)ds.\end{aligned}$$

So,

$$\begin{aligned}V_{\sigma(t)}(x(t))\leq&e^{2\alpha_{\sigma(t_k)}(t-t_k)}e^{2\alpha_{\sigma(t_{k-1})}(t_k-t_{k-1})}\mu V_{\sigma(t_{k-1})}(x(t_{k-1}))\\&+\mu\int_{t_{k-1}}^{t_k}e^{2\alpha_{\sigma(t_{k-1})}(t_k-s)}w^T(s)W_{\sigma(t_{k-1})}w(s)ds\\&+\int_{t_k}^{t}e^{2\alpha_{\sigma(t_k)}(t-s)}w^T(s)W_{\sigma(t_k)}w(s)ds.\end{aligned}$$

If Grönwall's lemma and (5.22) is applied to (5.20) until $[0,t_1)$ recursively, we get

$$
\begin{aligned}
V_{\sigma(t)}(x(t)) \leq & e^{2\alpha_{\sigma(t_k)}(t-t_k)+2\alpha_{\sigma(t_{k-1})}(t_k-t_{k-1})+\ldots+2\alpha_{\sigma(0)}(t_1-0)}\mu^N V_{\sigma(0)}(x(0)) \\
& + \mu^N \int_0^{t_1} e^{2\alpha_{\sigma(t_k)}(t-t_k)+2\alpha_{\sigma(t_{k-1})}(t_k-t_{k-1})+\ldots+2\alpha_{\sigma(0)}(t_1-s)} \\
& \qquad \times w^T(s)W_{\sigma(0)}w(s)ds \\
& + \ldots \\
& + \int_{t_k}^{t} e^{2\alpha_{\sigma(t_k)}(t-s)} w^T(s)W_{\sigma(t_k)}w(s)ds \\
\leq & e^{2\alpha_{max}T_f}\mu^N \left(V_{\sigma(0)}(x(0)) + \int_0^{T_f} w^T(s)W_{\sigma(s)}w(s)ds \right) \\
\leq & e^{2\alpha_{max}T_f}\mu^N \left(V_{\sigma(0)}(x(0)) + \lambda_8 d \right)
\end{aligned}
\tag{5.23}
$$

where N denotes the switching number of $\sigma(t)$ over $(0,T_f)$. By Definition 1, it is possible to write

$$
V_{\sigma(t)}(x(t)) \leq e^{2\alpha_{max}T_f}\mu^{N_0+T_f/\tau_a}\left(V_{\sigma(0)}(x(0)) + \lambda_8 d\right) \tag{5.24}
$$

Moreover,

$$
\begin{aligned}
V_{\sigma(0)}(x(0)) = & x^T(0)P^{-1}_{\sigma(0)}x(0) + e^T(0)P^{-1}_{\sigma(0)}e(0) \\
& + \int_{-h_1}^{0} e^{-2\alpha_{\sigma(0)}s} x^T(s)\overline{Q}_{1\sigma(0)}x(s)ds \\
& + \int_{-h_2}^{0} e^{-2\alpha_{\sigma(0)}s} x^T(s)\overline{Q}_{2\sigma(0)}x(s)ds \\
& + \int_{-h_1}^{0}\int_{\theta}^{0} h_1 e^{-2\alpha_{\sigma(0)}s}\dot{x}^T(s)\overline{S}_{1\sigma(0)}\dot{x}(s)dsd\theta \\
& + \int_{-h_2}^{-h_1}\int_{\theta}^{0} h_{12} e^{-2\alpha_{\sigma(0)}s}\dot{x}^T(s)\overline{S}_{2\sigma(0)}\dot{x}(s)dsd\theta \\
& + \int_{-h(0)}^{0} e^{-2\alpha_{\sigma(0)}s} x^T(s)\overline{T}_{\sigma(0)}x(s)ds.
\end{aligned}
\tag{5.25}
$$

When the orders of the double integrals are changed, we have

$$
\begin{aligned}
V_{\sigma(0)}(x(0)) = & x^T(0)P^{-1}_{\sigma(0)}x(0) + e^T(0)P^{-1}_{\sigma(0)}e(0) \\
& + \int_{-h_1}^{0} e^{-2\alpha_{\sigma(0)}s} x^T(s)\overline{Q}_{1\sigma(0)}x(s)ds \\
& + \int_{-h_2}^{0} e^{-2\alpha_{\sigma(0)}s} x^T(s)\overline{Q}_{2\sigma(0)}x(s)ds \\
& + \int_{-h_1}^{0}\int_{-h_1}^{s} h_1 e^{-2\alpha_{\sigma(0)}s}\dot{x}^T(s)\overline{S}_{1\sigma(0)}\dot{x}(s)d\theta ds \\
& + \int_{-h_2}^{-h_1}\int_{-h_2}^{s} h_{12} e^{-2\alpha_{\sigma(0)}s}\dot{x}^T(s)\overline{S}_{2\sigma(0)}\dot{x}(s)d\theta ds \\
& + \int_{-h_1}^{0}\int_{-h_2}^{-h_1} h_{12} e^{-2\alpha_{\sigma(0)}s}\dot{x}^T(s)\overline{S}_{2\sigma(0)}\dot{x}(s)d\theta ds \\
& + \int_{-h(0)}^{0} e^{-2\alpha_{\sigma(0)}s} x^T(s)\overline{T}_{\sigma(0)}x(s)ds.
\end{aligned}
$$

Note that $e(0) = x(0)$. Then, if the upper bounds of definite integrals are written, we can write

$$\begin{aligned}V_{\sigma(0)}(x(0)) \leq & 2\lambda_{max}\left(\tilde{P}_{\sigma(0)}^{-1}\right)x^T(0)Rx(0) \\ & + \lambda_{max}\left(\tilde{Q}_{1\sigma(0)}\right)h_1 e^{2\alpha_{max}h_1} \sup_{s\in[-h_1,0]} \left\{x^T(s)Rx(s)\right\} \\ & + \lambda_{max}\left(\tilde{Q}_{2\sigma(0)}\right)h_2 e^{2\alpha_{max}h_2} \sup_{s\in[-h_2,0]} \left\{x^T(s)Rx(s)\right\} \\ & + \lambda_{max}\left(\tilde{S}_{1\sigma(0)}\right)h_1^3 e^{2\alpha_{max}h_1} \sup_{s\in[-h_1,0]} \left\{\dot{x}^T(s)R\dot{x}(s)\right\} \\ & + \lambda_{max}\left(\tilde{S}_{2\sigma(0)}\right)h_{12}^3 e^{2\alpha_{max}h_2} \sup_{s\in[-h_2,-h_1]} \left\{\dot{x}^T(s)R\dot{x}(s)\right\} \\ & + \lambda_{max}\left(\tilde{S}_{2\sigma(0)}\right)h_{12}^2 h_1 e^{2\alpha_{max}h_1} \sup_{s\in[-h_1,0]} \left\{\dot{x}^T(s)R\dot{x}(s)\right\} \\ & + \lambda_{max}\left(\tilde{T}_{\sigma(0)}\right)h_2 e^{2\alpha_{max}h_2} \sup_{s\in[-h_2,0]} \left\{x^T(s)Rx(s)\right\}.\end{aligned}$$

Here, an upper bound for $V_{\sigma(0)}(x(0))$ can be written as follows

$$\begin{aligned}V_{\sigma(0)}(x(0)) \leq & 2\lambda_2\delta + \lambda_3 h_1 e^{2\alpha_{max}h_1}\delta + \lambda_4 h_2 e^{2\alpha_{max}h_2}\delta + \lambda_5 h_1^3 e^{2\alpha_{max}h_1}\delta' \\ & + \lambda_6 h_{12}^2(h_1 e^{2\alpha_{max}h_1} + h_{12}e^{2\alpha_{max}h_2})\delta' + \lambda_7 h_2 e^{2\alpha_{max}h_2}\delta\end{aligned} \tag{5.26}$$

Since,

$$\begin{aligned}V_{\sigma(t)}(x(t)) & \geq x^T(t)P_i^{-1}x(t) = x^T(t)R^{1/2}\tilde{P}_i^{-1}R^{1/2}x(t) \\ & \geq \inf_{i\in I}\left(\lambda_{min}(\tilde{P}_i^{-1})\right)x^T(t)Rx(t) \\ & = \lambda_1 x^T(t)Rx(t).\end{aligned} \tag{5.27}$$

By (5.6b), (5.8), (5.24), (5.26) and (5.27) the following inequality holds

$$x^T(t)Rx(t) < \varepsilon \tag{5.28}$$

which tells that the closed-loop switched system (5.5) is FT bounded. □

5.2.1 Switched systems with unstable subsystems

Based on Lemma 5, a FT boundedness criterion for the switched system with time-delay is introduced.

