ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY

A DYNAMICAL SYSTEMS APPROACH TO THE INTERPLAY BETWEEN TOBACCO SMOKERS, ELECTRONIC-CIGARETTE SMOKERS AND SMOKING QUITTERS

M.Sc. THESIS

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SİGARA İÇENLER, ELEKTRONİK SİGARA İÇENLER VE SİGARAYI BIRAKANLAR ARASINDAKİ ETKİLEŞİME YÖNELİK BİR DİNAMİK SİSTEMLER YAKLAŞIMI

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Esmanur YILDIZ, a M.Sc. student of ITU Graduate School of Science Engineering and Technology 509171208 successfully defended the thesis entitled "A DYNAMICAL SYSTEMS APPROACH TO THE INTERPLAY BETWEEN TOBACCO SMOKERS, ELECTRONIC-CIGARETTE SMOKERS AND SMOKING QUITTERS", which he/she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

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To my family,

FOREWORD

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A DYNAMICAL SYSTEMS APPROACH TO THE INTERPLAY BETWEEN TOBACCO SMOKERS, ELECTRONIC-CIGARETTE SMOKERS AND SMOKING QUITTERS

SUMMARY

In this thesis, the effect of e-cigarettes on smoking cessation is studied using the tools of dynamical systems theory. The purpose here is to examine this efficacy by representing and analysing a non-linear ODE system modelling potential smokers, tobacco smokers, e-cigarette smokers and quitters. Fundamental theories required for the interpretation of the behaviour of dynamical systems are given and some epidemiological models are analyzed.

The natural behaviour of some linear physical systems is quite predictable. Contrary to that, many natural phenomena are unpredictable. So, we employ non-linear systems which are more complex and are not exactly suitable for the solution to the problem at hand as opposed to linear systems. Non-linear systems are ubiquitous throughout the natural world. As presented in this work, biological systems can be represented by non-linear systems. For instance, several disease models are generally investigated by using non-linear mathematical models.

From a wider perspective, mathematical modelling is significant in describing the smoking cessation models. These models have been examined using ODE systems in view of the fact that we can analyse the spread and control of smoking with these systems.

It is well known that smoking is a common social phenomenon in today's world. Since smoking is an addiction, some individuals see the use of electronic cigarette as a way of quitting tobacco smoking. We also know that the prevalence of smoking extremely affects the social behaviour of people in a population. Therefore, peer pressure is quite substantial in starting or quitting the act of smoking.

This thesis consists of three chapters which are shaped by the above information.

In the first chapter, necessary elementary definitions and examples about stability analysis of dynamical systems are given. The classification of the equilibrium points is listed for two- and three-dimensional systems.

Chapter 2 covers the basic epidemiological models. Mathematical analysis of these epidemiological models is done all in detail. Importance of the basic reproduction number is examined with several infectious disease models. These models are diversified by adding different compartments or parameters. The analysis of these epidemic models is mostly done by non-dimensionalisation.

In Chapter 3, the proposed model to analyze the effect of electronic cigarettes on smoking cessation is given and described in detail. The standard term of "peer pressure" is used in this model. As part of the analysis, some theoretical results are obtained by the Next Generation Matrix Method and the Lyapunov Function Method.

Furthermore, some numerical simulations are plotted in the Mathematica using data obtained from the literature. Using this data, we verified our theoretical and numerical results by only slightly changing the parameters. We changed these parameters in a way to ensure that the equilibrium points are biologically meaningful.

In the conclusions section, the significance of the basic reproduction number is theoretically confirmed in numerical results. Theoretically observed bifurcation is confirmed in numerical illustrations. Thanks to the graphics used in our work, we noticed that the effect of using electronic cigarette on smoking cessation takes a long time to emerge.

As a conclusion of our work, we are under the impression that using e-cigarettes is quite effective to decrease the number of tobacco smokers, but our analysis indicates that it does not have a remarkable effect on the number of quitters. We conclude that e-cigarette is not a tool to quit tobacco smoking. We therefore recommend that, the society should be made conscious about the correct methods of quitting smoking and create awareness about the right methods of smoking cessation.

Keywords: Linear stability analysis, Lyapunov function, Dynamical systems, Epidemiological models.

SİGARA İÇENLER, ELEKTRONİK SİGARA İÇENLER VE SİGARAYI BIRAKANLAR ARASINDAKİ ETKİLEŞİME YÖNELİK BİR DİNAMİK SİSTEMLER YAKLAŞIMI

ÖZET

Bu tezde, elektronik sigara kullanımının sigara bırakma üzerindeki etkisi matematiksel modelleme yöntemleriyle oluşturulan dinamik bir sistem üzerinden incelenmiştir. Buradaki asıl amaç; sigara içme potansiyeline sahip bireyler, sigara içenler, elektronik sigara içenler ve sigarayı bırakan bireyler arasında yer alan dinamiği lineer olmayan bir adi diferansiyel denklem modeli ile temsil ederek bu modelin analizini yapmaktır. Bu sebeple, dinamik sistemlerin davranışlarını yorumlamak için gerekli bilgiler verilmiş ve bu bilgiler ışığında bazı salgın hastalık modellerinin dinamik yapıları incelenmiştir.

Hepimizin bildiği gibi, lineer denklemlerle modellenen fiziksel sistemlerin davranışları oldukça tahmin edilebilirken, birçok doğa olayı lineer olmayan bir modellemeyi gerekli kılar. Lineer olmayan bu sistemler, lineer sistemlere göre çok daha karmaşıktır ve bu denklemleri kesin çözümlerini elde etmek çoğu zaman mümkün değildir. Buna rağmen, bazı metotlar yardımıyla lineer olmayan dinamik sistemlerin davranışları hakkında tahminler yapılabilmektedir.

Lineer olmayan sistemler, yeryüzünde gerçekleşen doğal olayların hemen hemen hepsinde yer alır. Bu çalışmada da yer verildiği üzere, biyolojik sistemler lineer olmayan dinamik sistemler aracılığıyla temsil edilebilirler. Birçok bulaşıcı hastalık matematiksel modelleme ile temsil edilmektedir. Örneğin, veba ve grip bu yolla modellendirilmiş salgın hastalıklardandır.

Daha geniş bir perspektiften bakacak olursak, matematiksel modelleme yöntemi sadece salgın hastalıkların ve sigara bırakma modellerinin temsilinde değil farklı dinamiklerin modellenmesinde de kullanılmaktadır. Bu modellemelere ek olarak; alkol, eroin, uyuşturucu madde kullanımları, iklim-bitki örtüsü ve av-avcı dinamikleri de benzer metotlarla oluşturulmaktadır. Ayrıca, oluşturulan bu adi diferansiyel denklem sistemleriyle bahsedilen salgın modellerinin kontrolünün analizi yapılabilmektedir.

Günümüz dünyasında sigara kullanımının yaygın bir sosyal olay olduğu herkes tarafından iyi bilinmektedir. Sigara içmek, vücuttaki hemen hemen her organa zarar verir ve çeşitli hastalıklara sebep olur. Sigara içmek bir bağımlılıktır ve bu bağımlılıktan kurtulabilmek için çeşitli tedavi yöntemleri vardır. Sigara bıraktırma programları, nikotin çikletleri şeklindeki tedavi yöntemleri var olduğu gibi toplumdaki bazı bireyler elektronik sigara kullanmayı da sigarayı bırakmak için bir yol olarak görmektedir. Tütün sigarası nikotin, arsenik ve karbonmonoksit de dahil olmak üzere 7000'den fazla bileşene sahip iken elektronik sigara yalnızca nikotin içermektedir. Diğer bir ifadeyle elektronik sigara, tütün sigarasına oranla daha az zararlıdır. Bu sebepler doğrultusunda, bu çalışmadaki modelde elektronik sigara kullanımının sigara bırakma üzerindeki etkisi analiz edilmiştir.

Akran baskısı, bireylerin sigara içme deneyimini erken yaşta gerçekleştirmelerinde etkili olduğu gibi ilerleyen yaşlarda sigara içmeye devam etme istekleri üzerinde de

etkili olan önemli bir faktör olarak kabul edilmektedir. Diğer bir ifadeyle, sigara içme yaygınlığının artışında toplumdaki bireylerin sosyal davranışlarının etkisinin büyük olduğu oldukça açıktır. Bu sebepler doğrultusunda, bu çalışmada önerilen model "akran baskısı" göz önünde bulundurularak oluşturulmuştur.

Yukarıda bahsedildiği gibi lineer olmayan sistemlerdeki dinamik geçişlerin incelendiği bu tez çalışması üç bölümden oluşmaktadır.

Birinci bölümde, dinamik sistemlerin kararlılık analizini yapabilmek için gerekli olan temel tanımlar verilmiş ve birkaç diferansiyel denklem sisteminin kararlılık analizi yapılmıştır. İki boyutlu ve üç boyutlu sistemlere ait olan denge noktalarının sınıflandırılması yapılmıştır. Ayrıca Lyapunov Fonksiyon Metodu hakkında bilgi verilmiş ve uygun Lyapunov fonksiyonunun seçilmesiyle basit bir diferansiyel denklem sisteminin kararlılık analizi yapılmıştır.

İkinci bölüm, temel salgın hastalık modellerini kapsamaktadır. Bu bölümde SI, SIS, SIR, SIRS ve SEIR isimleri ile nitelendirilen temel hastalık modelleri verilmiş ve bu modellerin detaylı analizleri yapılmıştır. Bu modellemelere ek olarak, sadece doğal yollarla gerçekleşen ölümlerin yer aldığı ve hem doğal hem de hastalığa bağlı yollarla gerçekleşen ölümlerin yer aldığı temel SIR modelleri üzerinde çalışılmış ve yine bu modellerin kararlı olma durumları analiz edilmiştir. Salgın hastalıkların kontrol metotlarından biri olan aşılama yöntemi göz önünde bulundurularak oluşturulan bir SIR modelinin detayları verilmiştir. Bu modelleme ile salgın hastalıkların yayılımını kontrol etmek için etkili olduğu düşünülen bazı stratejilerden bahsedilmiştir. Bahsedilen hastalık modellerinin tümünün incelenmesiyle birlikte temel üreme oranının önemi ortaya konmuştur. Dinamik geçişlerin incelendiği bu hastalık modellerinin kararlılık analizi yapılırken boyutsuzlaştırma yöntemi kullanılmıştır.

Üçüncü bölümde, elektronik sigaranın sigarayı bırakma üzerindeki etkisini analiz etmek için önerilen model tüm detaylarıyla birlikte verilmiştir. Bu modelde, sigara içenlerin olduğu gruptan elektronik sigara içenlerin olduğu gruba geçiş “akran baskısı” terimiyle modellenmiştir. Yapılan kararlılık analizinde Lyapunov Fonksiyon Yöntemi’nden yararlanılmış ve temel üreme oranının saptanmasında bir metot olarak kullanılan Yeni Nesil Matris Yöntemi ile sistemin temel üreme oranı ayrıca gösterilmiştir. Ayrıca; bazı nümerik simülasyonlar, literatürde yer alan bazı veriler kullanılarak Mathematica’da çizdirilmiştir. Bu veriler üzerinde oldukça küçük değişiklikler yapılmış ve bu verilerle oluşturulan grafiklerle de teorik sonuçların doğruluğu gösterilmiştir. Sistemdeki denge noktalarının biyolojik olarak anlamlı olmasını sağlamak amacıyla bu veriler üzerinde oldukça küçük değişiklikler yapılmıştır.

Sonuçların verildiği kısımda ise temel üreme oranının öneminin hem teorik hem de nümerik olarak gözlemlendiğinden bahsedilmiştir. Yapılan nümerik simülasyonların, sistemde çatallanma olduğuna dair ortaya koyduğumuz teorik sonuçları doğruladığı görülmüştür. Çalışmalarımızda elde edilen grafikler sayesinde, elektronik sigara kullanımının sigarayı bırakma üzerinde kayda değer bir etki bırakması için çok uzun bir zaman geçmesi gerektiği gözlemlenmiştir.

Çalışmalarımızın bir sonucu olarak, elektronik sigara kullanımının tütün sigara kullanımını bir dereceye kadar azalttığını ancak sigarayı bırakanların sayısını her zaman artırmadığını gözlemledik. Bu nedenle; elektronik sigara kullanımının, tütün sigara kullanımını kontrol etmek için bir araç olarak görülmemesi gerektiği

sonucuna vardık. Dolayısıyla; elektronik sigara kullanımının bir tedavi yöntemi olarak görülmesinin önüne geçebilmek için, toplumun sigarayı bırakma metotları hakkında doğru şekilde bilinçlendirilmesi gerektiğini düşünüyor ve bununla ilgili farkındalık oluşturulacak çalışmalarda bulunulmasını tavsiye ediyoruz.

Anahtar Kelimeler: Lineer kararlılık analizi, Lyapunov fonksiyonu, Dinamik sistemler, Salgın hastalık modelleri.

1. INTRODUCTION

In this chapter, we give the necessary elementary definitions and examples about stability analysis of dynamical systems. Theories on dynamical systems have been well established in many fundamental works. For more details, see [3,4,18–20,24–29].

1.1 Basic Definitions

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and consider the ODE

$$\frac{dx}{dt} = f(x), \tag{1.1}$$

with the initial condition

$$x(0) = x_0 \tag{1.2}$$

In (1.1), the time derivative is also represented as $\dot{x} = \frac{dx}{dt}$. Besides, a differential equation of the form (1.1) is called **autonomous** since the independent variable t can not be explicitly found.

From Basic Theory of ODE's, we know that if f is Lipschitz continuous in a neighbourhood of x_0, then (1.1) and (1.2) has a unique solution

$$x : I \to \mathbb{R}^n, \quad x(0) = x_0$$

where $I \subset \mathbb{R}$ is the maximal interval of existence. Here we will assume that $[0,\infty) \subset I$. This ensures that we can take the limit of the solution as t goes to infinity.

Let $x(t,x_0)$ be the **unique solution** of (1.1) and (1.2) and x^* be an **equilibrium solution** of (1.1) that is

$$f(x^*) = 0.$$

This means x^* does not change in time. Also note that for an equilibrium solution

$$x(t,x^*) = x^*, \qquad \forall t \geq 0.$$

An equilibrium solution is also known as a **fixed point**, **critical point**, **singular point**, or **steady-state solution**. Moreover, it is common to use the term "fixed point" when referring to a map and "equilibrium" when referring to a flow.

Example 1.1.1 *As an example, let us find the fixed points of the differential equation*

$$\dot{x} = x^2 - 4.$$

Solution 1.1.1 *The differential equation, which is given above, can be written as*

$$\frac{dx}{dt} = (x-2)(x+2).$$

Since the definition of a fixed point is $\frac{dx}{dt} = 0$*, we get*

$$(x-2)(x+2) = 0$$

Thus, the differential equation has two fixed points which are $x_1^* = -2$ *and* $x_2^* = 2$*.*

Definition 1.1.1 *The derivative of a map* $f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ *at an x is written as* $Df(x)$ *and organized into an* $m \times n$ *matrix called the partial derivative matrix or* ***Jacobian matrix*** *of* f *at x.*

$$Df(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Definition 1.1.2 *Let* A *be an* $n \times n$ *square matrix.* λ *is called an* ***eigenvalue*** *of* $A = (a_{ij})$ *if there exists a nonzero column vector x providing*

$$Ax = \lambda x$$

or, equivalently,

$$(A - \lambda I)x = 0, \tag{1.3}$$

where λ *is scalar and I is* $n \times n$ *identity matrix.*

The nonzero vector x is called an **eigenvector** of A corresponding to the eigenvalue λ. By definition, for an eigenvector x, Ax must be a scalar multiple of x. The equation given by (1.3), has a solution if and only if the characteristic polynomial of A is a singular matrix, that is, equivalently

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} - \lambda & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} - \lambda \end{vmatrix} = 0.$$

In other words, the roots of the characteristic polynomial are exactly the eigenvalues of A.

1.2 Stability Analysis of the Equilibrium Points

We are interested in interpreting the long time behaviour of fixed points of the dynamical systems. Particularly, the real objective is trying to understand whether a given equilibrium solution is stable or not. We make the following definitions, see also [21].

Definition 1.2.1 *The equilibrium solution x^* of* (1.1) *is called* ***neutrally stable*** *if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if*

$$\|x_0 - x^*\| < \delta \implies \|x(t, x_0) - x^*\| < \varepsilon, \quad \forall t \geq 0.$$

Definition 1.2.2 *The equilibrium solution x^* of* (1.1) *is called* ***unstable*** *if the solution is not neutrally stable.*

Definition 1.2.3 *The equilibrium solution x^* of* (1.1) *is* ***locally asymptotically stable*** *if*

1. *It is neutrally stable, and*
2. *There exists a $\delta(t_0) > 0$ such that*

$$\|x_0 - x^*\| < \delta \implies \lim_{t \to \infty} \|x(t) - x^*\| = 0.$$

As in [22] and [23], the above definitions of stability can be paraphrased as follows.

- An equilibrium point x^* is **neutrally stable** if any solution with an initial condition close enough to x^* will remain close to x^* for all future times.
- An equilibrium point x^* is called **locally asymptotically stable** if, in addition to neutral stability, any solution starting from any nearby initial condition actually approaches to x^* as t goes to infinity.
- An equilibrium point is said to be **unstable** if it is not neutrally stable.

We note here that the above definitions are local in character. That is if the stability is determined with respect to small perturbations.

1.2.1 Linear stability analysis

The first step in the determination of stability of an equilibrium point $x = x^*$ is by a linear analysis.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and consider

$$\dot{x} = f(x) \tag{1.4}$$

which is a nonlinear system.
We also consider that $A = (a_{ij})$ is an $n \times n$ matrix which is the Jacobian matrix of f at the equilibrium point x^*. That is

$$A = Df(x^*). \tag{1.5}$$

Now we consider the linearized system

$$\dot{y} = Ay, \quad y \in \mathbb{R}^n. \tag{1.6}$$

of (1.4) around $x = x^*$.

The eigenvalue problem of A is given by

$$Ay = \lambda y.$$

The eigenvalues of A are roots of the characteristic polynomial $p(\lambda)$.

$$p(\lambda) = \det(A - \lambda I) = 0$$

Theorem 1.2.1 *Let $Df(x^*)$ be the Jacobian matrix at the equilibrium solution x^* of (1.4):*

- *An equilibrium point $x = x^*$ is **locally asymptotically stable** if*

$$Re(\lambda) < 0, \quad \text{for all eigenvalues } \lambda \text{ of } Df(x^*)$$

- *An equilibrium point $x = x^*$ is **unstable**, if $Re(\lambda) > 0$ for at least one eigenvalue.*

We particularly demonstrate the case where A is a 2×2 matrix.

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

It can be easily seen that

$$tr(A) = a + b \quad and \quad \det(A) = ad - bc$$

in the definitions of the trace and determinant A. The eigenvalues of A are roots of the characteristic polynomial $p(\lambda)$. Let us construct the characteristic polynomial:

$$\begin{aligned} p(\lambda) = \det(A - \lambda I) &= \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} \\ &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda + ad - bc \\ &= \lambda^2 - tr(A)\lambda + \det(A). \end{aligned}$$

If we use the quadratic formula, then the eigenvalues of A matrix can be written as

$$\lambda_{1,2} = \frac{tr(A) \mp \sqrt{tr(A)^2 - 4\det(A)}}{2}.$$

The classification of the equilibrium points are listed below and shown in Figure (1.1).

- The equilibrium point is a **stable node** if λ_1 and λ_2 are real and $\lambda_2 < \lambda_1 < 0$.
- The equilibrium point is an **unstable node** if λ_1 and λ_2 are real and $\lambda_2 > \lambda_1 > 0$.
- The equilibrium point is a **saddle** if λ_1 and λ_2 are real and $\lambda_1 < 0 < \lambda_2$.
- The equilibrium point is a **center** if $\lambda_1 = \overline{\lambda}_2 = i\mu$ with $\mu \in \mathbb{R} \setminus \{0\}$.
- The equilibrium point is a **stable focus** if λ_1 and λ_2 are complex-conjugate and $Re(\lambda_{1,2}) < 0$.
- The equilibrium point is a **unstable focus** if λ_1 and λ_2 are complex-conjugate and $Re(\lambda_{1,2}) > 0$.

The stability analysis of the equilibrium points is given in Table (1.1).

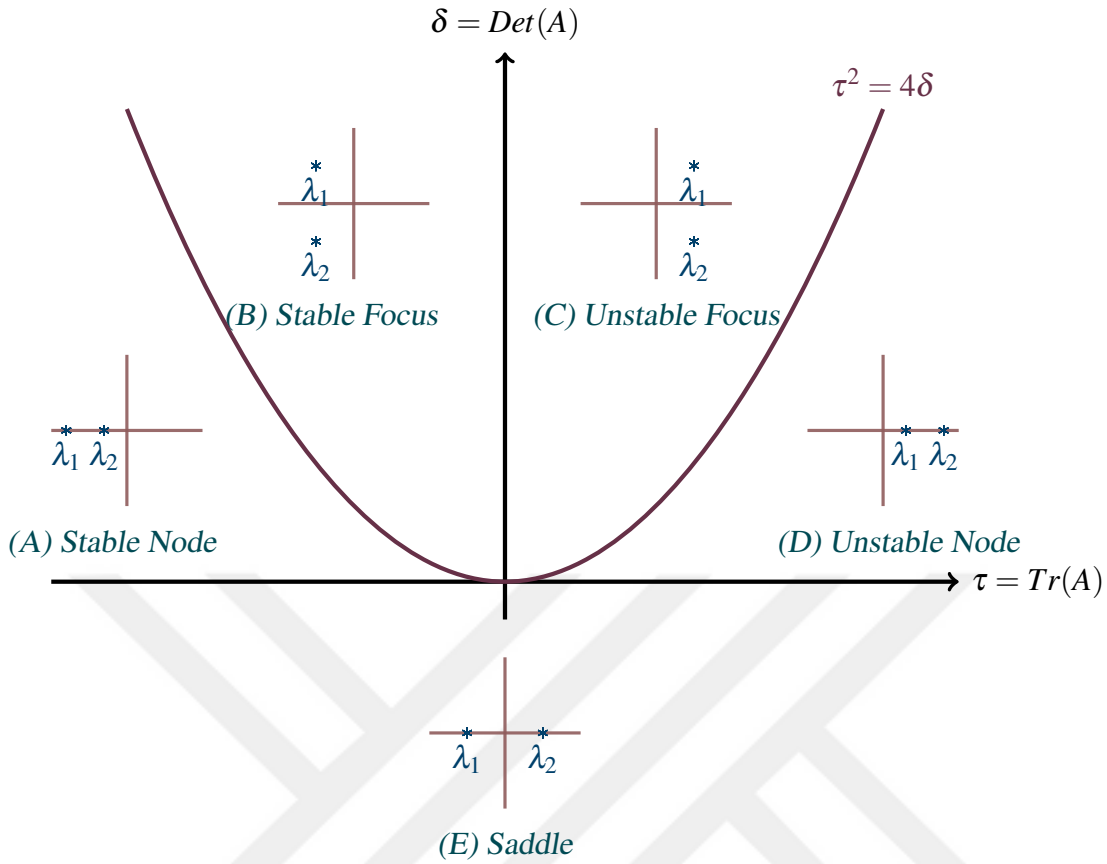


Figure 1.1 : Classification of phase portraits in the (τ, δ)-plane.

Example 1.2.1 *As an example, let us classify the equilibrium points of the following non-linear system*

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x) = \begin{pmatrix} x_1^2 - x_2^2 - 1 \\ 2x_2 \end{pmatrix}.$$

Solution 1.2.1 *The system has two equilibrium points which are* $(-1,0)$ *and* $(1,0)$*. It can be easily seen below*

$$f(x_1, x_2) = (0,0) \iff \quad x_1^2 - x_2^2 - 1 = 0 \quad \textit{and} \quad 2x_2 = 0.$$

The Jacobian matrix of $f(x)$ *can be calculated as follows:*

$$Df(x) = \begin{pmatrix} 2x_1 & -2x_2 \\ 0 & 2 \end{pmatrix}.$$

Then the Jacobian matrices can be evaluated at the equilibrium points

$$(-1,0) \quad \textit{and} \quad (1,0),$$

respectively

$$J(x)\,|_{(-1,0)} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} \quad \textit{and} \quad J(x)\,|_{(1,0)} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Thus, it can be obviously seen that

- *The eigenvalues, which corresponds to* $J(x)\mid_{(-1,0)}$*, are* $\lambda_1 = -2$ *and* $\lambda_2 = 2$*. The equilibrium point is a* ***saddle****.*
- *The eigenvalues, which corresponds to* $J(x)\mid_{(1,0)}$*, are* $\lambda_{1,2} = 2$*. The equilibrium point is a* ***source****.*

1.2.2 Global stability analysis

In this section we will discuss the global stability of equilibria.

Definition 1.2.4 *The equilibrium point* x^* *of* (1.1) *is called* ***globally asymptotically stable****, if*

$$\lim_{t\to\infty}\|x(t,x_0)-x^*\| = 0, \qquad \forall x_0 \in \mathbb{R}^n$$

Thus, unlike local stability, a globally asymptotically stable fixed point is a fixed point which is asymptotically stable with respect to any perturbation $\|x_0 - x^*\|$.

Global stability of an equilibrium point means that the system will achieve the equilibrium point from any possible beginning point.

We will now discuss a technique about determining the global stability known as **Lyapunov Function Method**. Before continuing on to this method, we will firstly make the definition of Lyapunov Function.

Lyapunov functions are non-negative definite functions that decrease in time along the orbits of a dynamical system:

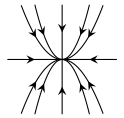
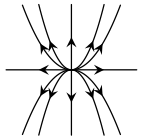
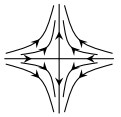
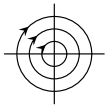
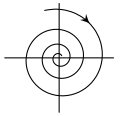
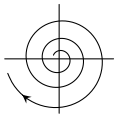
Definition 1.2.5 *Let a function* $V(x)$ *be continuously differentiable in an open set* U *in* R^n*,* $V : U \to \mathbb{R}$*. The function* $V(x)$ *is called the* ***Lyapunov function*** *for an autonomous system*

$$\dot{x} = f(x), \tag{1.7}$$

if the following conditions are satisfied:

1. $V(x) > 0$ *for all* $x \in U\backslash\{x_0\}$*,*
2. $V(x_0) = 0$*,*
3. $\dot{V} \leq 0$ *for all* $x \in U$*.*

Table 1.1 : Stability Analysis of Equilibrium Points of the Nonlinear System (1.4).

Eigenvalues of the Jacobian matrix	**Behaviour**	**Stability**	**Figure**
$\lambda_2 < \lambda_1 < 0$	Sink/Stable node	Locally asymptotically stable	
$0 < \lambda_1 < \lambda_2$	Source/Unstable node	Unstable	
$\lambda_1 < 0 < \lambda_2$	Saddle	Unstable	
$\lambda_1 = \overline{\lambda}_2 = i\mu$ with $\mu \in \mathbb{R} \setminus \{0\}$	Center	Can not be determined by linear stability analysis	
λ_1 and λ_2 are complex and satisfy $\lambda_1 = \overline{\lambda}_2$ with $Re(\lambda_{1,2}) < 0$	Stable Focus	Stable	
λ_1 and λ_2 are complex and satisfy $\lambda_1 = \overline{\lambda}_2$ with $Re(\lambda_{1,2}) > 0$	Unstable Focus	Unstable	

Theorem 1.2.2 *Let $f : \tilde{U} \subset \mathbb{R}^n \mapsto \mathbb{R}^n$ and x_0 be an equilibrium point of* (1.4) *and $V : U \subset \tilde{U} \mapsto \mathbb{R}^n$ be a Lyapunov function. Then,*

- *x_0 is **stable** if $\dot{V}(x) \leq 0$ for all $x \in U$.*
- *x_0 is **asymptotically stable** if $\dot{V}(x) < 0$ for all $x \in U \backslash \{x_0\}$. Moreover, U is the **basin of attraction** of the equilibrium x_0. That is*

$$\lim x(t, \tilde{x}) = x_0, \qquad \forall \tilde{x} \in U \tag{1.8}$$

*In particular, if $U = \tilde{U}$ then we say that x_0 is **globally asymptotically stable**.*

Example 1.2.2 *As an example, let us determine the stability of*

$$\begin{aligned} \dot{x_1} &= -x_2^3 \\ \dot{x_2} &= x_1^3 \end{aligned} \tag{1.9}$$

Solution 1.2.2 *Firstly, we obtain the equilibrium points of the system* (1.9).