Theorem 7. *The switched system* (5.1) *is FT bounded with respect to (*$\delta, \varepsilon, T_f, d, R$*), if there exist a set of symmetric matrices for every* i^{th} *subsystem* $P_i > 0$, $Q_{1i} > 0$, $Q_{2i} > 0$, $S_{1i} > 0$, $S_{2i} > 0$, $T_i > 0$, $W_i > 0$, Y_i, M_{1i}, M_{2i}, N_{1i}, N_{2i} *and scalars* $\alpha_i \geq 0$ *and* $\mu \geq 1$

satisfying

$$\Upsilon_i = \begin{bmatrix} \Omega_i & -M_i & -N_i & Z_i \\ * & -e^{2\alpha_i h_2} S_{2i} & 0 & 0 \\ * & * & -e^{2\alpha_i h_2} S_{2i} & 0 \\ * & * & * & -I \end{bmatrix} < 0 \tag{5.29a}$$

$$e^{2\alpha_{max} T_f} \eta' \leq \lambda_1 \varepsilon \tag{5.29b}$$

$$P_j \leq \mu P_i,\ Q_{kj} \leq \mu Q_{ki},\ S_{kj} \leq \mu S_{ki},\ T_j \leq \mu T_i, \tag{5.29c}$$

for $i, j \in \mathscr{I}$ *and* $k = 1, 2$*, where*

$$\Omega_i = \begin{bmatrix} \Omega_{11,i} & \Omega_{12,i} & \Omega_{13,i} & -N_{1i} & B_i Y_i & B_{wi} & \Omega_{17,i} \\ * & \Omega_{22,i} & M_{2i} & -N_{2i} & P_i A_{di}^T & 0 & P_i A_{di}^T \\ * & * & \Omega_{33,i} & 0 & 0 & 0 & 0 \\ * & * & * & \Omega_{44,i} & 0 & 0 & 0 \\ * & * & * & * & \Omega_{55,i} & B_{wi} & Y_i^T B_i^T \\ * & * & * & * & * & -W_i & B_{wi}^T \\ * & * & * & * & * & * & \Omega_{77,i} \end{bmatrix} \tag{5.30}$$

with entries

$$\begin{aligned}
\Omega_{11,i} =& A_i P_i + P_i A_i^T - B_i Y_i - Y_i^T B_i^T + Q_{1i} + Q_{2i} - e^{2\alpha_i h_1} S_{1i} - 2\alpha_i P_i + T_i, \\
\Omega_{12,i} =& A_{di} P_i - M_{1i} + N_{1i},\ \Omega_{13,i} = e^{2\alpha_i h_1} S_{1i} + M_{1i},\ \Omega_{17,i} = P_i A_i^T - Y_i^T B_i^T, \\
\Omega_{22,i} =& N_{2i} + N_{2i}^T - M_{2i} - M_{2i}^T - (1 - h_d) e^{2\alpha_i h_2} T_i,\ \Omega_{33,i} = -e^{2\alpha_i h_1}(Q_{1i} + S_{1i}), \\
\Omega_{44,i} =& -e^{2\alpha_i h_2} Q_{2i},\ \Omega_{55,i} = A_i P_i + P_i A_i^T - 2\alpha_i P_i,\ \Omega_{77,i} = h_1^2 S_{1i} + h_{12}^2 S_{2i} - 2P_i, \\
M_i =& [M_{1i}\ M_{2i}\ 0\ 0\ 0\ 0\ 0]^T,\ N_i = [N_{1i}\ N_{2i}\ 0\ 0\ 0\ 0\ 0]^T,\ Z_i = [0\ 0\ 0\ 0\ C_i P_i\ 0\ 0]^T
\end{aligned}$$

and

$$\begin{aligned}
\eta' =& 2\lambda_2 \delta + \lambda_3' h_1 e^{2\alpha_{max} h_1} \delta + \lambda_4' h_2 e^{2\alpha_{max} h_2} \delta + \lambda_5' h_1^3 e^{2\alpha_{max} h_1} \delta' \\
& + \lambda_6' h_{12}^2 (h_1 e^{2\alpha_{max} h_1} + h_{12} e^{2\alpha_{max} h_2}) \delta' + \lambda_7' h_2 e^{2\alpha_{max} h_2} \delta + \lambda_8 d.
\end{aligned} \tag{5.31}$$

with matrix transformations

$$\begin{aligned}
\hat{Q}_{1i} =& R^{1/2} Q_{1i} R^{1/2},\ \hat{Q}_{2i} = R^{1/2} Q_{2i} R^{1/2}, \\
\hat{S}_{1i} =& R^{1/2} S_{1i} R^{1/2},\ \hat{S}_{2i} = R^{1/2} S_{2i} R^{1/2},\ \hat{T}_i = R^{1/2} T_i R^{1/2}.
\end{aligned} \tag{5.32}$$

and

$$\begin{aligned}
\lambda_3' =& \sup_{i \in \mathscr{I}} \{\lambda_{max}(\tilde{P}_i^{-1} \hat{Q}_{1i} \tilde{P}_i^{-1})\},\ \lambda_4' = \sup_{i \in \mathscr{I}} \{\lambda_{max}(\tilde{P}_i^{-1} \hat{Q}_{2i} \tilde{P}_i^{-1})\}, \\
\lambda_5' =& \sup_{i \in \mathscr{I}} \{\lambda_{max}(\tilde{P}_i^{-1} \hat{S}_{1i} \tilde{P}_i^{-1})\},\ \lambda_6' = \sup_{i \in \mathscr{I}} \{\lambda_{max}(\tilde{P}_i^{-1} \hat{S}_{2i} \tilde{P}_i^{-1})\}, \\
\lambda_7' =& \sup_{i \in \mathscr{I}} \{\lambda_{max}(\tilde{P}_i^{-1} \hat{T}_i \tilde{P}_i^{-1})\}.
\end{aligned}$$

Then the ADT of the switching signal satisfies

$$\tau_a > \tau_a^* = \frac{T_f \ln \mu}{\ln(\lambda_1 \varepsilon) - \ln \eta' - 2\alpha_{max} T_f - N_0 \ln \mu} \tag{5.33}$$

The gain matrices K_i and L_i of controller and observer are perceived as

$$K_i = Y_i P_i^{-1},\ L_i = -\frac{1}{2} P_i C_i^T \tag{5.34}$$

Proof. Let us consider the inequalities (5.6a) in Lemma 5 and define

$$\begin{aligned} Q_{1i} =& P_i \overline{Q}_{1i} P_i,\ Q_{2i} = P_i \overline{Q}_{2i} P_i,\ S_{1i} = P_i \overline{S}_{1i} P_i,\ S_{2i} = P_i \overline{S}_{2i} P_i, \\ M_{1i} =& P_i \overline{M}_{1i} P_i,\ M_{2i} = P_i \overline{M}_{2i} P_i,\ N_{1i} = P_i \overline{N}_{1i} P_i,\ N_{2i} = P_i \overline{N}_{2i} P_i \\ T_i =& P_i \overline{T}_i P_i. \end{aligned} \tag{5.35}$$

By pre- and post-multiplying both sides of the inequalities in (5.6a) by $\mathscr{D}_i = diag\{P_i, P_i, P_i, P_i, P_i, I, P_i, P_i, P_i\}$ and using Schur complement lemma the LMIs in (5.29a) are obtained.

Now, substitute the matrices in (5.35) into $V_{\sigma(0)}(x(0))$ in (5.25) as follows

$$\begin{aligned} V_{\sigma(0)}(x(0)) = & 2x^T(0) P_{\sigma(0)}^{-1} x(0) \\ & + \int_{-h_1}^{0} e^{-2\alpha_{\sigma(0)} s} x^T(s) P_{\sigma(0)}^{-1} Q_{1\sigma(0)} P_{\sigma(0)}^{-1} x(s) ds \\ & + \int_{-h_2}^{0} e^{-2\alpha_{\sigma(0)} s} x^T(s) P_{\sigma(0)}^{-1} Q_{2\sigma(0)} P_{\sigma(0)}^{-1} x(s) ds \\ & + \int_{-h_1}^{0} \int_{\theta}^{0} h_1 e^{-2\alpha_{\sigma(0)} s} \dot{x}^T(s) P_{\sigma(0)}^{-1} S_{1\sigma(0)} P_{\sigma(0)}^{-1} \dot{x}(s) ds d\theta \\ & + \int_{-h_2}^{-h_1} \int_{\theta}^{0} h_{12} e^{-2\alpha_{\sigma(0)} s} \dot{x}^T(s) P_{\sigma(0)}^{-1} S_{2\sigma(0)} P_{\sigma(0)}^{-1} \dot{x}(s) ds d\theta \\ & + \int_{-h(0)}^{0} e^{-2\alpha_{\sigma(0)} s} x^T(s) P_{\sigma(0)}^{-1} T_{\sigma(0)} P_{\sigma(0)}^{-1} x(s) ds \end{aligned} \tag{5.36}$$

From (5.32), each matrix in the equation (5.36) can be written as,

$$P_{\sigma(0)}^{-1} Q_{1\sigma(0)} P_{\sigma(0)}^{-1} = R^{1/2} \tilde{P}_{\sigma(0)}^{-1} \hat{Q}_{1\sigma(0)} \tilde{P}_{\sigma(0)}^{-1} R^{1/2} \leq \lambda_{max}(\tilde{P}_{\sigma(0)}^{-1} \hat{Q}_{1\sigma(0)} \tilde{P}_{\sigma(0)}^{-1}) R \leq \lambda_3' R$$

The upper bound for $V_{\sigma(0)}(x(0))$ is obtained as follows

$$\begin{aligned} V_{\sigma(0)}(x(0)) \leq & 2\lambda_2 \delta + \lambda_3' h_1 e^{2\alpha_{max} h_1} \delta + \lambda_4' h_2 e^{2\alpha_{max} h_2} \delta + \lambda_5' h_1^3 e^{2\alpha_{max} h_1} \delta' \\ & + \lambda_6' h_{12}^2 (h_1 e^{2\alpha_{max} h_1} + h_{12} e^{2\alpha_{max} h_2}) \delta' + \lambda_7' h_2 e^{2\alpha_{max} h_2} \delta. \end{aligned} \tag{5.37}$$

By the equations (5.24), (5.26) and (5.37) the inequality $x^T(t) R x(t) < \varepsilon$ is obtained, which tells that the switched system (5.1) is FT bounded. Then, for $\mu = 1$ the inequality in (5.29b) and for $\mu > 1$ the ADT bound in (5.33) are calculated. □

In the following corollary, lower bound for time-delay is not considered and the similar Lyapunov-Krasovskii functional as in [40] is used to compare the results given in Theorem 7.