$f(x_0) = 0$ implies that

$$-x_2^3 = 0 \quad \textit{and} \quad x_1^3 = 0.$$

It is easy to see that $x_1 = 0$ and $x_2 = 0$. So, we have one equilibrium point as $x_0 = (0,0)$. Let us construct the Lyapunov Function which satisfies $V(x_0) = 0$ as follows:

$$V(x) = x_1^4 + x_2^4$$

It can be clearly seen that $V(x) > 0$. We now find the derivative of $V(x)$.

$$\begin{aligned} \dot{V}(x) &= 4x_1^3\dot{x}_1 + 4x_2^3\dot{x}_2 \\ &= 4x_1^3(-x_2^3) + 4x_2^3(x_1^3) \\ &= 4(-x_1^3x_2^3 + x_1^3x_2^3) \\ &= 0 \end{aligned}$$

Then $\dot{V}(x) = 0$ implies $V(x) = c$. So, the solution lies on the closed curves

$$x_1^4 + x_2^4 = c^2.$$

Thus, the origin is a stable equilibrium point according to Theorem (1.2.2).

1.2.3 Stability of three-dimensional systems

We have mostly discussed about stability of the two dimensional system. However, we will study on the stability of the three-dimensional system on Chapter 3. For this reason, we study on the stability of the Jacobian matrix of a three-dimensional system.

The Jacobian matrix of a three-dimensional system has three eigenvalues. One of these eigenvalues must be a real number and the other two eigenvalues types can change. The classification of the system depends on the types and signs of the eigenvalues. (See Table 1.2).

The classification of the equilibrium points are given below:

- The equilibrium point is a **stable node** if all eigenvalues are real and negative.
- The equilibrium point is a **unstable node** if all eigenvalues are real and positive.
- The equilibrium point is a **saddle** if all eigenvalues are real and at least one of them is positive and at least one is negative.
- The equilibrium point is a **stable focus-node** if it has one real eigenvalue and a pair of complex-conjugate eigenvalues with all eigenvalues having negative real parts.
- The equilibrium point is a **unstable focus-node** if it has one real eigenvalue and a pair of complex-conjugate eigenvalues with all eigenvalues having positive real parts.
- The equilibrium point is a **saddle-focus** if it has one real eigenvalue with the sign opposite to the sign of the real parts of a pair of complex-conjugate eigenvalues.

Table 1.2 : Stability analysis of hyperbolic (that is those with eigenvalues with non-zero real part) equilibrium points of the three-dimensional system.

Classification	**Types of Roots**		**Sign of Roots**	**Behaviour**	**Stability**
	Real Root	Complex Root			
Node	$\lambda_1, \lambda_2, \lambda_3$	-	$\lambda_1, \lambda_2, \lambda_3 < 0$	Stable Node	Stable
	$\lambda_1, \lambda_2, \lambda_3$	-	$\lambda_1, \lambda_2, \lambda_3 > 0$	Unstable Node	Unstable
Focus-node	λ_1	$\lambda_2 = \bar{\lambda}_3$	$\lambda_1, Re(\lambda_2) < 0$, $Re(\lambda_3) < 0$	Stable-focus node	Stable
	λ_1	$\lambda_2 = \bar{\lambda}_3$	$\lambda_1, Re(\lambda_2) > 0$, $Re(\lambda_3) > 0$	Unstable-focus node	Unstable
Saddle	$\lambda_1, \lambda_2, \lambda_3$	–	At least one of them is positive and at least one of them is negative	Saddle	Unstable
Saddle-focus	λ_1	$\lambda_2 = \bar{\lambda}_3$	λ_1 has opposite sign of $Re(\lambda_2)$ and $Re(\lambda_3)$	Saddle-focus	Unstable

2. BASIC EPIDEMIOLOGICAL MODELS

In this section, we summarize the well-known results on several epidemiological models in increasing complexity. For detailed treatment of the subject matter, see [5, 6, 29, 34, 35]. We will start with the most basic model which is the SI model.

2.1 The SI Model

In this model, we divide the population into two distinct classes:

- **Susceptible:** $S(t)$, denotes the people who can catch the disease.
- **Infective**: $I(t)$, denotes the people who have the disease and can transmit it.

In the classical models we consider the size of the population as a constant, $N(t)$, that is $S(t)+I(t)=N(t)$.

The transfer diagram for SI model is as in Figure (2.1).

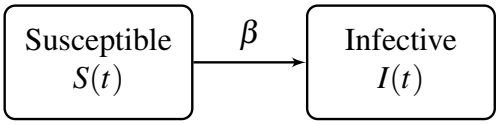


Figure 2.1 : Flowchart of SI model.

The SI model can be written as the following ordinary differential equation(ODE):

$$\begin{aligned} \frac{dS}{dt} &= -\beta SI \\ \frac{dI}{dt} &= \beta SI = \beta I(N-I) \end{aligned} \tag{2.1}$$

where $\beta > 0$ is the **infection rate**. Here, the infected population obey the logistic equation.

It can be clearly seen that the disease is contagious. Once infected, without treatment individuals stay infected for the rest of their lives. We can also see from the logistic equation that this epidemic will always spread and will eventually infect all susceptible individuals.

Mathematically, the system has two equilibria.

- The **disease-free equilibrium** is

$$E_0 = (S,I) = (N,0).$$

- The **endemic equilibrium** is

$$E_e = (S,I) = (0,N).$$

If we view the sign pattern of the (2.1), then we achieve the following results:

- $\frac{dS}{dt} < 0$, which shows that the numbers of susceptible individuals always decrease,
- $\frac{dI}{dt} > 0$, which shows that the numbers of infected individuals always increase,

when $S > 0$ and $I > 0$.

The corresponding phase plane to the SI model is given by Figure (2.2).

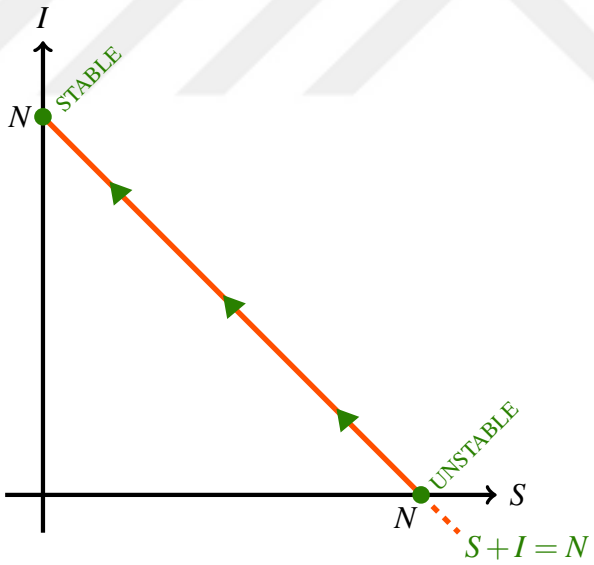


Figure 2.2 : The phase plane of the SI model.

2.2 The SIS Model

Similar to the SI model, we have two distinct classes as $S(t)$ and $I(t)$. This model is convenient for diseases which commonly have repeating infections. The SIS model does not include recovery. The infective individuals can be susceptible again. The progress of individuals is shown by the transfer diagram which is given by Figure (2.3).

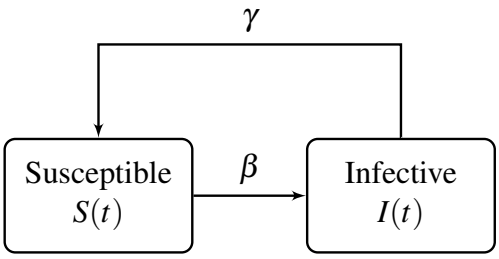


Figure 2.3 : Flowchart of SIS model.

The SIS model can be written as the following system:

$$\begin{aligned} \frac{dS}{dt} &= -\beta SI + \gamma I \\ \frac{dI}{dt} &= \beta SI - \gamma I \end{aligned} \tag{2.2}$$

where β is the **infection rate** and γ is the **rate of recovery**. $(\gamma, \beta > 0)$

2.2.1 Analysis of the SIS model

Let us note that

$$R_0 = \frac{\beta N}{\gamma}.$$

Theorem 2.2.1 *[5, p. 88]*

If $R_0 < 1$, the disease dies out, but if $R_0 > 1$, it remains in the population.

Remark 1 *According to the [30, p. 419], Basic Reproduction Number "the average number of secondary infecteds, produced by one typical primary infected person in a completely uninfected population". In other words, it shows the measurement of the transmission potential of a disease. It is denoted by R_0 and sometimes termed **basic reproductive rate** or **basic reproductive ratio**. This is the most important parameter of a disease because it shows that the magnitude of the epidemic.*

The proof of the theorem in (2.2.1) is given following steps.

Step 1: As a first step in analysing the SIS model we could simplify the equations by non-dimensionalisation by defining

$$u = \frac{S}{N}, \quad v = \frac{I}{N}, \quad \tau = \gamma t.$$

The system becomes

$$\begin{aligned} \frac{du}{d\tau} &= -(R_0u-1)v \\ \frac{dv}{d\tau} &= (R_0u-1)v \end{aligned} \tag{2.3}$$

where $R_0 = \frac{\beta N}{\gamma}$.

Step 2: (Invariant of the system)

The new system is to be solved on the one-dimensional simplex

$$\Omega = \{(u,v) \mid 0 \leq u \leq 1, 0 \leq v \leq 1, u+v=1\}.$$

In the non-dimensionalised system,

$$R_0 = \frac{\beta N}{\gamma}$$

is the basic reproduction number. The interpretation of this number is as follows:

- βN represents the rate at which an infected individual can infect a susceptible population of N capacity with random contacts.
- $\frac{1}{\gamma}$ represents the expected duration of time in which such an infectivity becomes contagious.

Under the interpretations listed above, we can say that R_0 is the expected number of infectious contacts made by such an infective individual.

Step 3: We should find the equilibrium points of the system (2.3).

$$\frac{du}{d\tau} = 0 \quad \Longrightarrow \quad (R_0u-1)v = 0.$$

We have seen that

$$v = 0 \quad \text{or} \quad u = \frac{1}{R_0}$$

from the last equation. Using

$$u+v=1$$

we obtain **two equilibrium points** such as

$$E_0 = (1,0) \quad \text{and} \quad E_1 = (\frac{1}{R_0}, 1-\frac{1}{R_0})$$

E_0 is called the **disease-free equilibrium** and E_1 is called the **endemic equilibrium**. It can be obviously seen that the endemic equilibrium does not exist when $R_0 < 1$.

We can now substitute $v = 1 - u$ in the first equation of the system (2.3) and the equation becomes

$$\frac{du}{d\tau} = -(R_0 u - 1)(1 - u). \tag{2.4}$$

There are two cases we need to examine here. We criticize the possible sign patterns for the cases $R_0 < 1$ and $R_0 > 1$.

Case 1: Let us assume that $R_0 < 1$.

It can be clearly seen that

$$\frac{du}{d\tau} > 0 \quad \text{for} \quad 0 \leq u < 1.$$

The solution trajectories can be sketched in $(1,0)$ as in Figure (2.4).

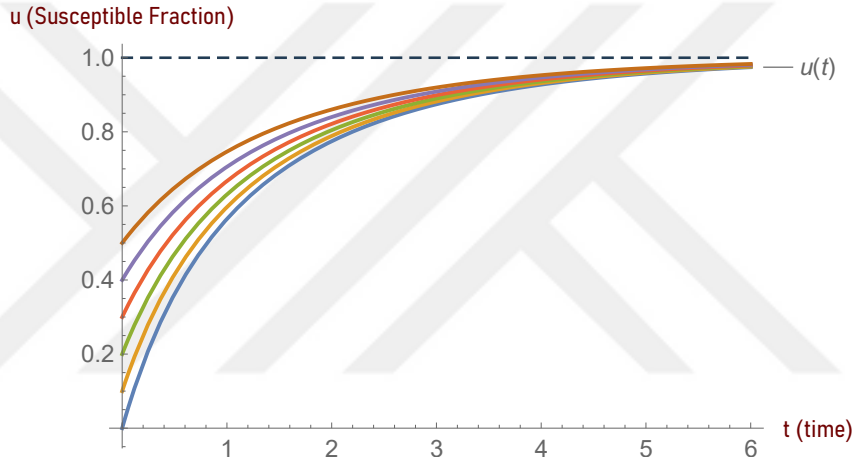


Figure 2.4 : SIS Epidemic, $R_0 < 1$.

This figure shows that the infection will die out eventually.

The corresponding phase plane to the SIS model when $R_0 < 1$ is given by Figure (2.5).

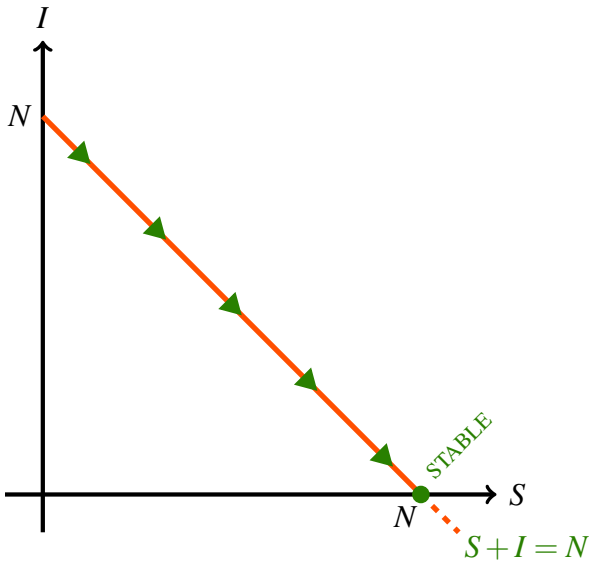


Figure 2.5 : The phase plane of the SIS model when $R_0 < 1$.

Case 2: Let us assume that $R_0 > 1$.

If we view the sign pattern of the (2.4), then we achieve the following results:

- $\frac{du}{d\tau} > 0 \quad \text{if} \quad 0 \leq u < \frac{1}{R_0}$
- $\frac{du}{d\tau} < 0 \quad \text{if} \quad \frac{1}{R_0} < u \leq 1.$

Under the conditions listed above, the solution trajectories can be sketched as seen in (Figure 2.6).

(Figure 2.6) shows that the infection will reach an equilibrium.