Corollary 1. *The switched system* (5.1)*) is FT bounded with respect to ($\delta, \varepsilon, T_f, d, R$), if there exist a set of symmetric matrices for every i^{th} subsystem $P_i > 0$, $T_i > 0$, $W_i > 0$, Y_i and scalars $\alpha_i \geq 0$ and $\mu \geq 1$ satisfying*

$$\Omega'_i = \begin{bmatrix} \Omega'_{11,i} & A_{di}P_i & B_iY_i & B_{wi} & 0 \\ * & \Omega'_{22,i} & P_iA_{di}^T & 0 & 0 \\ * & * & \Omega'_{33,i} & B_{wi} & P_iC_i^T \\ * & * & * & -W_i & 0 \\ * & * & * & * & -I \end{bmatrix} < 0, \tag{5.38a}$$

$$e^{2\alpha_{max}T_f}\eta'' \leq \lambda_1\varepsilon \tag{5.38b}$$

$$P_j \leq \mu P_i, T_j \leq \mu T_i \tag{5.38c}$$

for arbitrary $i, j \in \mathscr{I}$, where

$$\Omega'_{11,i} = A_iP_i + P_iA_i^T - B_iY_i - Y_i^TB_i^T - 2\alpha_iP_i + T_i,$$
$$\Omega'_{22,i} = -(1-h_d)e^{2\alpha_ih_2}T_i,\ \Omega'_{33,i} = A_iP_i + P_iA_i^T - 2\alpha_iP_i,$$

and

$$\eta'' = 2\lambda_2\delta + \lambda'_7h_2e^{2\alpha_{max}h_2}\delta + \lambda_8d. \tag{5.39}$$

Then the ADT of the switching signal satisfies

$$\tau_a > \tau_a^* = \frac{T_f \ln\mu}{\ln(\lambda_1\varepsilon) - \ln\eta'' - 2\alpha_{max}T_f - N_0\ln\mu} \tag{5.40}$$

The gain matrices K_i and L_i of controller and observer are perceived as in (5.34)

Proof. The proof is similar to Lemma 5 and Theorem 7 by taking the Lyapunov-Krasovskii functional in (5.9) and (5.10) as $V_i = V_{1i} + V_{6i}$. □

5.2.2 Switched systems with mixed stable subsystems

In this section, we suppose that A_1, A_2,..., A_r, ($1 \leq r < N$) in system (5.1) are Hurwitz stable and the remaining matrices are unstable. Let us define

$$\psi_i = \begin{cases} -\alpha_i & i \in \mathscr{I}_{st} \\ \alpha_i & i \in \mathscr{I}_{un} \end{cases}$$

where $\mathscr{I}_{st}$ and $\mathscr{I}_{un}$ are the index set of all Hurwitz stable and unstable subsystems, respectively. Note that $\mathscr{I} = \mathscr{I}_{st} \cup \mathscr{I}_{un}$. For a given switching sequence Σ, the total activation times of stable and unstable subsystems are defined as T^- and T^+, respectively in a finite interval $[0, T_f]$. Thus, $T_f = T^+ + T^-$.

Theorem 8. *Consider the switched system* (5.1) *with r Hurwitz stable and $N-r$ unstable subsystems. The system* (5.1) *is FT bounded with respect to $(\delta, \varepsilon, T_f, d, R)$, for given constants $\alpha_i \geq 0$, $\mu \geq 1$, $T^+ > 0$ and $T^- > 0$ such that $T_f = T^+ + T^-$, if there exist a set of symmetric matrices for every i^{th} subsystem $P_i > 0$, $Q_{1i} > 0$, $Q_{2i} > 0$, $S_{1i} > 0$, $S_{2i} > 0$, $T_i > 0$, $W_i > 0$, Y_i, M_{1i}, M_{2i}, N_{1i}, N_{2i} satisfying*

$$\overline{\Upsilon}_i = \begin{bmatrix} \overline{\Omega}_i & -M_i & -N_i & Z_i \\ * & -e^{2\psi_i h_2} S_{2i} & 0 & 0 \\ * & * & -e^{2\psi_i h_2} S_{2i} & 0 \\ * & * & * & -I \end{bmatrix} < 0 \tag{5.41a}$$

$$e^{2\alpha^+_{max} T^+} \eta'_+ \leq \lambda_1 e^{2\alpha^-_{min} T^-} \varepsilon \tag{5.41b}$$

$$P_j \leq \mu P_i,\ Q_{kj} \leq \mu Q_{ki}, S_{kj} \leq \mu S_{ki}, T_j \leq \mu T_i, \tag{5.41c}$$

for $i, j \in \mathscr{I}$ and $k = 1, 2$, where

$$\overline{\Omega}_{11,i} = A_i P_i + P_i A_i^T - B_i Y_i - Y_i^T B_i^T + Q_{1i} + Q_{2i} - e^{2\psi_i h_1} S_{1i} - 2\psi_i P_i + T_i,$$

$$\overline{\Omega}_{13,i} = e^{2\psi_i h_1} S_{1i} + M_{1i},$$

$$\overline{\Omega}_{22,i} = N_{2i} + N_{2i}^T - M_{2i} - M_{2i}^T - (1 - h_d) e^{2\psi_i h_2} T_i,$$

$$\overline{\Omega}_{33,i} = -e^{2\psi_i h_1}(Q_{1i} + S_{1i}), \quad \overline{\Omega}_{44,i} = -e^{2\psi_i h_2} Q_{2i},\ \overline{\Omega}_{55,i} = A_i P_i + P_i A_i^T - 2\psi_i P_i$$

and the remaining entries of $\overline{\Omega}_i$ are of $\overline{\Omega}_{jk,i} = \Omega_{jk,i}$.

Then the ADT of the switching signal satisfies

$$\tau_a > \tau_a^* = \frac{T_f \ln \mu}{\ln(\lambda_1 \varepsilon) - \ln \eta'_+ - 2\alpha^+_{max} T^+ + 2\alpha^-_{min} T^- - N_0 \ln \mu} \tag{5.42}$$

where $\alpha^+_{max} = \max_{i \in \mathscr{I}_{un}}\{\alpha_i\}$, $\alpha^-_{min} = \min_{i \in \mathscr{I}_{st}}\{\alpha_i\}$ and

$$\begin{aligned} \eta'_+ = & 2\lambda_2 \delta + \lambda'_3 h_1 e^{2\alpha^+_{max} h_1} \delta + \lambda'_4 h_2 e^{2\alpha^+_{max} h_2} \delta + \lambda'_5 h_1^3 e^{2\alpha^+_{max} h_1} \delta' \\ & + \lambda'_6 h_{12}^2 (h_1 e^{2\alpha^+_{max} h_1} + h_{12} e^{2\alpha^+_{max} h_2}) \delta' + \lambda'_7 h_2 e^{2\alpha^+_{max} h_2} \delta + \lambda_8 d. \end{aligned} \tag{5.43}$$

The gain matrices K_i and L_i of controller and observer can be obtained as it is in (5.34).

Proof. The inequality in (5.41a) is obtained by taking the Lyapunov-Krasovskii functional (5.9) and (5.10) in Lemma 5 by replacing α_i with ψ_i.