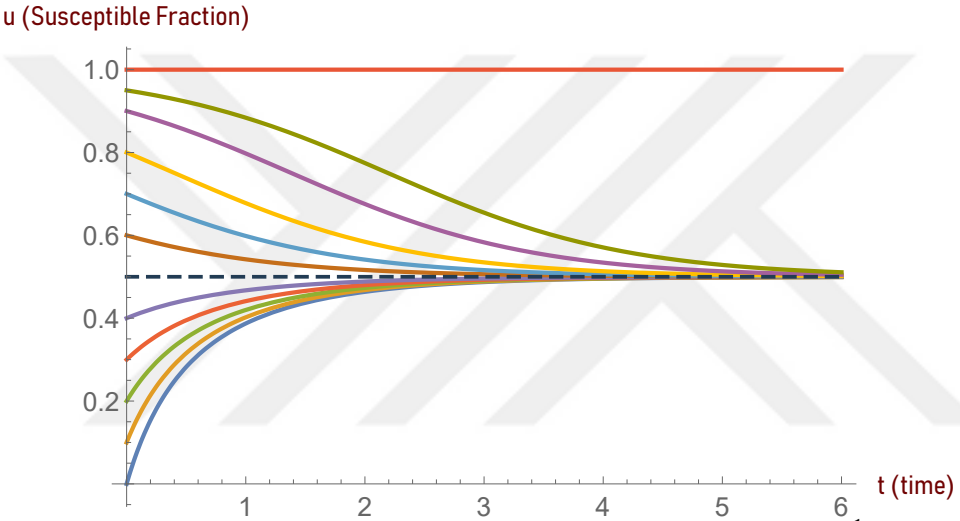


Figure 2.6 : SIS Epidemic, $R_0 > 1$. Dashed line shows that $\frac{1}{R_0}$.

The corresponding phase plane to the SIS model when $R_0 > 1$ is given by Figure (2.7).

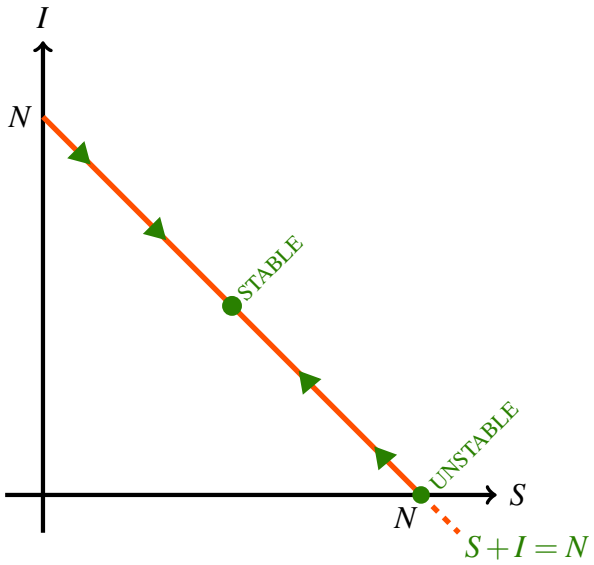


Figure 2.7 : The phase plane of the SIS model when $R_0 > 1$.

2.3 The SIR Model

The pioneer work in epidemiology has been done by Kernack and McKendrick, [31]. Unlike the other models, the SIR model consist of three compartments labeled S, I and R. As we know, $S(t)$ and $I(t)$ represent the **susceptibles** and the **infectives**, respectively. In addition to that, $R(t)$ represents the number of individuals who have been infected and then recovered from the disease or who are immune, dead, or otherwise.

The dynamics of S(t), I(t) and R(t) are shown by the following system:

$$\begin{aligned} \frac{dS}{dt} &= -\beta SI \\ \frac{dI}{dt} &= \beta SI - \gamma I \\ \frac{dR}{dt} &= \gamma I \end{aligned} \tag{2.5}$$

where β is the **infection rate** and γ is the **recovery rate**. $(\gamma, \beta > 0)$

We assume here that we can neglect natural birth or death. Therefore, the population size N is a constant and $N(t) = S(t) + I(t) + R(t)$.

The structure of the SIR model represented by the transfer diagram as in Figure (2.8).

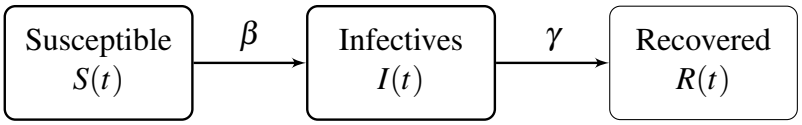


Figure 2.8 : Flowchart of the SIR model.

The mathematical formulation of the SIR model is completed with the following initial conditions

$$S(0) = S_0, \quad I(0) = I_0, \quad R(0) = 0.$$

2.3.1 Analysis of the SIR model

Step 1: Let us non-dimensionalise the system by defining

$$u = \frac{S}{N}, \quad v = \frac{I}{N}, \quad w = \frac{R}{N}, \quad \tau = \gamma t.$$

If we consider the first equation of the system (2.5),

$$\frac{1}{N}\frac{1}{\gamma}\frac{dS}{dt} = \frac{-\beta SI}{N}\frac{N}{N}\frac{1}{\gamma} \quad \Longrightarrow \quad \frac{d(\frac{S}{N})}{d(\gamma t)} = \frac{-\beta N}{\gamma}\frac{S}{N}\frac{I}{N}$$

then the equation becomes

$$\frac{du}{d\tau} = -R_0 uv$$

where $R_0 = \frac{\beta N}{\gamma}$ is the basic reproduction number.

As in the first equation we can non-dimensionalise the second and third equations as follows:

$$\frac{dI}{dt} = I(\beta S - \gamma) \implies \frac{dv}{d\tau} = v(R_0 u - 1)$$
$$\frac{dR}{dt} = \gamma I \implies \frac{dw}{d\tau} = v$$

Namely, the system becomes

$$\begin{aligned} \frac{du}{d\tau} &= -R_0 uv \\ \frac{dv}{d\tau} &= v(R_0 u - 1) \\ \frac{dw}{d\tau} &= v \end{aligned} \tag{2.6}$$

where $R_0 = \frac{\beta N}{\gamma}$.

Step 2: (Invariant of the system)

The equations are to be solved on the two-dimensional simplex $u+v+w=1$. We can find the simplex under the conditions listed below:

- We simply see that $0 \leq u \leq 1$, $0 \leq v \leq 1$ and $0 \leq w \leq 1$.
- Adding together the equations in the (2.6) system , we obtain

  $$\frac{du}{d\tau} + \frac{dv}{d\tau} + \frac{dw}{d\tau} = 0.$$

  We integrate this equation and find

  $$u(\tau) + v(\tau) + w(\tau) = u(0) + v(0) + w(0) = 1$$

  Now, we can simply construct the two-dimensional simplex as

  $$\Omega = \{(u,v,w) \mid 0 \leq u \leq 1, 0 \leq v \leq 1, 0 \leq w \leq 1, u+v+w = 1\}.$$

Step 3: Let us find the equilibrium points of the system (2.6).

Equilibria of the system are given by $u^* = c$, $v^* = 0$, $w^* = 1-c$ and c is any number in $[0,1]$. Thus there are infinitely many equilibria given by

$$E_c = (c, 0, 1-c), \qquad 0 \leq c \leq 1. \tag{2.7}$$

In this model, the endemic equilibrium does not show up and there are infinitely many disease-free equilibria as given below:

$$E_0 = (u^*, 0, 1-u^*).$$

It demonstrates that the u-axis is a nullcline for (2.6), namely, any point on it is a steady state.

We have two cases now:

Case 1: Let us assume that $R_0 > 1$.

We can easily see that

$$\frac{du}{d\tau} < 0$$

and

$$\frac{dv}{d\tau} > 0 \qquad \text{if } u > \frac{1}{R_0} \tag{2.8}$$

$$\frac{dv}{d\tau} < 0 \qquad \text{if } u < \frac{1}{R_0} \tag{2.9}$$

since $0 \leq u \leq 1, 0 \leq v \leq 1$.

Without calculation, we can estimate the qualitative behaviour by the solution trajectories as seen in Figure (2.9).

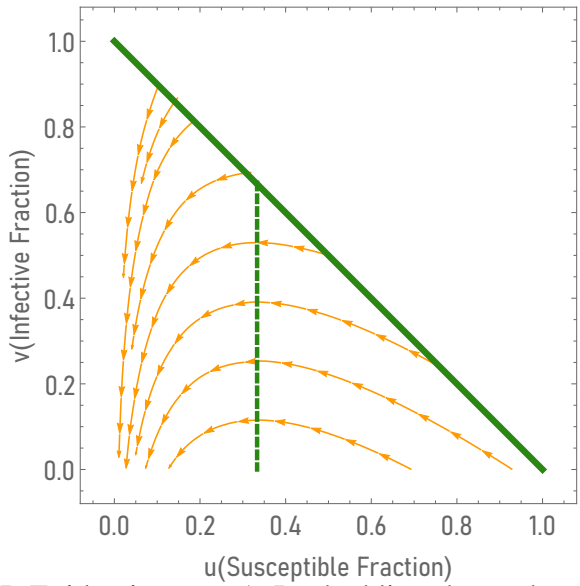


Figure 2.9 : SIR Epidemic, $R_0 > 1$. Dashed line shows that the value of $\frac{1}{R_0}$.

We can interpret the case $R_0 > 1$ by the above figure as given below:

- If $c < \frac{1}{R_0}$ then all the equilibria are **neutrally stable**. Namely, the disease dies out when $c < \frac{1}{R_0}$.

- If $c > \frac{1}{R_0}$ then all the equilibria are **unstable**.

Case 2: Let us consider that $R_0 < 1$.

In the similar way, it is clear to see that

$$\frac{du}{d\tau} < 0 \quad \text{and} \quad \frac{dv}{d\tau} < 0.$$

Under these circumstances, the solution trajectories is sketched as in Figure (2.10).

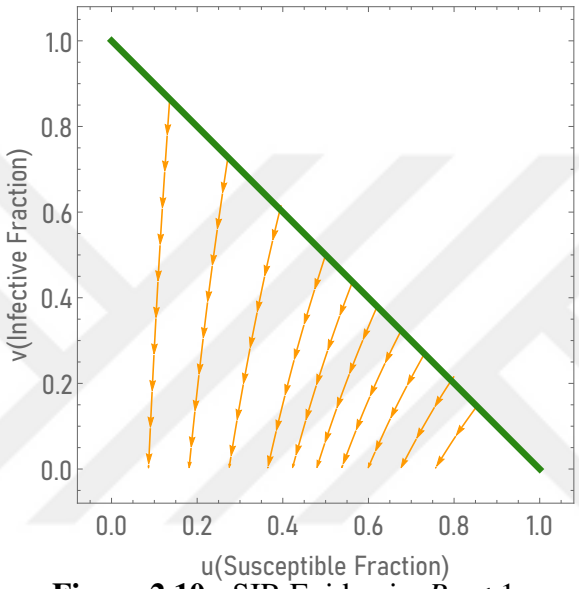


Figure 2.10 : SIR Epidemic, $R_0 < 1$.

We see that all the equilibria are **neutrally stable** when $R_0 < 1$.

2.4 The SIRS Model

In this model, the total population N is classified into three compartments as it is in the SIR model. **Susceptibles** are denoted by S, **infectives** by I and **recovereds** by R. As in the SIR model, susceptible individuals infected by infective individiuals gain immunity after getting infected and recover. Yet, for some infected individuals, this gained immunity may disappear after some time. In other words, the recovered compartment will be free of infection and rejoin the susceptible individuals department.

The structure of the SIRS model is shown in Figure (2.11).

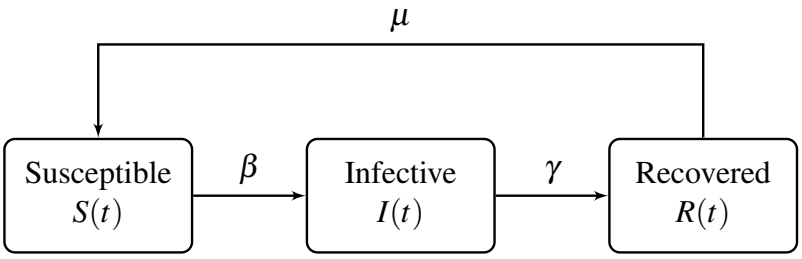


Figure 2.11 : Flowchart of SIRS model.

The dynamics of $S(t)$, $I(t)$ and $R(t)$ are written by the following differential equations

$$\begin{aligned} \frac{dS}{dt} &= -\beta SI + \mu R \\ \frac{dI}{dt} &= \beta SI - \gamma I \\ \frac{dR}{dt} &= \gamma I - \mu R \end{aligned} \tag{2.10}$$

where β is the **infectious rate**, γ is the **cured rate** and μ is the **transfer rate** from recovered individuals to susceptible individuals. $(\gamma, \beta, \mu > 0)$

The total population N is a constant because we ignored the natural birth and death. In other words, it is represented by $N(t) = S(t) + I(t) + R(t)$.

Further, we have initial conditions

$$S(0) = S_0, \quad I(0) = I_0 \quad \text{and} \quad R(0) = 0$$

corresponding to the SIRS system.

Let us now review the stability of the SIRS model.

2.4.1 Analysis of the SIRS model

Step 1: In order to simplify, we non-dimensionalise the system (2.10) by defining

$$u = \frac{S}{N}, \quad v = \frac{I}{N}, \quad w = \frac{R}{N}, \quad \tau = t(\gamma + \mu).$$

Primarily, we consider the first equation of the system (2.10). Let us divide the system by N and $(\gamma + \mu)$. Then, the equation becomes

$$\frac{d(\frac{S}{N})}{d((\gamma+\mu)t)} = \frac{-\beta}{(\gamma+\mu)} \frac{S}{N} I + \frac{\mu}{(\gamma+\mu)} \frac{R}{N}.$$

This new equation is still dimensional. This is the reason, why we divide and multiply the first term in the right hand side of the equation by N, respectively. Now, the last

equation becomes

$$\frac{d\left(\frac{S}{N}\right)}{d((\gamma+\mu)t)} = \frac{-\beta N}{(\gamma+\mu)}\frac{S}{N}\frac{I}{N} + \frac{\mu}{(\gamma+\mu)}\frac{R}{N}.$$

We now substitute the u, v and w terms and we get

$$\frac{du}{d\tau} = \frac{-\beta N}{(\gamma+\mu)}uv + \frac{\mu}{(\gamma+\mu)}w.$$

In the similar method, we can non-dimensionalise the second and third equations of the system (2.10) as below:

$$\frac{d\left(\frac{I}{N}\right)}{d((\gamma+\mu)t)} = \frac{\beta N}{(\gamma+\mu)}\frac{S}{N}\frac{I}{N} - \frac{\gamma}{(\gamma+\mu)}\frac{I}{N} \quad \Longrightarrow \quad \frac{dv}{d\tau} = \frac{\beta N}{(\gamma+\mu)}uv - \frac{\gamma}{(\gamma+\mu)}v$$
$$\frac{d\left(\frac{R}{N}\right)}{d((\gamma+\mu)t)} = \frac{\gamma}{(\gamma+\mu)}\frac{I}{N} - \frac{\mu}{(\gamma+\mu)}\frac{I}{N} \quad \Longrightarrow \quad \frac{dw}{d\tau} = \frac{\gamma}{(\gamma+\mu)}v - \frac{\mu}{(\gamma+\mu)}w$$

Namely, we can indicate the non-dimensionalised system with the following differential equations:

$$\begin{aligned}
\frac{du}{d\tau} &= -\frac{\beta N}{(\gamma+\mu)}uv + \frac{\mu}{(\gamma+\mu)}w \\
\frac{dv}{d\tau} &= \frac{\beta N}{(\gamma+\mu)}uv - \frac{\gamma}{(\gamma+\mu)}v \\
\frac{dw}{d\tau} &= \frac{\gamma}{(\gamma+\mu)}v - \frac{\mu}{(\gamma+\mu)}w
\end{aligned} \tag{2.11}$$

Step 2: We should find an **invariant set** for the system so that all solutions remain sensible. As seen before

$$0 \le u \le 1, \quad 0 \le v \le 1, \quad 0 \le w \le 1.$$

The constant population size is constructed by the system (2.11), by adding the equations

$$\frac{du}{d\tau} + \frac{dv}{d\tau} + \frac{dw}{d\tau} = 0 \quad \Longrightarrow \quad u(\tau) + v(\tau) + w(\tau) = 1$$

We can now clearly show the two-dimensional simplex as

$$\Omega = \{(u,v,w) \mid 0 \le u,v,w \le 1, u+v+w = 1\}.$$

Step 3: We should calculate the equilibrium points.