In order to obtain upper bound for $V_{\sigma(t)}(x(t))$ substitute ψ_i for α_i in (5.23). So, the following inequality holds.

$$\begin{aligned} V_{\sigma(t)}(x(t)) \leq & e^{2\psi_{\sigma(t_k)}(t-t_k)+2\psi_{\sigma(t_{k-1})}(t_k-t_{k-1})+\ldots+2\psi_{\sigma(0)}(t_1-0)}\mu^N V_{\sigma(0)}(x(0)) \\ & +\mu^N \int_0^{t_1} e^{2\psi_{\sigma(t_k)}(t-t_k)+2\psi_{\sigma(t_{k-1})}(t_k-t_{k-1})+\ldots+2\psi_{\sigma(0)}(t_1-s)} \\ & \qquad \times w^T(s)W_{\sigma(0)}w(s)ds \\ & +\ldots \\ & +\int_{t_k}^{t} e^{2\psi_{\sigma(t_k)}(t-s)}w^T(s)W_{\sigma(t_k)}w(s)ds \end{aligned} \tag{5.44}$$

By considering the activation times T^- and T^+ for stable and unstable subsystems, respectively, the inequality (5.44) can be written as follows:

$$V_{\sigma(t)}(x(t)) \leq e^{2\alpha^+_{max}T^+-2\alpha^-_{min}T^-}\mu^N\big(V_{\sigma(0)}(x(0))+\lambda_8 d\big). \tag{5.45}$$

$V_{\sigma(0)}(x(0))$ in (5.45) is obtained by substituting ψ_i for α_i in (5.25). Then, from (5.25) and the facts

$$\begin{aligned} \sup_{s\in[-h_1,0]}\{e^{-2\psi_{\sigma(0)}s}\} &= \sup_{s\in[-h_1,0]}\{\max\{e^{-2\alpha_{\sigma(0)}s},e^{2\alpha_{\sigma(0)}s}\}\} = e^{2\alpha^+_{max}h_1}, \\ \sup_{s\in[-h_2,0]}\{e^{-2\psi_{\sigma(0)}s}\} &= \sup_{s\in[-h_2,0]}\{\max\{e^{-2\alpha_{\sigma(0)}s},e^{2\alpha_{\sigma(0)}s}\}\} = e^{2\alpha^+_{max}h_2}, \\ \sup_{s\in[-h(0),0]}\{e^{-2\psi_{\sigma(0)}s}\} &\leq \sup_{s\in[-h_2,0]}\{e^{-2\psi_{\sigma(0)}s}\} = e^{2\alpha^+_{max}h_2} \end{aligned} \tag{5.46}$$

$V_{\sigma(0)}(x(0))$ is obtained as follows

$$\begin{aligned} V_{\sigma(0)}(x(0)) \leq & 2\lambda_2\delta+\lambda_3' h_1 e^{2\alpha^+_{max}h_1}\delta+\lambda_4' h_2 e^{2\alpha^+_{max}h_2}\delta+\lambda_5' h_1^3 e^{2\alpha^+_{max}h_1}\delta' \\ & +\lambda_6' h_{12}^2(h_1e^{2\alpha^+_{max}h_1}+h_{12}e^{2\alpha^+_{max}h_2})\delta'+\lambda_7' h_2 e^{2\alpha^+_{max}h_2}\delta. \end{aligned} \tag{5.47}$$

By the equations (5.27), (5.45) and (5.47) the inequality $x^T(t)Rx(t) < \varepsilon$ is obtained, which tells that the switched system (5.1) is FT bounded. Then, for $\mu = 1$ the inequality in (5.41b) and for $\mu > 1$ the ADT bound in (5.42) are calculated. □

Remark 6. *Note that the condition* (5.29b) *contains the constants* λ_1, λ_2, λ_3', λ_4', λ_5', λ_6', λ_7' *and* λ_8. *The existance of these constants depends on the solutions of the following* *inequalities*

$$\begin{aligned}
&\lambda_1 I < \tilde{P}_i^{-1} < \lambda_2 I,\\
&0 < \tilde{P}_i^{-1}\hat{Q}_{1i}\tilde{P}_i^{-1} < \lambda_3' I,\ 0 < \tilde{P}_i^{-1}\hat{Q}_{2i}\tilde{P}_i^{-1} < \lambda_4' I,\\
&0 < \tilde{P}_i^{-1}\hat{S}_{1i}\tilde{P}_i^{-1} < \lambda_5' I,\ 0 < \tilde{P}_i^{-1}\hat{S}_{2i}\tilde{P}_i^{-1} < \lambda_6' I,\\
&0 < \tilde{P}_i^{-1}\hat{T}_i\tilde{P}_i^{-1} < \lambda_7' I,\ \ 0 < W_i < \lambda_8 I.
\end{aligned} \tag{5.48}$$

For more details see [47].

To solve the inequalities in (5.48)*, it is necessary to put them into LMIs form. Thus, consider* $0 < \tilde{P}_i^{-1}\hat{Q}_{1i}\tilde{P}_i^{-1} < \lambda_3' I$*, write it as* $-\lambda_3' I + \tilde{P}_i^{-1}\hat{Q}_{1i}\tilde{P}_i^{-1} < 0$ *and use Schur Complement*

$$\begin{bmatrix} -\lambda_3' I & \tilde{P}_i^{-1} \\ * & -\hat{Q}_{1i}^{-1} \end{bmatrix} \leq 0 \iff \begin{bmatrix} -\lambda_3' I & J_i \\ * & -E_{1i} \end{bmatrix} \leq 0 \tag{5.49}$$

where $J_i := \tilde{P}_i^{-1}$ *and* $E_{1i} := \hat{Q}_{1i}^{-1}$ *(or equivalently* $J_i\tilde{P}_i = I$ *and* $E_{1i}\hat{Q}_{1i} = I$*). By applying same procedure to the other nonlinear inequalities from* (5.48) *and defining the matrices* E_{2i}*,* F_{1i}*,* F_{2i} *and* G_i *for the matrix inverse approximates of* $\hat{Q}_{2i}$*,* $\hat{S}_{1i}$*,* $\hat{S}_{2i}$ *and* $\hat{T}_i$*, the following inequalities can be stated in terms of cone-complementarity algorithm given in [48].*

$$\begin{aligned}
&\lambda_1 I < J_i < \lambda_2 I,\ 0 \leq \begin{bmatrix} \tilde{P}_i & I \\ * & J_i \end{bmatrix},\\
&\begin{bmatrix} -\lambda_3' I & J_i \\ * & -E_{1i} \end{bmatrix} \leq 0,\ 0 \leq \begin{bmatrix} \hat{Q}_{1i} & I \\ * & E_{1i} \end{bmatrix},\ \begin{bmatrix} -\lambda_4' I & J_i \\ * & -E_{2i} \end{bmatrix} \leq 0,\ 0 \leq \begin{bmatrix} \hat{Q}_{2i} & I \\ * & E_{2i} \end{bmatrix},\\
&\begin{bmatrix} -\lambda_5' I & J_i \\ * & -F_{1i} \end{bmatrix} \leq 0,\ 0 \leq \begin{bmatrix} \hat{S}_{1i} & I \\ * & F_{1i} \end{bmatrix},\ \begin{bmatrix} -\lambda_6' I & J_i \\ * & -F_{2i} \end{bmatrix} \leq 0,\ 0 \leq \begin{bmatrix} \hat{S}_{2i} & I \\ * & F_{2i} \end{bmatrix},\\
&\begin{bmatrix} -\lambda_7' I & J_i \\ * & -G_i \end{bmatrix} \leq 0,\ 0 \leq \begin{bmatrix} \hat{T}_i & I \\ * & G_i \end{bmatrix},\ 0 < W_i < \lambda_8 I.
\end{aligned} \tag{5.50}$$

Remark 7. *In the following, a general algorithm is defined for solving the sufficient conditions given in Theorem 7, 8 and Corollary 1. Note that,* $\Omega_{55,i}$ *of Theorem 7,* $\Omega'_{33,i}$ *of Corollary 1,* $\overline{\Omega}_{55,i}$ *of Theorem 8 contains the terms* $A_iP_i + P_iA_i^T - 2\alpha_iP_i$ *or* $A_iP_i + P_iA_i^T - 2\psi_iP_i$*. It should be stressed out that, positive definite solutions for* P_i *satisfying the corresponding LMIs can only be found, if* $(A_i - \alpha_iI)$ *or* $(A_i - \psi_iI)$ *are Hurwitz stable, respectively. In other words, it is not possible to find positive definite* P_i *satisfying the corresponding LMIs, unless* $(A_i - \alpha_iI)$ *or* $(A_i - \psi_iI)$ *are Hurwitz stable, respectively. So, the following algorithm should be considered in that way.*

Algorithm 2. *This algorithm is derived for Theorem 8.*

- ***Step 1:*** *Define constants ψ_i, for each $i \in \mathscr{I}$ such that all $(A_i - \psi_i I)$ are Hurwitz stable.*
- ***Step 2:*** *Find a feasible set*

$$(P_i^0, Q_{1i}^0, Q_{2i}^0, S_{1i}^0, S_{2i}^0, T_i^0, J_i^0, E_{1i}^0, E_{2i}^0, F_{1i}^0, F_{2i}^0, G_i^0, W_i^0, T_i^0, M_{1i}^0, M_{2i}^0, N_{1i}^0, N_{2i}^0)$$

satisfying the inequalities in (5.41a), (5.41b), (5.41c) *and* (5.50). *Set $k = 0$.*

- ***Step 3:*** *Solve the following LMI problem for the variables*

$$(P_i, Q_{1i}, Q_{2i}, S_{1i}, S_{2i}, T_i, J_i, E_{1i}, E_{2i}, F_{1i}, F_{2i}, G_i, W_i, T_i, M_{1i}, M_{2i}, N_{1i}, N_{2i})$$

according to the following minimization problem

$$\begin{aligned} \textit{minimize } tr\Bigg(\sum_{i\in\mathscr{I}} & J_i^k \tilde{P}_i + J_i \tilde{P}_i^k + E_{1i}^k \hat{Q}_{1i} + E_{1i}\hat{Q}_{1i}^k + E_{2i}^k \hat{Q}_{2i} + E_{2i}\hat{Q}_{2i}^k \\ & + F_{1i}^k \hat{S}_{1i} + F_{1i}\hat{S}_{1i}^k + F_{2i}^k \hat{S}_{2i} + F_{2i}\hat{S}_{2i}^k + G_i^k \hat{T}_i + G_i \hat{T}_i^k \Bigg) \\ \textit{subject to } & (5.41a),\ (5.41b),\ (5.41c) \textit{ and } (5.50) \end{aligned}$$

- ***Step 4:*** *If a stopping criteria is satisfied, then exit. Otherwise, set*

$$\tilde{P}_i^k = \tilde{P}_i, \hat{Q}_{1i}^k = \hat{Q}_{1i}, \hat{Q}_{2i}^k = \hat{Q}_{2i}, \hat{S}_{1i}^k = \hat{S}_{1i}, \hat{S}_{2i}^k = \hat{S}_{2i}, \hat{T}_i^k = \hat{T}_i,$$
$$J_i^k = J_i, E_{1i}^k = E_{1i}, E_{2i}^k = E_{2i}, F_{1i}^k = F_{1i}, F_{2i}^k = F_{2i}, G_i^k = G_i$$

and set $k = k+1$ and go to Step 3.

Remark 8. *The previous algorithm can be stated for Theorem 7 as well as Corollary 1 by defining constants α_i, for each $i \in \mathscr{I}$ such that all $(A_i - \alpha_i I)$ are Hurwitz stable in Step 1. Note that, $Q_{1i}, Q_{2i}, S_{1i}, S_{2i}, E_{1i}, E_{2i}, F_{1i}, F_{2i}$ do not exist in Corollary 1; the feasible sets and objective functions of the minimization problems are going to be altered in that manner. Also, inequality constraints of the LMI problem and the minimization problem will be* (5.29a), (5.29b), (5.29c) *and* (5.48) *for Theorem 7 and* (5.38a), (5.38b), (5.38c) *and* (5.48) *for Corollary 1.*

5.3 Numerical Examples

Some numerical examples are presented in order to show the effects of the Algorithm 2 for Theorem 7 and Corollary 1.

Example 11. *Consider the time-delay switched system* (5.1) *with matrices as in [40]*

$$A_1 = \begin{bmatrix} -1.9 & -1.5 & -1.2 \\ 0.7 & -1.6 & 0.5 \\ -1.3 & 0.5 & -1.1 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & -1.2 & -1.5 \\ 0.2 & -1.5 & 0.4 \\ -0.7 & 1.1 & -1.2 \end{bmatrix},$$

$$A_{d1} = \begin{bmatrix} 0.2 & 0 & 0.1 \\ 0.1 & 0.3 & 0.1 \\ 0.3 & 0.1 & 0.2 \end{bmatrix}, A_{d2} = \begin{bmatrix} 0.2 & 0 & 0 \\ 0.1 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 \\ 0.5 \\ 2 \end{bmatrix}, B_2 = \begin{bmatrix} 0.5 \\ 0.7 \\ 1.5 \end{bmatrix}, B_{w1} = \begin{bmatrix} 0.3 \\ 0.5 \\ 0.2 \end{bmatrix}, B_{w2} = \begin{bmatrix} 0.4 \\ 0.2 \\ 0.3 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} -1.2 & 0.5 & 0.9 \end{bmatrix}, C_2 = \begin{bmatrix} -1 & 1.2 & 0.5 \end{bmatrix}.$$

In order to apply Algorithm 2, following constants are given

$$\alpha_1 = \alpha_2 = 0.01,\ h_1 = 0,\ h_2 = 0.2,\ h_d = 0.01,\ R = I,\ \delta = 1,$$

$$\delta' = 1,\ \varepsilon = 30,\ \mu = 1.001,\ d = 0.01,\ T_f = 10 \text{ and } N_0 = 0.$$

By solving the minimization problem described in Algorithm 2 for Theorem 7 in LMI Toolbox of MATLAB, we get the controller and observer gains

$$K_1 = \begin{bmatrix} 3.2191 & -3.6818 & 4.1374 \end{bmatrix}, K_2 = \begin{bmatrix} 2.3492 & -0.4347 & 3.3150 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} 0.0126 & -0.0007 & -0.0077 \end{bmatrix}^T, L_2 = \begin{bmatrix} 0.0103 & -0.0047 & -0.0028 \end{bmatrix}^T$$

and ADT is obtained as $\tau_a^* = 0.0481$. *Also by solving the minimization problem in Algorithm 2 for Corollary 1, we get the controller and observer gains*

$$K_1 = \begin{bmatrix} 90.5968 & 6.9892 & 98.5752 \end{bmatrix}, K_2 = \begin{bmatrix} 57.6511 & 31.5745 & 65.3609 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} 0.0002 & 0 & -0.0001 \end{bmatrix}^T, L_2 = \begin{bmatrix} 0.0002 & -0.0001 & 0 \end{bmatrix}^T.$$

with the ADT $\tau_a^* = 0.2851$. *Comparison table is given in Table 5.1.*

Table 5.1 : τ_a^* Comparison

	τ_a^*
In [40]	1.8219
Algorithm 2 for Theorem 7	0.0481
Algorithm 2 for Corollary 1	0.2851

For various μ, α *and* h_2, τ_a^* *values are presented in Table 5.2, 5.3, 5.4 and 5.5 for the problem in Algorithm 2 for Theorem 7. As it can be seen in Table 5.2, as* μ *decreases,* τ_a^* *also decreases.*

From Table 5.3, it is seen that for small values of α_i *of* $i = 1,2$ *smaller ADTs are obtained.*

Table 5.2 : τ_a^* Comparison for Various μ.

	$\mu = 1.1$	$\mu = 1.05$	$\mu = 1.01$	$\mu = 1.005$	$\mu = 1.001$
τ_a^*	4.5908	2.3501	0.4793	0.2402	0.0481

Table 5.3 : τ_a^* Comparison for Various α_i's.

	$\alpha_i = 0.001$	$\alpha_i = 0.01$	$\alpha_i = 0.1$	$\alpha_i = 1$
τ_a^*	0.0445	0.0481	0.3059	Infeas.

If the lower bound h_1 of delay is fixed, the system (5.1) *can adapt to frequent switchings among the subsystems which is presented in Table 5.4 and 5.5.*
On the other hand, if the upper bound of delay h_2 is fixed, an increase in the lower bound h_1 forces the system (5.1) *to stay more in the subsystems, which can be seen in Table 5.6.*

Remark 9. *We should note that all of the subsystems in Example 1 are Hurwitz stable. Corresponding study in [40], switching among unstable subsystems and among mixed stable subsystems are not considered, although there is a solid restriction on the choice of α_i, which takes part in the exponential coefficient of the Lyapunov-Krasovskii functional's integral term.*

In the following examples, all the possible cases are proposed to demonstrate the validity of the theorems/algorithms.

Example 12. *Consider the switched system with time delay* (5.1) *with Hurwitz unstable subsystems*

$$A_1 = \begin{bmatrix} 0.01 & 0 \\ 0 & -0.05 \end{bmatrix}, A_2 = \begin{bmatrix} 0.02 & 0.3 \\ -0.1 & 0 \end{bmatrix},$$
$$A_{d1} = \begin{bmatrix} -0.06 & 0 \\ 0.06 & -0.03 \end{bmatrix}, A_{d2} = \begin{bmatrix} -0.03 & 0 \\ -0.09 & -0.12 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} -1 & 0.3 \end{bmatrix}^T, B_2 = \begin{bmatrix} -0.7 & 1 \end{bmatrix}^T, B_{w1} = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}^T, B_{w2} = \begin{bmatrix} 0.15 & 0.3 \end{bmatrix}^T,$$
$$C_1 = \begin{bmatrix} -0.5 & 1.3 \end{bmatrix}, C_2 = \begin{bmatrix} 0.6 & 0.7 \end{bmatrix}.$$

with constants

$$\alpha_1 = 0.72,\ \alpha_2 = 0.90,\ h_1 = 0.1,\ h_2 = 0.3,\ h_d = 0.01,\ R = I,\ \delta = 1$$
$$\delta' = 1,\ \varepsilon = 16,\ \mu = 1.001,\ d = 0.01,\ T_f = 1 \text{ and } N_0 = 0.$$

Table 5.4 : For Fixed $h_1 = 0$ and Various h_2.

	$h_1 = 0$ $h_2 = 0.2$	$h_1 = 0$ $h_2 = 0.4$	$h_1 = 0$ $h_2 = 0.6$	$h_1 = 0$ $h_2 = 0.8$	$h_1 = 0$ $h_2 = 1.0$
τ_a^*	0.0481	0.0578	0.0701	0.0866	0.1087

Table 5.5 : For Fixed $h_1 = 0.1$ and Various h_2.

	$h_1 = 0.1$ $h_2 = 0.3$	$h_1 = 0.1$ $h_2 = 0.5$	$h_1 = 0.1$ $h_2 = 0.7$	$h_1 = 0.1$ $h_2 = 0.9$	$h_1 = 0.1$ $h_2 = 1.1$
τ_a^*	0.0545	0.0648	0.0780	0.0958	0.1194

satisfying $(A_i - \alpha_i I) < 0$. *By Algorithm 2 for Theorem 7, we get a feasible solution with controller and observer gains*

$$K_1 = \begin{bmatrix} -0.0991 & 0.0926 \end{bmatrix}, \; K_2 = \begin{bmatrix} -0.3813 & 0.1678 \end{bmatrix},$$
$$L_1 = \begin{bmatrix} 0.0346 & -0.0898 \end{bmatrix}^T, \; L_2 = \begin{bmatrix} -0.0416 & -0.0484 \end{bmatrix}^T$$

with the ADT $\tau_a^* = 0.1842$.