Since

$$\frac{\beta N}{(\gamma+\mu)}uv - \frac{\gamma}{(\gamma+\mu)}v = 0$$

we get

$$v^* = 0 \quad \text{or} \quad u^* = \frac{\gamma}{\beta N}.$$

- If we substitute $v^* = 0$ in the third equation of the system (2.11) then we obtain

$$w^* = 0.$$

Namely, we find

$$E_0 = (u^*, 0, 0).$$

Since

$$u + v + w = 1,$$

we see that the **first equilibrium point** is

$$E_0 = (1, 0, 0).$$

And it's called the **disease-free equilibrium**.

- If we substitute $u^* = \frac{\gamma}{\beta N}$ in the first equation of the system (2.11) then we find

$$v^* = \frac{\mu}{\gamma} w^*.$$

So, we get

$$E_1 = \left(\frac{\gamma}{\beta N}, \frac{\mu}{\gamma} w^*, w^* \right).$$

Due to the fact we have proved above which is $u + v + w = 1$, then the **second equilibrium point** becomes

$$E_1 = \left(\frac{\gamma}{\beta N}, \frac{\mu}{\mu + \gamma} \left(1 - \frac{\gamma}{\beta N} \right), \frac{\gamma}{\mu + \gamma} \left(1 - \frac{\gamma}{\beta N} \right) \right).$$

It's called the **endemic equilibrium point**. It can be obviously seen that the endemic equilibrium point exists only if all the terms are positive. This implies that the following condition must hold:

$$R_0 = \frac{\beta N}{\gamma} \geq 1.$$

Step 4: Let us reduce the system to two dimensions by substituting $w = 1 - u - v$.

$$\begin{aligned} \frac{du}{d\tau} &= -\frac{\beta N}{(\gamma + \mu)} uv + \frac{\mu}{(\gamma + \mu)} (1 - u - v) \\ \frac{dv}{d\tau} &= \frac{\beta N}{(\gamma + \mu)} uv - \frac{\gamma}{(\gamma + \mu)} v \end{aligned} \tag{2.12}$$

Step 5: Let us construct **the Jacobian matrix** of the system (2.12).

$$J = \begin{pmatrix} -\frac{\beta N}{(\gamma+\mu)} v - \frac{\mu}{(\gamma+\mu)} & -\frac{\beta N}{(\gamma+\mu)} u - \frac{\mu}{(\gamma+\mu)} \\ \frac{\beta N}{(\gamma+\mu)} v & \frac{\beta N}{(\gamma+\mu)} u - \frac{\gamma}{(\gamma+\mu)} \end{pmatrix}$$

The Jacobian matrix at $E_0 = (1,0,0)$ is calculated as seen below:

$$J\,|_{(1,0,0)} = \begin{pmatrix} -\frac{\mu}{(\gamma+\mu)} & -\frac{\beta N}{(\gamma+\mu)} - \frac{\mu}{(\gamma+\mu)} \\ 0 & \frac{\beta N}{(\gamma+\mu)} - \frac{\gamma}{(\gamma+\mu)} \end{pmatrix}$$

We now calculate the $tr(J)$ and $\det(J)$ for the disease-free equilibrium.

$$tr(J) = \frac{1}{(\gamma+\mu)}(\beta N - \gamma - \mu), \qquad \det(J) = -\frac{\mu}{(\gamma+\mu)^2}(\beta N - \gamma).$$

Thus,

- if $R_0 < 1$, namely $\gamma \geq \beta N$, then

  $$tr(J) < 0 \quad \text{and} \quad \det(J) > 0$$

  and the disease-free equilibrium is **stable**.

- if $R_0 > 1$, namely $\beta N \geq \gamma$, then

  $$det(J) < 0$$

  and the disease-free equilibrium is **unstable**.

In a similar way, we can construct the Jacobian matrix at (u^*, v^*, w^*) as follows:

$$J\,|_{(u^*,v^*,w^*)} = \begin{pmatrix} \frac{\mu}{(\gamma+\mu)}\left(-\frac{\beta N}{(\gamma+\mu)} + \frac{\gamma}{(\gamma+\mu)} - 1\right) & -1 \\ \frac{\mu}{(\gamma+\mu)^2}(\beta N - \gamma) & 0 \end{pmatrix}$$

The trace and determinant of the Jacobian for the endemic equilibrium point is as follows:

$$tr(J) = -\frac{\mu}{(\gamma+\mu)^2}(\beta N + \mu), \quad det(J) = \frac{\mu}{(\gamma+\mu)^2}(\beta N - \gamma).$$

- if $R_0 < 1$, then the endemic equilibrium point **does not exist**.

- if $R_0 > 1$, then

  $$tr(J) < 0 \quad \text{and} \quad \det(J) > 0$$

  which shows that the endemic equilibrium is **stable**.

2.5 The SEIR Model

Unlike previous models, this model has a different compartment which is the so called **exposed class** represented by $E(t)$. In other words, the SEIR model consists of four compartments:

1. **Susceptible class** $S(t)$, which refers to the people who can catch the virus and become infectious if exposed.

2. **Exposed class** $E(t)$, which shows the number of people who are exposed to the virus or infected but not yet infectious. Namely, although the people are infected, the symptoms of the virus are not still visible [33].

3. **Infectious class** $I(t)$, which refers to the number of infective individuals who are able to transmit the disease by contacting with susceptible individuals.

4. **Recovered class** $R(t)$, which denotes the people who gain immunity from the disease.

We can easily understand the process of this model. Ebola virus transmission process can be given as an example for this model [32, 36]. After transmission of the virus, susceptible individuals go into the exposed compartment before they become infectious and then they either recover or die. Yet, we ignore new births and deaths because of the fact that we only study on the simplest SEIR model. Moreover, we assume here that the total size of the population is a constant and this assumption is written by $N(t) = S(t) + E(t) + I(t) + R(t)$.

The basic SEIR model's progress can be shown with the transfer diagram as seen in Figure (2.12).

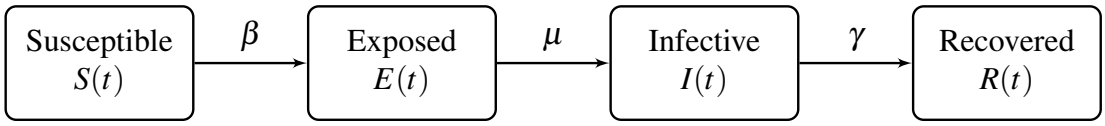


Figure 2.12 : Flowchart of SEIR model.

In this transfer diagram,

- The **transmission rate**, $\beta \geq 0$, shows the rate of spread which denotes the probability of carrying disease among a group of susceptible and infectious people.
- The **incubation rate**, $\mu \geq 0$, is the rate of latent people becoming infectious.
- The **recovery rate**, $\gamma \geq 0$, is the rate of recovery by infectious people.

Therefore, the dynamics of the SEIR model are defined by the following differential equations:

$$\begin{aligned} \frac{dS}{dt} &= -\beta SI \\ \frac{dE}{dt} &= \beta SI - \mu E \\ \frac{dI}{dt} &= \mu E - \gamma I \\ \frac{dR}{dt} &= \gamma I \end{aligned} \tag{2.13}$$

with initial conditions

$$S(0) = S_0 > 0, \quad E(0) = E_0 \geq 0, \quad I(0) = I_0, \quad R(0) = 0. \tag{2.14}$$

2.5.1 Analysis of the SEIR model

Step 1: For simplicity reasons, we non-dimensionalise the system by

$$x = \frac{S}{N}, \quad y = \frac{E}{N}, \quad z = \frac{I}{N}, \quad w = \frac{R}{N}, \quad \tau = t(\gamma+\mu).$$

Firstly, let us study on the first equation of (2.13). If we divide that equation by N and $(\gamma+\mu)$, we get

$$\frac{d\left(\frac{S}{N}\right)}{d((\gamma+\mu)t)} = \frac{-\beta}{(\gamma+\mu)}\frac{S}{N}I.$$

We can easily see that the new form is still dimensional. For this reason, we firstly divide and then multiply with N in the right hand side of the last equation. So, we have

$$\frac{d\left(\frac{S}{N}\right)}{d((\gamma+\mu)t)} = \frac{-\beta}{(\gamma+\mu)}\frac{S}{N}\frac{I}{N}N.$$

Now substituting x, z and τ into the last equation yields

$$\frac{dx}{d\tau} = \frac{-\beta N}{(\gamma+\mu)}xz.$$

Similarly, we obtain the non-dimensionalised form of the second, third and fourth equations of the system (2.13).

$$\begin{aligned}
\frac{d\left(\frac{E}{N}\right)}{d((\gamma+\mu)t)} &= \frac{\beta N}{(\gamma+\mu)}\frac{S}{N}\frac{I}{N} - \frac{\mu}{(\gamma+\mu)}\frac{E}{N} \quad &\Longrightarrow \quad \frac{dy}{d\tau} &= \frac{\beta N}{(\gamma+\mu)}xz - \frac{\mu}{(\gamma+\mu)}y \\
\frac{d\left(\frac{I}{N}\right)}{d((\gamma+\mu)t)} &= \frac{\mu}{(\gamma+\mu)}\frac{E}{N} - \frac{\gamma}{(\gamma+\mu)}\frac{I}{N} \quad &\Longrightarrow \quad \frac{dz}{d\tau} &= \frac{\mu}{(\gamma+\mu)}y - \frac{\gamma}{(\gamma+\mu)}z \\
\frac{d\left(\frac{R}{N}\right)}{d((\gamma+\mu)t)} &= \frac{\gamma}{(\gamma+\mu)}\frac{I}{N} \quad &\Longrightarrow \quad \frac{dw}{d\tau} &= \frac{\gamma}{(\gamma+\mu)}z
\end{aligned}$$

We can now demonstrate the non-dimensionalised form of the (2.13) by the following differential equations:

$$\begin{aligned}
\frac{dx}{d\tau} &= \frac{-\beta N}{(\gamma+\mu)}xz \\
\frac{dy}{d\tau} &= \frac{\beta N}{(\gamma+\mu)}xz - \frac{\mu}{(\gamma+\mu)}y \\
\frac{dz}{d\tau} &= \frac{\mu}{(\gamma+\mu)}y - \frac{\gamma}{(\gamma+\mu)}z \\
\frac{dw}{d\tau} &= \frac{\gamma}{(\gamma+\mu)}z
\end{aligned} \tag{2.15}$$

Step 2: Let us determine the feasible region for (2.15).

From (2.15), we easily see that

$$\frac{d}{d\tau}[x(\tau)+y(\tau)+z(\tau)+w(\tau)] = 0.$$

Namely, the population size N is always constant:

$$x(\tau)+y(\tau)+z(\tau)+w(\tau) = 1$$

for any $\tau \geq 0$.

We obtain the three-dimensional simplex as

$$\Omega = \{(x,y,z,w) \mid 0 \leq x,y,z,w \leq 1, x(\tau)+y(\tau)+z(\tau)+w(\tau) = 1\}.$$

The **positive invariant set** shows us that any solution starting in Ω does not leave this region. We can also write the simplex as

$$\Omega = \{(x,y,z) \mid 0 \leq x,y,z \leq 1, x(\tau)+y(\tau)+z(\tau) \leq 1\}.$$

using the relation $w = 1-x-y-z$.

Step 3: Let us find the equilibrium points of the system (2.15).

Since

$$\frac{dz}{d\tau} = 0$$

we get

$$y^* = \frac{\gamma}{\mu} z^*.$$

If we substitute $y^* = \frac{\gamma}{\mu} z^*$ in the second equation of the system (2.15), we obtain

$$z^* = 0 \quad \text{or} \quad x^* = \frac{\gamma}{\beta N}.$$

- If $z^* = 0$ then it is easy to see that the $y^* = 0$.