The initial condition function is taken as $\phi(t) = x_0 = [0.6\ 0.5]^T$ *for all* $t \in [-h_2, 0]$ *and* $\hat{x}_0 = [0.55\ 0.55]^T$, *the time varying delay is taken as* $h(t) = h_1 + h_{12}\sin\big((h_d/h_{12})t\big)$ *to satisfy* (4.3) *and the disturbance is taken as* $w(t) = 0.04\sin(t)$ *to satisfy* (4.4). *Simulation is made under a periodic switching shown in Figure 5.1.*

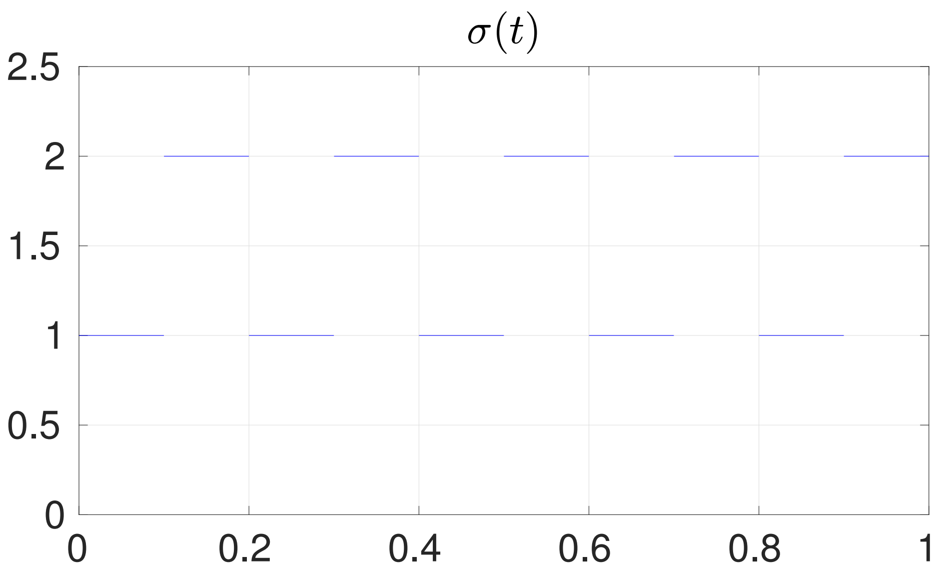

Figure 5.1 : Switching Signal $\sigma(t)$.

The phase portraits of the switched system and the observer are presented in Figure 5.2.

Table 5.6 : τ_a^* Comparison for Fixed $h_2 = 1$ and Various h_1.

	$h_1 = 0$ $h_2 = 1.0$	$h_1 = 0.2$ $h_2 = 1.0$	$h_1 = 0.4$ $h_2 = 1.0$	$h_1 = 0.6$ $h_2 = 1.0$	$h_1 = 0.8$ $h_2 = 1.0$
τ_a^*	0.1087	0.1054	Infeas.	Infeas.	0.1197

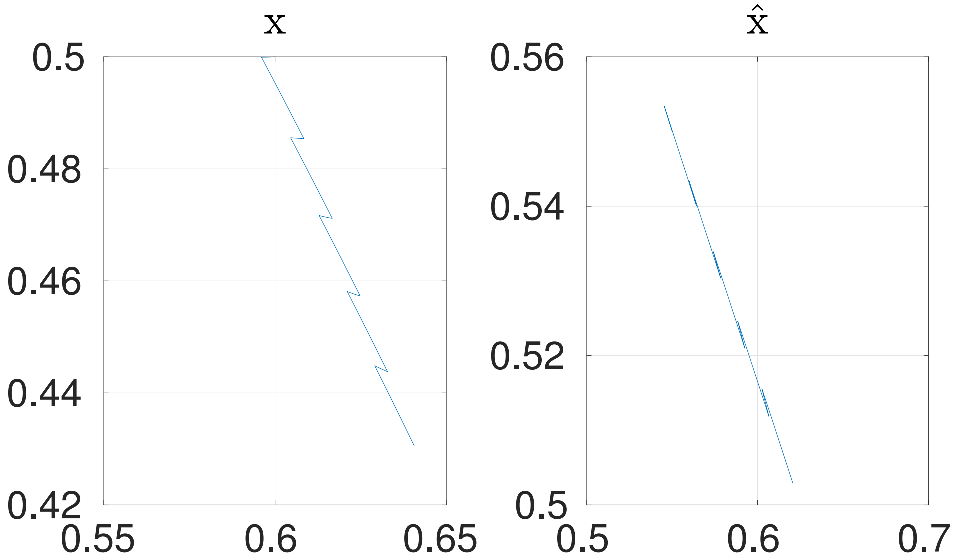

Figure 5.2 : Phase Portraits of the Switched System and the Observer.

It is seen that for $\sup_{s\in[-h_2,0]}\{x^T(s)Rx(s)\} = 0.61 < \delta = 1$, $\sup_{t\in[0,T_f]}\{x^T(t)Rx(t)\} = 0.61 < \varepsilon = 16$, *so FT boundedness is satisfied.*

Example 13. *Consider the switched system with time delay* (5.1) *with two subsystems*

$$A_1 = \begin{bmatrix} 0.4 & 0 \\ 0 & -0.34 \end{bmatrix},\ A_2 = \begin{bmatrix} -1.6 & 0 \\ 0 & -0.14 \end{bmatrix},$$

$$A_{d1} = \begin{bmatrix} -0.06 & 0 \\ 0.06 & -0.03 \end{bmatrix},\ A_{d2} = \begin{bmatrix} -0.03 & 0 \\ -0.69 & -0.12 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.4 & 0.1 \end{bmatrix}^T,\ B_2 = \begin{bmatrix} 0.3 & 0.15 \end{bmatrix}^T,\ B_{w1} = \begin{bmatrix} 0.1 & 0.4 \end{bmatrix}^T,\ B_{w2} = \begin{bmatrix} 0.15 & 0.3 \end{bmatrix}^T,$$

$$C_1 = \begin{bmatrix} -1.2 & 0.9 \end{bmatrix},\ C_2 = \begin{bmatrix} -1 & 0.5 \end{bmatrix}.$$

Note that, A_1 *is Hurwitz unstable and* A_2 *is Hurwitz stable. The activation times of the unstable and unstable subsystems are chosen as* $T^+ = 0.6$ *and* $T^- = 1.4$*, respectively. The constants*

$$\psi_1 = 0.5,\ \psi_2 = -0.05,\ h_1 = 0,\ h_2 = 0.1,\ h_d = 0.01,\ R = I,\ \delta = 4$$

$$\delta' = 4,\ \varepsilon = 25,\ \mu = 1.01,\ d = 0.01,\ T_f = 2,\ N_0 = 0.$$

are chosen to satisfy $(A_i - \psi_i I) < 0$*. So, by Algorithm 2 for Theorem 8, we get a feasible solution with controller and observer gains*

$$K_1 = \begin{bmatrix} 2.8156 & 3.1708 \end{bmatrix},\ K_2 = \begin{bmatrix} -21.3321 & 6.0683 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} 0.0035 & -0.0050 \end{bmatrix}^T,\ L_2 = \begin{bmatrix} 0.0029 & -0.0028 \end{bmatrix}^T$$

with the ADT $\tau_a^* = 0.4526$.

The initial condition function is taken as $\phi(t) = x_0 = [1.0\ 1.0]^T$ for all $t \in [-h_2, 0]$ and $\hat{x}_0 = [0.1\ 0.1]^T$, the time varying delay is taken as $h(t) = h_1 + h_{12}\sin\big((h_d/h_{12})t\big)$ to satisfy (4.3) *and the disturbance is taken as $w(t) = 0.04\sin(t)$ to satisfy* (4.4)*. Simulation is made under a non-periodic switching shown in Figure 5.3.*

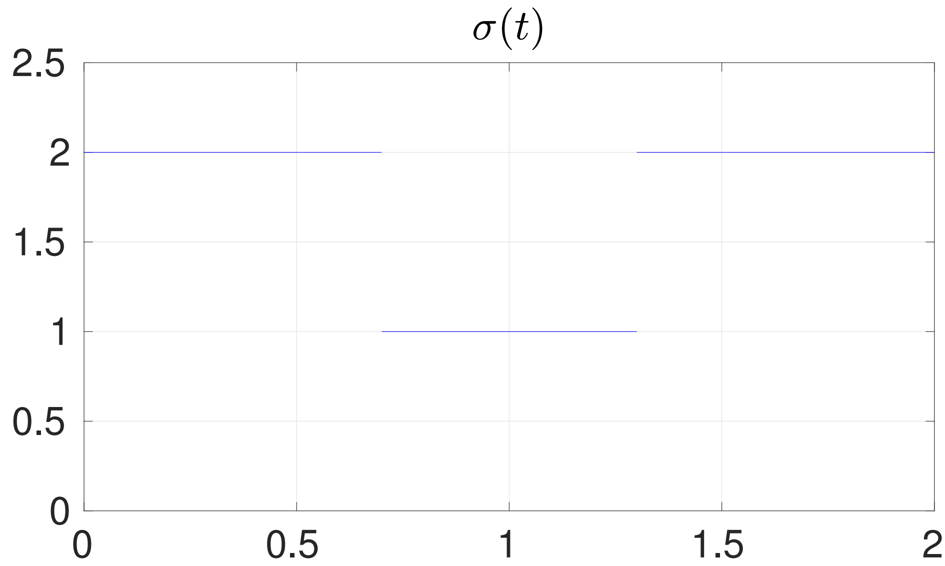

Figure 5.3 : Switching Signal $\sigma(t)$.

The state responses of the switched system and the observer are presented in Figure 5.4.

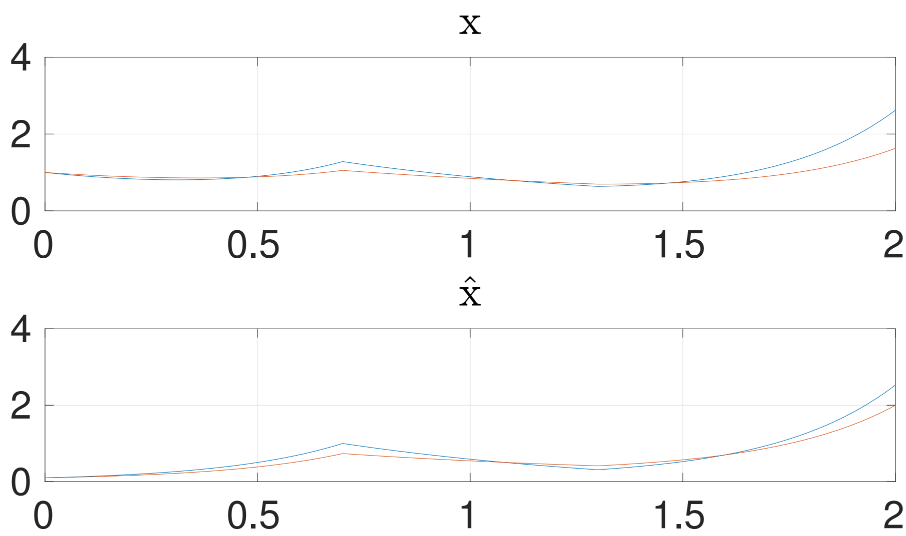

Figure 5.4 : State Responses of the Switched System and the Observer.

It is seen that for $\sup_{s\in[-h_2,0]}\left\{x^T(s)Rx(s)\right\} = 2 < \delta = 4$, $\sup_{t\in[0,T_f]}\left\{x^T(t)Rx(t)\right\} = 9.5105 < \varepsilon = 25$, so FT boundedness is satisfied.

6. OBSERVER-BASED CONTROL FOR H$_\infty$ FINITE-TIME BOUNDEDNESS

In this part, observer-based controller is designed for H$_\infty$ finite-time boundedness of interval time-delay switched systems having mixed stable matrices of the state vector. A numerical example is given for the effectiveness and validity of the proposed solutions.

6.1 Problem Statement

Consider a switched linear system with an interval time-varying delay in the state vector, where

$$\begin{aligned} \dot{x}(t) =& A_{\sigma(t)}x(t) + A_{d\sigma(t)}x(t-h(t)) + B_{\sigma(t)}u(t) + B_{w\sigma(t)}w(t), \\ y(t) =& C_{\sigma(t)}x(t) \\ z(t) =& D_{\sigma(t)}x(t) \end{aligned} \tag{6.1}$$

with the initial conditon function

$$x(t) = \phi(t),\ t \in [-h_2, 0]. \tag{6.2}$$

Here $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^m$ the control input, $y(t) \in \mathbf{R}^q$ and $z(t) \in \mathbf{R}^r$ the measurement output and the controlled output respectively. $A_{\sigma(t)}$, $A_{d\sigma(t)}$, $B_{\sigma(t)}$, $B_{w\sigma(t)}$, $C_{\sigma(t)}$ and $D_{\sigma(t)}$ are real constant matrices of appropriate dimensions, $\phi \in \mathscr{C}([-h_2, 0], \mathbf{R})$ is the initial function, h is the delay function satisfying (4.3) and $w(t)$ is the exogenous disturbance satisfying (4.4). Consider the observer based feedback controller

$$\begin{aligned} \dot{\hat{x}}(t) &= A_{\sigma(t)}\hat{x}(t) + B_{\sigma(t)}u(t) + L_{\sigma(t)}(y(t) - \hat{y}(t)), \\ \hat{y}(t) &= C_{\sigma(t)}\hat{x}(t), \\ \hat{x}(t) &= 0,\ \forall t \in [-h_2, 0], \end{aligned} \tag{6.3}$$

and the control law

$$u(t) = -K_{\sigma(t)}\hat{x}(t). \tag{6.4}$$

Here, $K_{\sigma(t)}$ and $L_{\sigma(t)}$ are controller and observer gains, respectively. Define an error vector $e(t) = x(t) - \hat{x}(t)$. The closed-loop system will be

$$\begin{aligned} \dot{x}(t) =& A_{K\sigma(t)}x(t) + A_{d\sigma(t)}x(t-h(t)) + B_{\sigma(t)}K_{\sigma(t)}e(t) + B_{w\sigma(t)}w(t), \\ \dot{e}(t) =& A_{L\sigma(t)}e(t) + A_{d\sigma(t)}x(t-h(t)) + B_{w\sigma(t)}w(t) \end{aligned} \tag{6.5}$$

where

$$\begin{aligned} A_{K\sigma(t)} &= A_{\sigma(t)} - B_{\sigma(t)}K_{\sigma(t)}, \\ A_{L\sigma(t)} &= A_{\sigma(t)} - L_{\sigma(t)}C_{\sigma(t)}. \end{aligned} \tag{6.6}$$

For a prescribed scalar $\gamma > 0$, the following performance index is defined

$$J_{T_f} = \int_0^{T_f} (z^T z - \gamma^2 w^T w) d\tau. \tag{6.7}$$

6.2 H∞ FT Stabilization

In this section, H∞ FT stabilization of time-delay switched system with observer-based control is considered.

Theorem 9. *The switched system* (6.1) *is H_∞ FT bounded with respect to $(\delta, \varepsilon, T_f, d, R)$, if there exist a set of symmetric matrices for every i^{th} subsystem $P_i > 0$, $Q_{1i} > 0$, $Q_{2i} > 0$, $S_{1i} > 0$, $S_{2i} > 0$, $T_i > 0$, M_{1i}, M_{2i}, N_{1i} and N_{2i} and scalars $\alpha_i \geq 0$ satisfying*

$$\Lambda_i = \begin{bmatrix} \Theta_i & -M_i & -N_i & Z_{1i}^T & Z_{2i}^T \\ * & -e^{2\psi_i h_2}S_{2i} & 0 & 0 & 0 \\ * & * & -e^{2\psi_i h_2}S_{2i} & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -I \end{bmatrix} < 0 \tag{6.8a}$$

$$\gamma^2 d \leq \lambda_1 e^{-2\alpha_{max}T_f}\varepsilon \tag{6.8b}$$

$$P_j \leq \mu P_i,\ Q_{kj} \leq \mu Q_{ki}, S_{kj} \leq \mu S_{ki}, T_j \leq \mu T_i, \tag{6.8c}$$

for $i, j \in \mathscr{I}$ and $k = 1,2$, where

$$\Theta_i = \begin{bmatrix} \Omega_{11,i} & \Omega_{12,i} & \Omega_{13,i} & -N_{1i} & B_iY_i & B_{wi} & \Omega_{17,i} \\ * & \Omega_{22,i} & M_{2i} & -N_{2i} & P_iA_{di}^T & 0 & P_iA_{di}^T \\ * & * & \Omega_{33,i} & 0 & 0 & 0 & 0 \\ * & * & * & \Omega_{44,i} & 0 & 0 & 0 \\ * & * & * & * & \Omega_{55,i} & B_{wi} & Y_i^T B_i^T \\ * & * & * & * & * & -\gamma^2 I & B_{wi}^T \\ * & * & * & * & * & * & \Omega_{77,i} \end{bmatrix} \tag{6.9}$$

with

$$Z_{1i} = [D_iP_i\ 0\ 0\ 0\ 0\ 0\ 0]^T,\ Z_{2i} = [0\ 0\ 0\ 0\ C_iP_i\ 0\ 0]^T$$

and the ADT of the switching signal satisfies

$$\tau_a > \tau_a^* = \frac{T_f \ln\mu}{\ln(\lambda_1\varepsilon) - \ln(\gamma^2 d) - 2\alpha_{max}T_f - N_0\ln\mu} \tag{6.10}$$

The gain matrices K_i and L_i of controller and observer are same as (5.34).