  Since

  $$x + y + z = 1$$

  we obtain

  $$x^* = 1.$$

  Namely, the **disease-free equilibrium point** is

  $$E_0 = (1, 0, 0).$$

- If $x^* = \frac{\gamma}{\beta N}$ and $y^* = \frac{\gamma}{\mu} z^*$ then the endemic equilibrium is

  $$E_1 = (\frac{\gamma}{\beta N}, \frac{\gamma}{\mu} z^*, z^*).$$

  Since $x + y + z = 1$, the **endemic equilibrium** becomes

  $$E_1 = \left(\frac{\gamma}{\beta N}, \frac{\gamma}{\gamma + \mu} \left(1 - \frac{\gamma}{\beta N} \right), \frac{\mu}{\gamma + \mu} \left(1 - \frac{\gamma}{\beta N} \right) \right).$$

It is easy to see that the endemic equilibrium point exists only if all the terms are positive. This implies that the following condition must hold:

$$R_0 = \frac{\beta N}{\gamma} \geq 1.$$

Step 4: Let us reduce the system to three dimensions by substituting $w = 1 - x - y - z$.

$$\begin{aligned} \frac{dx}{d\tau} &= \frac{-\beta N}{(\gamma + \mu)} xz \\ \frac{dy}{d\tau} &= \frac{\beta N}{(\gamma + \mu)} xz - \frac{\mu}{(\gamma + \mu)} y \\ \frac{dz}{d\tau} &= \frac{\mu}{(\gamma + \mu)} y - \frac{\gamma}{(\gamma + \mu)} z \end{aligned} \tag{2.16}$$

Step 5: Let us construct the Jacobian matrix of the system (2.16).

$$J = \begin{pmatrix} \frac{-\beta N}{(\gamma+\mu)}z & 0 & \frac{-\beta N}{(\gamma+\mu)}x \\ \frac{\beta N}{(\gamma+\mu)}z & \frac{-\mu}{(\gamma+\mu)} & \frac{\beta N}{(\gamma+\mu)}x \\ 0 & \frac{\mu}{(\gamma+\mu)} & \frac{-\gamma}{(\gamma+\mu)} \end{pmatrix}$$

The Jacobian matrix at the E_0 is calculated as seen below:

$$J\mid_{E_0} = \begin{pmatrix} 0 & 0 & \frac{-R_0\gamma}{(\gamma+\mu)} \\ 0 & \frac{-\mu}{(\gamma+\mu)} & \frac{R_0\gamma}{(\gamma+\mu)} \\ 0 & \frac{\mu}{(\gamma+\mu)} & \frac{-\gamma}{(\gamma+\mu)} \end{pmatrix}$$

The characteristic polynomial is $det(J\mid_{E_0}) = 0$. Solving this polynomial, the eigenvalues become:

$$\lambda_1 = 0, \quad \lambda_{2,3} = \frac{-1}{2(\gamma+\mu)}\left(\gamma+\mu \mp \sqrt{\gamma^2+2\gamma\mu(2R_0-1)+\mu^2}\right).$$

We now investigate whether the real parts of $\lambda_{2,3}$ are negative or not. For simplicity, let us assume:

$$D = \gamma^2+2\gamma\mu(2R_0-1)+\mu^2.$$

- If $D<0$ the eigenvalues $\lambda_{2,3}$ are complex with

  $$Re(\lambda_{2,3}) = \frac{-1}{2}\left(\frac{\gamma+\mu}{\gamma+\mu}\right) = \frac{-1}{2} \leq 0.$$

  So, all the eigenvalues are zero or negative. Thus, the stability of the endemic equilibrium can not be determined since at least one eigenvalue is zero.

- If $D>0$, since $R_0>1$, it is easy to see that

  $$(\gamma+\mu) < \sqrt{D}$$

  by using

  $$\sqrt{\gamma^2+2\gamma\mu(2R_0-1)+\mu^2} > \sqrt{(\gamma+\mu)^2} = \mid \gamma+\mu \mid .$$

  For simplicity, we can write the $\lambda_{2,3}$ as follows:

  $$\lambda_{2,3} = \frac{-1}{2(\gamma+\mu)}(\gamma+\mu \mp \sqrt{D})$$

  Then it is obviously seen that the λ_2 and λ_3 have different signs and it implies that the endemic equilibrium is **saddle**.

The Jacobian matrix at the E_1 is calculated as follows:

$$J\mid_{E_1}=\begin{pmatrix} \frac{(1-R_0)\gamma\mu}{(\gamma+\mu)^2} & 0 & \frac{-\gamma}{(\gamma+\mu)} \\ \frac{(R_0-1)\gamma\mu}{(\gamma+\mu)^2} & \frac{-\mu}{(\gamma+\mu)} & \frac{\gamma}{(\gamma+\mu)} \\ 0 & \frac{\mu}{(\gamma+\mu)} & \frac{-\gamma}{(\gamma+\mu)} \end{pmatrix}$$

In this analysis, we use a different method which is mentioned-below as **Routh-Hurtwitz stability criterion**. The third-order polynomial

$$P(s) = s^3 + a_2 s^2 + a_1 s + a_0$$

has only roots in the open left half plane if and only if

$$a_2, a_0 \quad \text{are positive and} \quad a_2 a_1 > a_0.$$

The criterion provides a way to determine if the behaviour of a physical system has only a stable solution, without solving the system directly.

The characteristic polynomial of $J\mid_{E_1}$ matrix is constructed by $det(J\mid_{E_1} -\lambda I) = 0$. Solving this polynomial, the coefficients are determined to be as follows:

$$a_0 = \frac{(R_0-1)\gamma^2\mu^2}{(\gamma+\mu)^4}, \quad a_1 = \frac{(R_0-1)\gamma\mu}{(\gamma+\mu)^2}, \quad a_2 = \frac{\gamma^2+(R_0+1)\gamma\mu+\mu^2}{(\gamma+\mu)^2}, \quad a_3 = 1.$$

In here, it is easy to see that a_2 and a_0 are positive for $R_0 > 1$. And after some algebraic computations we see that

$$\frac{a_2 a_1}{a_0} = \frac{\gamma^2+(R_0+1)\gamma\mu+\mu^2}{\gamma\mu} > 1 \quad \Longrightarrow \quad a_2 a_1 > a_0.$$

Since a_2 and a_0 are positive and $a_2 a_1 > a_0$ the conditions of Routh-Hurtwitz criterion are satisfied. Hence, the endemic equilibrium point is **stable** when $R_0 > 1$.

2.6 SIR Endemics

In the simple SIR model our objective is to study on the endemic disease's short-term behaviour. In this section, we study on an endemic disease's long-term behaviour according to [5, 37, 38]. And for this reason ignoring birth and death rate is no longer meaningful. Namely, we take into consideration the birth and death rate from now on. And we also examine the death rate in two different models: **No disease-related death** and **including disease-related death**.

We will insert birth and death rates into the SIR model from now on. Namely, the total population is no longer closed, and the total population size N will only be constant

under additional assumptions on the birth and death rates. In line with the information given here, we can investigate the models.

2.6.1 No disease-related death

The diagrammatic representation of the disease is shown in Figure (2.13).

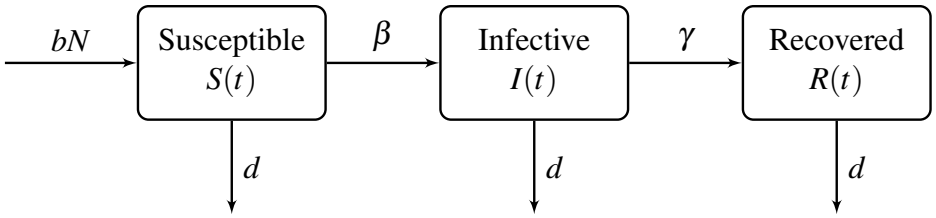


Figure 2.13 : SIR endemic with no-disease related death.

In this model, we assume that there is no vertical-transmission. Namely, there is no direct transmission from parent to an embriyo, fetus or baby during pregnancy. Therefore, **all births**, which are denoted by B, are assumed to enter the susceptible compartment. We consider $B = bN$, because B is not per capita birth rate. We assume that $b = d$ in here and d is a constant which denotes the **disease-unrelated death rate**. So, the population size N is constant.

The transfer diagram leads to the following system of ordinary differential equations:

$$\begin{aligned} \frac{dS}{dt} &= bN - \beta SI - dS \\ \frac{dI}{dt} &= \beta SI - \gamma I - dI \\ \frac{dR}{dt} &= \gamma I - dR \end{aligned} \tag{2.17}$$

Substituting $b = d$ into the (2.17) gives

$$\begin{aligned} \frac{dS}{dt} &= bN - \beta SI - bS \\ \frac{dI}{dt} &= \beta SI - \gamma I - bI \\ \frac{dR}{dt} &= \gamma I - bR \end{aligned} \tag{2.18}$$

Let us analyze the system now:

Step 1: We non-dimensionalise the system by defining

$$u = \frac{S}{N}, \quad v = \frac{I}{N}, \quad w = \frac{R}{N}, \quad \tau = t(\gamma + b).$$

Firstly, we examine the first equation of the system (2.18). Let us divide the equation by N and $(\gamma+b)$. We obtain

$$\frac{d\left(\frac{S}{N}\right)}{d((\gamma+b)t)} = \frac{b}{(\gamma+b)} - \frac{\beta}{(\gamma+b)}\frac{I}{N}S - \frac{b}{\gamma+b}\frac{S}{N}.$$

It can clearly be seen that the equation still dimensional. We firstly divide and then multiply the second term in the right hand side of the last equation by N. Then we get

$$\frac{d\left(\frac{S}{N}\right)}{d((\gamma+b)t)} = \frac{b}{(\gamma+b)} - \frac{\beta}{(\gamma+b)}\frac{I}{N}\frac{S}{N}N - \frac{b}{\gamma+b}\frac{S}{N}.$$

Substituting u,v and w into the last equation gives

$$\frac{du}{d\tau} = \frac{b}{(\gamma+b)} - \frac{\beta N}{(\gamma+b)}uv - \frac{b}{\gamma+b}u.$$

By using the same technique, we can find the non-dimensionalized form of the second and third equations of the system (2.18).

$$\frac{d\left(\frac{I}{N}\right)}{d((\gamma+b)t)} = \frac{\beta}{(\gamma+b)\frac{I}{N}\frac{S}{N}N - \frac{\gamma}{(\gamma+b)}\frac{I}{N}} - \frac{b}{(\gamma+b)}\frac{I}{N} \quad \Longrightarrow \quad \frac{dv}{d\tau} = v\left(\frac{\beta N}{(\gamma+b)}u - 1\right)$$

$$\frac{d\left(\frac{R}{N}\right)}{d((\gamma+b)t)} = \frac{\gamma}{(\gamma+b)}\frac{I}{N} - \frac{b}{(\gamma+b)}\frac{R}{N} \quad \Longrightarrow \quad \frac{dw}{d\tau} = \frac{\gamma}{(\gamma+b)}v - \frac{b}{(\gamma+b)}w$$

Thus, the non-dimensionalized form of the system (2.18) is as follows:

$$\begin{aligned}
\frac{du}{d\tau} &= \frac{b}{(\gamma+b)}(1-u) - R_0uv \\
\frac{dv}{d\tau} &= v(R_0u-1) \\
\frac{dw}{d\tau} &= \frac{\gamma}{(\gamma+b)}v - \frac{b}{(\gamma+b)}w
\end{aligned} \tag{2.19}$$

where $R_0 = \frac{\beta N}{\gamma+b}$ is the basic reproduction number.

Step 2: Here, we find a suitable boundary region for (2.19).

$$u(\tau)+v(\tau)+w(\tau)+=1 \quad \Longrightarrow \quad \frac{du}{d\tau}+\frac{dv}{d\tau}+\frac{dw}{d\tau}=0$$

Then, the **positive invariant set** is

$$\Omega = \{(u,v,w) \mid 0 \leq u,v,w \leq 1, u+v+w=1\}.$$

Step 3: Let us find the equilibrium points of the system.

We obtain $u^* = \frac{1}{R_0}$ and $v^* = 0$ since $v(R_0u-1)=0$. The system has two equilibrium points. The first equilibrium point is the disease-free equilibrium:

- If we substitute $v^* = 0$ in the third equation of the system (2.19) we get

$$w^* = 0.$$

Additional, since

$$u + v + w = 1$$

we find

$$u^* = 1.$$

Namely, the **disease-free equilibrium point** is

$$E_0 = (1, 0, 0).$$

The second equilibrium point is the endemic equilibrium:

- If we substitute $u^* = \frac{1}{R_0}$ in the first equation of the system (2.19) we obtain

$$v^* = \frac{b}{(\gamma + b)} \left(1 - \frac{1}{R_0}\right).$$

And it is easy to calculate

$$w^* = \frac{\gamma}{\beta N}(R_0 - 1)$$

from the third equation of the system (2.19). Then the **endemic equilibrium point** is

$$E_e = \left(\frac{1}{R_0}, \frac{b}{(\gamma + b)} \left(1 - \frac{1}{R_0}\right), \frac{\gamma}{(\gamma + b)} \left(1 - \frac{1}{R_0}\right)\right).$$

Clearly, the endemic equilibrium exists only if $v^* > 0$ which means that, the basic reproduction number must be greater than 1:

$$R_0 = \frac{\beta N}{\gamma + b} > 1.$$

Step 4: Let us reduce the system two dimensions by substituting $w = 1 - u - v$.

$$\begin{aligned} \frac{du}{d\tau} &= \frac{b}{(\gamma + b)}(1 - u) - R_0 uv \\ \frac{dv}{d\tau} &= v(R_0 u - 1) \end{aligned} \tag{2.20}$$

Step 5: Let us find the Jacobian matrix of (2.20).

$$J = \begin{pmatrix} -\frac{b}{(\gamma + b)} - R_0 v & -R_0 u \\ R_0 v & R_0 u - 1 \end{pmatrix}$$

The Jacobian matrix which is evaluated at the disease-free equilibrium point $(1,0,0)$ is

$$J\mid_{(1,0,0)}=\begin{pmatrix} -\frac{b}{(\gamma+b)} & -R_0 \\ 0 & R_0-1 \end{pmatrix} \tag{2.21}$$

It is clearly seen

$$tr(J)=R_0-1-\frac{b}{(\gamma+b)},\quad \det(J)=(1-R_0)\frac{b}{(\gamma+b)}.$$

for the Jacobian matrix which is represented by (2.21). Thus,

- if $R_0<1$, then

  $$trace(J)<0 \quad \text{and} \quad \det(J)>0$$

  which shows that the disease-free equilibrium is **stable**.

- if $R_0>1$, then

  $$\det(J)>0$$

  which shows that the disease-free equilibrium point is a **saddle**.

The Jacobian matrix evaluated at the endemic equilibrium point (u^*,v^*,w^*) is

$$J\mid_{(u^*,v^*,w^*)}=\begin{pmatrix} -\frac{R_0 b}{(\gamma+b)} & -1 \\ \frac{b}{(\gamma+b)}(R_0-1) & 0 \end{pmatrix}.$$

It is clearly seen from

$$tr(J)=-\frac{R_0 b}{(\gamma+b)},\quad \det(J)=\frac{b}{(\gamma+b)}(R_0-1)$$

- if $R_0<1$, then the endemic equilibrium point **does not exists**.

- if $R_0>1$, then $trace(J)<0$, $det(J)>0$ and the endemic equilibrium point is **stable**.

2.6.2 Including disease-related death

Let us illustrate the dynamics of this epidemic model by the diagram as in Figure (2.14).