Proof. From Lemma 5, the conditions (5.6a) and (5.6b) provides that the switched system (6.1) is FT bounded with respect to $(\delta, \varepsilon, T_f, d, R)$. We choose the Lyapunov-Krasovskii functional as it is in (5.9) and (5.10) and define

$$\Gamma(t) = -z^T(t)z(t) + \gamma^2 w^T(t)w(t). \tag{6.11}$$

By derivation, we get

$$\dot{V}_i(x(t)) - 2\alpha_i V_i(x(t)) \leq \xi^T(t)\Pi_i\xi(t) + \Gamma(t) \tag{6.12}$$

where

$$\Pi_i = \begin{bmatrix} \Pi_{11,i} & \Pi_{12,i} & \Pi_{13,i} & -\overline{N}_{1i} & \Pi_{15,i} & P_i^{-1}B_{wi} & A_{Ki}P_i^{-1} \\ * & \Pi_{22,i} & \overline{M}_{2i} & -\overline{N}_{2i} & A_{di}^TP_i^{-1} & 0 & A_{di}^TP_i^{-1} \\ * & * & \Pi_{33,i} & 0 & 0 & 0 & 0 \\ * & * & * & \Pi_{44,i} & 0 & 0 & 0 \\ * & * & * & * & \Pi_{55,i} & P_i^{-1}B_{wi} & \Pi_{57,i} \\ * & * & * & * & * & -\gamma^2 I & B_{wi}^TP_i^{-1} \\ * & * & * & * & * & * & \Pi_{77,i} \end{bmatrix} \tag{6.13}$$

with

$$\Pi_{11,i} = P_i^{-1}A_{Ki} + A_{Ki}^TP_i^{-1} + \overline{Q}_{1i} + \overline{Q}_{2i} - e^{2\alpha_i h_1}\overline{S}_{1i} - 2\alpha_i P_i^{-1} + \overline{T}_i + D_i^TD_i$$

and the remaining entries of Π_i are of $\Pi_{jk,i} = \Xi_{jk,i}$. By pre- and post-multiplying both sides with $\mathscr{D}_i$, using Schur Complement and applying $\Lambda_i < 0$ conditions in (6.8a) the following inequality is obtained.

$$\dot{V}_i(x(t)) - 2\alpha_i V_i(x(t)) \leq \Gamma(t). \tag{6.14}$$

By applying Grönwall's lemma on $t \in [t_k, t_{k+1})$,

$$V_{\sigma(t)}(x(t)) \leq e^{2\alpha_{\sigma(t_k)}(t-t_k)}V_{\sigma(t_k)}(x(t_k)) + \int_{t_k}^{t} e^{2\alpha_{\sigma(t_k)}(t-s)}\Gamma(s)ds. \tag{6.15}$$

From the inequality constraints in (6.8c) and by assuming $\sigma(t_k) = i$ and $\sigma(t_k^-) = j$, (5.22) is obtained. If Grönwall's lemma and (5.22) is applied until $[0,t_1)$ iteratively, we get

$$V_{\sigma(t)}(x(t)) \leq e^{2\alpha_{max}T_f}\mu^N \left(V_{\sigma(0)}(x(0)) + \int_0^t \Gamma(s)ds \right) \tag{6.16}$$

where N denotes the switching number of $\sigma(t)$ over $(0,T_f)$. By zero initial condition

$$0 \leq \int_0^t \Gamma(s)ds \tag{6.17}$$

and setting $t = T_f$

$$\int_0^{T_f} z^T(s)z(s)ds < \gamma^2 \int_0^{T_f} w^T(s)w(s)ds \tag{6.18}$$

which tells that the switched system (6.1) is H_∞ FT bounded. □

Remark 10. *Algorithm 2 can also be stated for Theorem 9. Note that, $\Omega_{55,i}$ of Theorem 9 contains the term $A_iP_i + P_iA_i^T - 2\psi_iP_i$ so that, in order to get positive definite solutions for P_i satisfying the corresponding LMIs, $(A_i - \psi_iI)$ should be Hurwitz stable. Also, inequality constraints of the LMI problem and the minimization problem will be* (6.8a), (6.8b), (6.8c) *and* (5.48) *for Theorem 9. Notice also, W_i does not exist in Theorem 9; the feasible sets, constraints and objective functions of the minimization problem will be altered in that manner.*

6.3 Numerical Example

Example 14. *Consider the H_∞-control problem for the time-delay switched system* (6.1) *with matrices*

$$A_1 = \begin{bmatrix} -1.75 & 1.32 \\ -2.78 & -3.44 \end{bmatrix}, A_2 = \begin{bmatrix} -1.52 & -1.44 \\ 2.35 & -3.34 \end{bmatrix},$$
$$A_{d1} = \begin{bmatrix} 0.44 & 0.49 \\ -0.38 & -0.56 \end{bmatrix}, A_{d2} = \begin{bmatrix} 0.34 & -0.38 \\ 0.29 & 0.42 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0.82 \\ -1.26 \end{bmatrix}, B_2 = \begin{bmatrix} -0.11 \\ 0.55 \end{bmatrix}, B_{w1} = \begin{bmatrix} 0.56 \\ 0.79 \end{bmatrix}, B_{w2} = \begin{bmatrix} 0.30 \\ -0.30 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} -0.57 \\ 1.33 \end{bmatrix}^T, C_2 = \begin{bmatrix} 0.62 \\ 0.73 \end{bmatrix}^T, D_1 = \begin{bmatrix} 0.31 \\ -1.30 \end{bmatrix}^T, D_2 = \begin{bmatrix} -0.43 \\ 0.34 \end{bmatrix}^T.$$

In order to solve this system with the constants

$$\alpha_1 = 0.42,\ \alpha_2 = 0.34,\ h_1 = 0.1,\ h_2 = 0.3,\ h_d = 0.01,\ R = I,\ \delta = 1,$$
$$\delta' = 1,\ \varepsilon = 25,\ \mu = 1.1,\ d = 0.01,\ \gamma = 1.6,\ T_f = 5 \text{ and } N_0 = 0$$

Algorithm 2 for Theorem 9 is applied. The following controller and observer gains are obtained.

$$K_1 = \begin{bmatrix} 4.0529 & 4.2057 \end{bmatrix}, \; K_2 = \begin{bmatrix} 10.0924 & -5.7680 \end{bmatrix},$$
$$L_1 = \begin{bmatrix} 0.1104 & -0.2539 \end{bmatrix}^T, \; L_2 = \begin{bmatrix} -0.1304 & -0.1370 \end{bmatrix}^T.$$

The ADT is obtained as $\tau_a^* = 0.0744$.

Note that, even for large μ*, small* τ_a^* *can be obtained for* H_∞ *case. The reason stands in the denominator of the ADT formula in* (6.10). *Since the conditions* (6.8b) *and* (6.8c) *are relaxed to choose* γ *and* d*, smaller* τ_a^* *can be obtained depending on these constants.*

7. CONCLUSIONS

This dissertation investigates the FT stability of switched systems, FT boundedness and H_∞ FT boundedness of the switched systems with and without interval time-delay and disturbances.

First, FT stability of switched systems were analyzed by using vector and matrix norms and the results were presented in the third chapter. Sufficient conditions for FT stability were obtained. These conditions include the spectral properties of the subsystems, which were obtained by using Jordan decomposition. Possible activation numbers of the subsystems were deduced from these conditions. ADT conditions were presented by considering that all the subsystems have negative, positive and mixed spectral norm bounds. Numerical examples presented at the end of the third chapter showed that the number of activations of the subsystems can be adjusted to ensure FT stability and the proposed ADT bounds for different types of systems ensure FT stability of the switched system.

Second, the FT boundedness of the switched systems with interval time-delay and disturbances were analyzed based on a state-feedback controller. Sufficient conditions were obtained for system vector. Due to the nonconvex elements on these conditions, a cone-complementarity linearization was made. A numerical example was presented at the end of the fourth chapter.

Third, observer-based controller was proposed to ensure FT boundedness of switched linear systems having interval time-delay. Sufficient conditions and ADT bounds were presented in case of unstable and mixed stable subsystems in the fifth chapter. Cone-complementarity linearization method and algorithm were proposed for the calculation of the variables in ADT bound having nonconvex elements. Given numerical example demonstrated that applied controller enables the system to switch more frequently among the subsystems. On the other hand, the controller was extended to be applied to unstable and mixed stable subsystems and all the possible cases were presented with a numerical example.

Finally, observer based H_∞ controller was designed for the H_∞ FT boundedness of switched linear systems with interval time delay in the presence of disturbance. In this chapter, sufficient conditions were again obtained for the subsystem matrices to be mixed stable and the effectiveness and validity of the proposed conditions were shown on a numerical example.

Extending proposed conditions by relaxing the subconditions via mode-dependent stabilization analysis can create new directions in the future. Concerning the matrix condition number minimization results in the literature, the estimations in the ADT bounds can be improved further. Last but not least, FT input-to-state (FTISS) stability notions can be investigated to propose new frameworks to analyze nonlinear systems.

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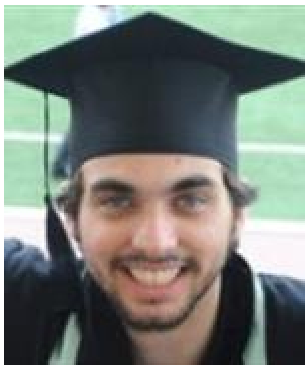
CURRICULUM VITAE



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PUBLICATIONS AND PRESENTATIONS ON THE THESIS:

- **Göksu, G.** and Başer, U. "Observer-Based Finite Time Control for Switched Linear Systems with Interval Time Delay" (In Turkish), 30^{th} *National Mathematics Symposium*, Atilim University, Ankara, Türkiye, 06.09.2017-09.09.2017.
- **Göksu, G.** and Başer, U. (2018). "Finite Time Control for Switched Linear Systems with Interval Time Delay", IFAC-PapersOnLine 51(14), 130-135.
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