Similar to the previous epidemic model, we assume that there is no vertical-transmission. In other words, **all births**, which is B, are assumed to enter the susceptible class. B is taken constant birth rate instead of constant per capita

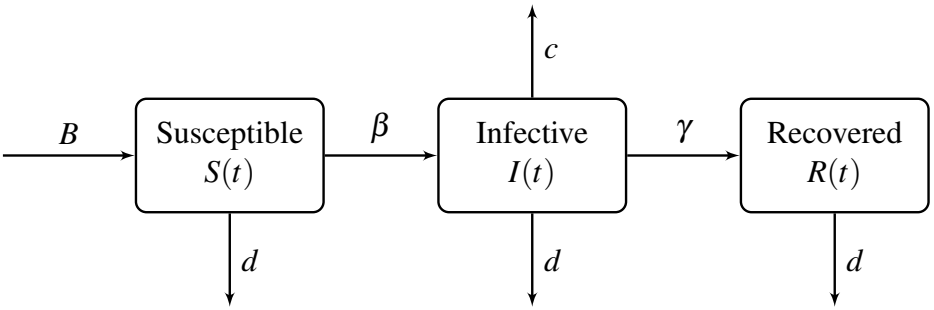


Figure 2.14 : SIR endemic including disease related death.

birth rate here. This is another way to say that the birth rate is not proportional with the population size. We take c as a constant and c denotes the disease-related death rate. Then the model is shown by the following ODE:

$$\begin{aligned} \frac{dS}{dt} &= B - \beta SI - dS \\ \frac{dI}{dt} &= \beta SI - \gamma I - cI - dI \\ \frac{dR}{dt} &= \gamma I - dR \end{aligned} \tag{2.22}$$

Adding these three equations, we obtain

$$\frac{dN}{dt} = B - cI - dN \tag{2.23}$$

We analyze the system using any three of the equations which are shown by (2.22) and (2.23) with $N = S + I + R$. We shall choose the (N,S,I) equations. We can not reduce the system to two equations as we have done before, since the population size is not constant.

We now find the equilibrium points. From $\frac{dI}{dt} = 0$, we obtain

$$I^* = 0 \quad \text{or} \quad S^* = \frac{(\gamma + c + d)}{\beta}.$$

- In the first case, if we substitute $I^* = 0$ in the first equation of the system (2.22) we get

  $$S^* = \frac{B}{d}.$$

  Finally, we obtain $N^* = \frac{B}{d}$ from (2.23) by using $I^* = 0$. So, the **first equilibrium point**

  $$E_0 = (N,S,I) = \left(\frac{B}{d}, \frac{B}{d}, 0\right).$$

  is the disease-free equilibrium.

- In the second case, we examine the endemic equilibrium point. Substituting S^* into the first equation of (2.22) and the last equation which is labeled (2.23) gives us

$$I^* = \frac{B - S^* d}{S^* \beta} \quad \text{and} \quad N^* = \frac{B}{d} - \frac{c}{d\beta S^*}(B - S^* d).$$

It is easy to see that the endemic equilibrium point exists only if all the terms are positive. This implies that the following condition must hold:

$$B - S^* d > 0 \quad \Longrightarrow \quad B > S^* d \quad \Longrightarrow \quad \frac{B}{S^* d} > 1$$

Substituting S^* into the condition gives us the basic reproduction number as follows:

$$R_0 = \frac{B\beta}{d(\gamma + c + d)} > 1.$$

Then the **endemic equilibrium point** can be written as

$$(N^*, S^*, I^*) = \left(\frac{1}{\beta}[(\gamma + d)R_0 + c], \frac{B}{\mathrm{d}R_0}, \frac{d(R_0 - 1)}{\beta} \right).$$

Let us construct the Jacobian matrix for the (N, S, I) system which is given below:

$$\begin{aligned} \frac{dN}{dt} &= B - cI - dN \\ \frac{dS}{dt} &= B - \beta SI - dS \\ \frac{dI}{dt} &= \beta SI - \gamma I - cI - dI \end{aligned}$$

The Jacobian matrix which corresponds to the last ODE system is

$$J = \begin{pmatrix} -d & 0 & -c \\ 0 & -\beta I - d & -\beta S \\ 0 & \beta I & \beta S - (\gamma + c + d) \end{pmatrix}.$$

The Jacobian matrix at the disease-free equilibrium is constructed as follows:

$$J\,|_{E_0} = \begin{pmatrix} -d & 0 & -c \\ 0 & -d & 0 \\ 0 & 0 & \frac{\beta B}{d} - (\gamma + c + d) \end{pmatrix}$$

The eigenvalues of an upper triangular matrix are the entries on its main diagonal as seen before. Then the eigenvalues are

$$\lambda_1 = \lambda_2 = -d \quad \text{and} \quad \lambda_3 = \frac{\beta B}{d} - (\gamma + c + d).$$

Let us substitute $R_0 = \frac{B\beta}{d(\gamma + c + d)}$ in $\lambda_3 = (\gamma + c + d)(R_0 - 1)$. Thus,

- if $R_0 < 1$ then all the eigenvalues are negative and the disease-free equilibrium point is **stable**.

- if $R_0 > 1$ then $\lambda_3 > 0$ and the disease-free equilibrium point is a **saddle**.

The Jacobian matrix at the endemic equilibrium is calculated as follows:

$$J\,|_{(N^*,S^*,I^*)} = \begin{pmatrix} -d & 0 & -c \\ 0 & -dR_0 & \frac{-\beta B}{dR_0} \\ 0 & d(R_0-1) & 0 \end{pmatrix}$$

Then the eigenvalues are

$$\lambda_1 = -d, \quad \lambda_{2,3} = \frac{-dR_0^2}{2R_0} \mp \frac{1}{2R_0}\sqrt{(dR_0^2)^2 - 4B\beta(R_0-1)R_0}.$$

We examine whether the real parts of $\lambda_{2,3}$ are negative or not. Let us take

$$D = (dR_0^2)^2 - 4B\beta(R_0-1)R_0.$$

- If $D < 0$ then the eigenvalues $\lambda_{2,3}$ are complex with

$$Re(\lambda_{2,3}) = \frac{-dR_0^2}{2R_0} < 0.$$

  Since the real parts of all the eigenvalues are negative the endemic equilibrium is **damped oscillation**.

- If $D > 0$, we investigate only when $R_0 > 1$. Because we know that the endemic equilibrium exists only if $R_0 > 1$. Thus, if $D > 0$ then since $R_0 > 1$, we get

$$\sqrt{D} < dR_0^2.$$

  If we rewrite $\lambda_{2,3}$ as

$$\lambda_{2,3} = \frac{-1}{2R_0}(dR_0^2 \mp \sqrt{D})$$

we can easily see that both λ_2 and λ_3 are negative. Thus, the endemic equilibrium is **stable** since all the eigenvalues are negative.

2.7 Eradication and Control

In today's world, diseases are one of the major problems of the society. Urbanization and other factors, which are making our lives easier, are causing diseases. They bring about epidemic and cause big loss of population. Thus, the modelling of infectious diseases is crucial in controlling and diminishing the effects of epidemics [39, 40]. The forecasting of diseases makes it possible to eradicate or at least control it. Such control methods might aim to reduce the effect of the basic reproduction number R_0. In the simplest models which we studied before $R_0 = \frac{\beta N}{\gamma}$, there are three convenient strategies according to [5, p. 101]:

1. Increase γ, the rate of recovery
2. Decrease β, the rate of transmission of disease from an infected person
3. Decrease the effective value of N, which should be interpreted as decreasing the initial susceptible population

2.7.1 Vaccination against an SIR epidemic

In this model, assume that we have a perfect vaccine against the disease, [5]. We focus on vaccinating new borns and p denotes the **fraction of vaccinated new borns**. We now study a population which is categorised into three group of individuals: the **susceptibles** (S), the **infectous** (I), and the **recovered** (R), whose dynamics are modelled under the following assumptions.

- The natural birth and death rates are included.
- Age, sex, social status, and race do not affect the probability of being infected.
- There is no disease-related death. In other words, members of the infective class leave either by recovery or due to natural death from their compartment.
- Recovered individuals keep their immunity. Namely, the vaccine gives long-term immunity against the infection which averts both transmission and disease.

In addition to above all, the aim is to find the proportion of the population we have to vaccinate in order to eradicate the disease. Putting all these assumptions together with the corresponding notations, the model can be shown by the schematic diagram which is given in Figure (2.15).

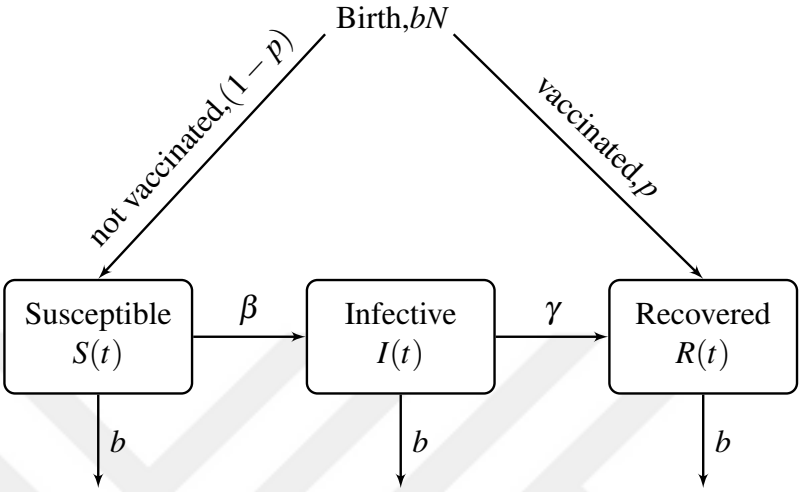


Figure 2.15 : Vaccination Against an SIR Epidemic Model.

Then the model can be written mathematically by the following system:

$$\begin{aligned} \frac{dS}{dt} &= b(1-p)N - \beta SI - bS \\ \frac{dI}{dt} &= \beta SI - bI - \gamma I \\ \frac{dR}{dt} &= bpN + \gamma I - bR \end{aligned} \tag{2.24}$$

where β is the **transmission rate**, γ is the **recovery rate** and b is the **natural death** or **birth rate** $(\beta, \gamma, b > 0)$. And the mathematical formulation is completed with the non-negativity requirements of the initial conditions:

$$S(0) \geq 0, \quad I(0) \geq 0, \quad R(0) \geq 0.$$

As in [41], we do the following analysis:

Step 1: We modify the sytem (2.24) using a simple change of variables:

$$S = (1-p)S_1, \quad I = (1-p)I_1, \quad R = (1-p)R_1 + pN.$$

Substituting the above-mentioned variables into the (2.24) gives us a new set of differential equations as follows:

$$\begin{aligned}(1-p)\frac{dS_1}{dt} &= b(1-p)N - \beta(1-p)^2 S_1 I_1 - b(1-p)S_1 \\ (1-p)\frac{dI_1}{dt} &= \beta(1-p)^2 S_1 I_1 - \gamma(1-p)I_1 - b(1-p)I_1 \\ (1-p)\frac{dR_1}{dt} &= \gamma(1-p)I_1 - b(1-p)R_1 - bpN + bpN\end{aligned} \tag{2.25}$$

If we divide the system (2.25) by $(1-p)$ then the system takes the following form:

$$\begin{aligned}\frac{dS_1}{dt} &= bN - \beta(1-p)S_1 I_1 - bS_1 \\ \frac{dI_1}{dt} &= \beta(1-p)S_1 I_1 - \gamma I_1 - bI_1 \\ \frac{dR_1}{dt} &= \gamma I_1 - bR_1\end{aligned} \tag{2.26}$$

Step 2: Let us study on the invariant set.

Adding the equations (2.24) together, we obtain $\frac{dN}{dt} = b(1-N)$. It shows that there is no invariant set because the total population size is open.

Step 3: Let us find the equilibrium points.

There are two equilibrium points that exists for (2.26) as follows:

- If $I_1^* = 0$, the second equation in (2.26) holds. Substituting $I_1^* = 0$ into the first and third equation in (2.26) gives us $S_1^* = N$ and $R_1^* = 0$, respectively. Thus, we have the equilibrium point

  $$E_0 = (S_1^*, I_1^*, R_1^*) = (N, 0, 0).$$

  This is the first equilibrium point which is called the **disease-free equilibrium point**.

  The second equilibrium point which is called the endemic equilibrium point can be found easily.

- If $I_1^* = 0$, then from the second equation in (2.26), we get $S_1^* = \frac{N}{R_0}$. Substituting this value of S_1^* into the first equation in (2.26) gives

  $$I_1^* = b\left(\frac{N}{\gamma + b} - \frac{1}{\beta(1-p)}\right).$$

  Before finding the value of R_1^* we define the basic reproduction number for simplicity. As we have seen before, the endemic equilibrium point exists only if

all the terms are positive which means that I_1^* must be greater than zero here. So, we get

$$R_0 = \frac{N\beta(1-p)}{\gamma+b} \quad \text{from} \quad \left(\frac{N}{\gamma+b} - \frac{1}{\beta(1-p)}\right) > 0.$$

Then the value of I_1^* becomes

$$I_1^* = \frac{b}{\beta(1-p)}(R_0-1).$$

If we substitute I_1^* in the third equation of the system (2.26) then we obtain

$$R_1^* = \frac{\gamma}{\beta(1-p)}(R_0-1).$$

Then the **endemic equilibrium point** is

$$E_e = \left(\frac{N}{R_0}, \frac{b}{\beta(1-p)}(R_0-1), \frac{\gamma}{\beta(1-p)}(R_0-1)\right).$$

Step 4: Let us determine the Jacobian matrix of the system (2.26).

$$J = \begin{pmatrix} -\beta(1-p)I_1 - b & -\beta(1-p)S_1 & 0 \\ \beta(1-p)I_1 & \beta(1-p)S_1 - \gamma - b & 0 \\ 0 & \gamma & -b \end{pmatrix}$$

The Jacobian matrix which is evaluated at the disease-free equilibrium point is

$$J\,|_{E_0} = \begin{pmatrix} -b & -\beta(1-p)N & 0 \\ 0 & \beta(1-p)N_1 - \gamma - b & 0 \\ 0 & \gamma & -b \end{pmatrix}.$$

If we compute the eigenvalues of this matrix we find

$$\lambda_1 = -b, \quad \lambda_2 = -b, \quad \text{and} \quad \lambda_3 = \beta N(1-p) - (\gamma+b).$$

We can easily see that λ_1 and λ_2 are negative. So, we investigate the sign of λ_3. Thus,

- if $\lambda_3 > 0$ which means that the basic reproduction number $R_0 > 1$, the disease-free equilibrium is a **saddle**.
- if $\lambda_3 < 0$ which means that the basic reproduction number $R_0 < 1$, the disease-free equilibrium is a **stable node**.

We now investigate the stability of the endemic equilibrium point. If we evaluate the Jacobian matrix at the endemic equilibrium we obtain

$$J\,|_{E_e} = \begin{pmatrix} -bR_0 & -(\gamma+\beta) & 0 \\ b(R_0-1) & 0 & 0 \\ 0 & \gamma & -b \end{pmatrix}.$$

After some algebraic computations, we find the eigenvalues of this Jacobian matrix. These eigenvalues are

$$\lambda_1 = -b, \quad \lambda_{2,3} = -\frac{R_0 b}{2} \mp \frac{\sqrt{b^2(R_0-2)^2 - 4b\gamma(R_0-1)}}{2}.$$

We now examine whether the real parts of $\lambda_{2,3}$ are negative or not. Let us

$$D = b^2(R_0-2)^2 - 4b\gamma(R_0-1).$$

- If $D < 0$ then the eigenvalues $\lambda_{2,3}$ are complex with

  $$Re(\lambda_{2,3}) = \frac{-bR_0}{2}.$$

  Thus, the endemic equilibrium is **damped oscillation** because the real parts of all the eigenvalues are negative.

- If $D > 0$, we study only when $R_0 > 1$ since the endemic equilibrium point exists only if $R_0 > 1$. For simplicity, we can write the value of $\lambda_{2,3}$ as follows:

  $$\lambda_{2,3} = \frac{-1}{2}(bR_0 \mp \sqrt{D}).$$

  There are two cases in here:

  - If $0 < (R_0 b)^2 < D$ then λ_2 and λ_3 have different signs. Thus, the endemic equilibrium is **saddle**.
  - If $0 < D < (R_0 b)^2$ then all the eigenvalues are negative. Thus, the endemic equilibrium is **stable**.

3. A DYNAMICAL SYSTEMS APPROACH TO THE INTERPLAY BETWEEN TOBACCO SMOKERS, ELECTRONIC-CIGARETTE SMOKERS AND SMOKING QUITTERS

3.1 Introduction

The natural behaviour of physical systems modelled by linear models is quite predictable [1, 2]. Contrary to that, many natural phenomena such as alcohol, heroin, drug transmission, epidemiological models, climate-vegetation, prey-predator and smoking cessation models are governed by non-linear systems and their behaviour is often unpredictable. For details see, [8–15].

In today's world, smoking is one of the most critical public-health issues. As well known, smoking damages nearly every organ of the body and causes diseases. In addition, smoking is an addiction, this means that quitting smoking is not very easy. Many smokers need support for quitting. In fact, there are some ways to help smokers quit smoking. Some of them are smoking cessation programs, nicotine gums or using electronic cigarettes for that matter. For this reason, some smokers are inclined to use e-cigarettes instead of tobacco cigarettes as using e-cigarettes is a method to quit smoking. According to [16], e-cigarettes are less harmful than tobacco cigarettes because e-cigarettes involve only nicotine contrary to tobacco cigarettes which involve more than 7000 chemicals such as arsenic(poison) and carbon monoxide (gas from car exhaust) [42]. Indeed, there are different views on whether using e-cigarette is beneficial or not from the medical point of view. We aim here to investigate the addictive behaviour of tobacco smoking and the effect of e-cigarettes as an aid in quitting smoking by taking into consideration the peer pressure and by using mathematical modelling.

In this study, we first propose the model in line with the articles [7] and [43]. These articles examined the effect of e-cigarette on smoking cessation using different mathematical models. In [43], Straughan concentrated on the efficiency of peer pressure term, which is non-linear term, in the transition from smoking to e-cigarette

smoking and studied a three compartmental model. This model consists of potential smokers, tobacco smokers and e-cigarette smokers. In addition to these compartments, Jung et al. considered another compartment which is the quitters' class [7]. The transition from smoking to e-cigarette smoking does not based on the peer pressure term as in [7]. Considering these two articles, we offer a model which consist of four compartments and considered the transition from smoking to e-cigarette smoking by incorporating a peer pressure term.

3.2 The Model

Based on the traditional epidemiological models, we propose a mathematical model to see the dynamics of the effectiveness of using e-cigarette on quitting smoking. The dynamics of smoking is similar to the traditional epidemiological models: a potential smoker makes contact with a smoker and starts smoking under the influence of the smokers.

3.2.1 The model description and its parameters

In this model, we classified the total population N into four distinct classes:

1. **Potential smoker** class $P(t)$, which represents the people who never smoke or smoke in some degree but might become smokers in the future.
2. **Smoker** class $S(t)$, which represents the people who smoke "everyday" or "some days".
3. **E-cigarette smoker** class $E(t)$,which represents the people who now use electronic cigarettes.
4. **Quitter** class $Q(t)$, which represents the people who quit smoking altogether.

In addition to above all, we have some assumptions in order to propose the dynamical system. The dynamics among $P(t)$, $S(t)$, $E(t)$ and $Q(t)$ are modelled under the following assumptions:

- $P(t)+S(t)+E(t)+Q(t)=N(t)$; the total population size N is always constant.
- The natural rates of birth and death are included.

- All of the natural death rates are equal to the natural birth rates.
- There is no mortality rate related to certain diseases caused by smoking.
- We take into account the effective contact rate, that is the probability of becoming a smoker because of influential contact with smokers, which is called **peer pressure**.
- The effective contact rates are constants.
- We indicate the effect of the peer pressure with a non-linear term in dynamics of smoking.
- Individuals in the quitter class may relapse after some time by making contact with smokers.

Putting all these assumptions together with the corresponding notations, the flow among those classes, which are mentioned-above, can be shown by the transfer diagram as seen in Figure (3.1).

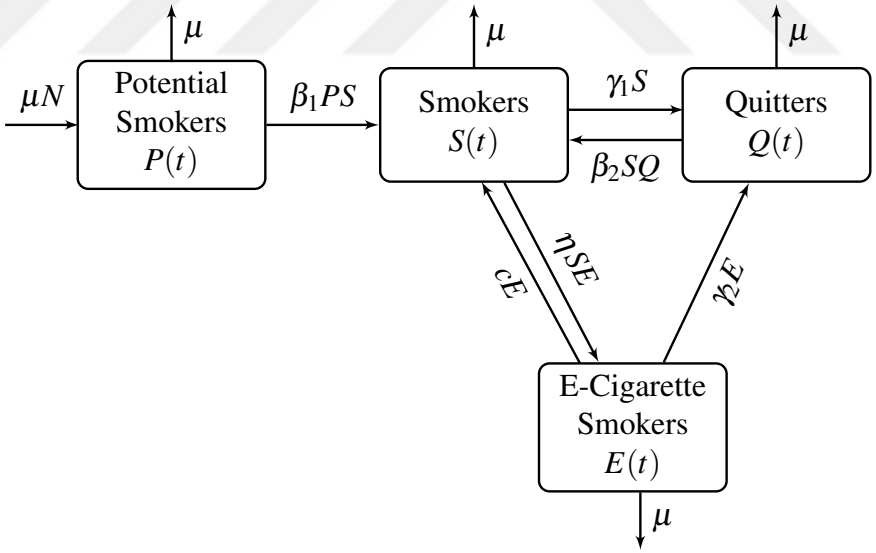


Figure 3.1 : Flowchart of the proposed model.

All the non-negative parameters in the transfer diagram are defined as follows:

- μ: The natural birth and death rates of the population.
- β_1: An effective contact rate that represents the probability of a potential smoker becoming a smoker by peer pressure.

- β_2: An effective contact rate that represents the probability of a quitter relapsing to become a smoker due to the peer influence.
- η: The transformation rate from a smoker to an e-cigarette smoker by peer pressure.
- γ_1: The transformation rate from a smoker to a quitter by their own will.
- γ_2: The rate of quitting smoking by using e-cigarette, per unit time.
- c: The return rate to smoker class, after using e-cigarettes, by their own will.

In line with the information given above, the proposed model can be written mathematically by a set of four non-linear differential equations as follows:

$$\begin{aligned}
\frac{dP}{dt} &= \mu N - \mu P - \beta_1 PS \\
\frac{dS}{dt} &= \beta_1 PS - \mu S - \gamma_1 S + \beta_2 SQ - \eta SE + cE \\
\frac{dE}{dt} &= \eta SE - cE - \mu E - \gamma_2 E \\
\frac{dQ}{dt} &= \gamma_1 S - \beta_2 SQ - \mu Q + \gamma_2 E
\end{aligned} \tag{3.1}$$

Further, we have initial conditions

$$P(0) = P_0 \geq 0, \quad S(0) = S_0 \geq 0, \quad E(0) = E_0 \geq 0, \quad Q(0) = Q_0 \geq 0.$$

corresponding to the proposed system.

3.2.2 Invariant region

In this section, we will construct an invariant set for the system (3.1) so that all solutions remain sensible. Because the system (3.1) indicates the dynamics among human population, it is logical to consider that the parameters are non-negative for all $t \geq 0$. To put it in a mathematical notation:

$$0 \leq P \leq N, \quad 0 \leq S \leq N, \quad 0 \leq E \leq N, \quad \text{and} \quad 0 \leq Q \leq N.$$

Adding together all of the equations in system (3.1) gives us

$$\frac{dP}{dt} + \frac{dS}{dt} + \frac{dE}{dt} + \frac{dQ}{dt} = 0.$$

And the last equation yields

$$\frac{d}{dt}[P(t) + S(t) + E(t) + Q(t)] = 0.$$

It can be clearly seen that $\frac{dN}{dt} = 0$ from the last equation. And this conclusion infers that the total population is constant. Thus, we can simply construct the positively invariant region as

$$\Omega = \{(P,S,E,Q) \mid 0 \leq P,S,E,Q \leq N, P(t)+S(t)+E(t)+Q(t) = N\}.$$

Before moving on to finding the equilibrium points of the system (3.1), we first obtain the reduced form of the system (3.1). We can reduce the system (3.1) by defining

$$\frac{P}{N} = p, \quad \frac{S}{N} = s, \quad \frac{E}{N} = e, \quad and \quad \frac{Q}{N} = q.$$

Substituting p, s, e and q into the system (3.1) gives us

$$\begin{aligned}
\frac{dp}{dt} &= \mu - \mu p - \beta_1 N p s \\
\frac{ds}{dt} &= \beta_1 N p s - \mu s - \gamma_1 s + \beta_2 N s q - \eta N s e + c e \\
\frac{de}{dt} &= \eta N s e - c e - \mu e - \gamma_2 e \\
\frac{dq}{dt} &= \gamma_1 s - \beta_2 N s q - \mu q + \gamma_2 e
\end{aligned} \tag{3.2}$$

We can rearrange the system by defining

$$\beta_1 N = \xi_1, \quad \beta_2 N = \xi_2, \quad \eta N = \theta$$

which yields:

$$\begin{aligned}
\frac{dp}{dt} &= \mu - \mu p - \xi_1 p s \\
\frac{ds}{dt} &= \xi_1 p s - \mu s - \gamma_1 s + \xi_2 s q - \theta s e + c e \\
\frac{de}{dt} &= \theta s e - c e - \mu e - \gamma_2 e \\
\frac{dq}{dt} &= \gamma_1 s - \xi_2 s q - \mu q + \gamma_2 e
\end{aligned} \tag{3.3}$$

Since the $p = 1 - s - e - q$, we can reduce the system (3.3) as seen below:

$$\begin{aligned}
\frac{ds}{dt} &= \xi_1 s(1 - s - e - q) - \mu s - \gamma_1 s + \xi_2 s q - \theta s e + c e \\
\frac{de}{dt} &= \theta s e - c e - \mu e - \gamma_2 e \\
\frac{dq}{dt} &= \gamma_1 s - \xi_2 s q - \mu q + \gamma_2 e
\end{aligned} \tag{3.4}$$

We note that the individuals, who have never smoked yet, are more curious about smoking than the individuals who quit smoking at least once. Mathematically,

$$\xi_1 \geq \xi_2. \tag{3.5}$$

To put it in other words, **the starting rate of smoking is higher than the rate of relapse**.

3.3 Existence of Steady-states

We can now observe the equilibrium points for the system (3.4). Based on the definition of the equilibrium point, which satisfies $\dot{s} = \dot{e} = \dot{q} = 0$, we can indicate the corresponding equations in the system (3.4) as follows:

$$\begin{aligned} &\xi_1 s(1-s-e-q) - \mu s - \gamma_1 s + \xi_2 sq - \theta se + ce = 0 \\ &\theta se - ce - \mu e - \gamma_2 e = 0 \\ &\gamma_1 s - \xi_2 sq - \mu q + \gamma_2 e = 0 \end{aligned} \tag{3.6}$$

From the second equation of (3.6), we get $e^* = 0$ or

$$s^* = \frac{c+\mu+\gamma_2}{\theta}. \tag{3.7}$$

3.3.1 Smoking-free equilibrium point

Firstly, we will investigate $e^* = 0$ case. If we substitute $e^* = 0$ into the third equation of (3.6), then we have

$$q^* = \frac{s^*\gamma_1}{\mu + s^*\xi_2}. \tag{3.8}$$

From the first equation of (3.6), and together with (3.8), we obtain

$$\frac{s^*\left(\gamma_1\left(\mu + s^*\xi_1\right) + \left(\mu + \left(s^*-1\right)\xi_1\right)\left(\mu + s^*\xi_2\right)\right)}{\mu + s^*\xi_2} = 0 \tag{3.9}$$

Simplifying and factorizing (3.9) leads to

$$s^*\left(\mu^2 + \mu\gamma_1 - \mu\xi_1 + s^*\mu\xi_1 + s^*\gamma_1\xi_1 + s^*\mu\xi_2 - s^*\xi_1\xi_2 + (s^*)^2\xi_1\xi_2\right) = 0 \tag{3.10}$$

Here, we have an explicit solution for (3.10) and that is $s_0^* = 0$. Now, it can be clearly seen that $q_0^* = 0$ from the third equation of (3.6) by using $e_0^* = 0$ and $s_0^* = 0$. We have shown that there is a **smoking-free equilibrium** which is represented by

$$E_0 = (0,0,0).$$

3.3.2 e-Cigarette smoking-free equilibrium

We know that there is another equilibrium point since we have the following quadratic equation in (3.10):

$$A(s^*)^2 + Bs^* + C = 0, \tag{3.11}$$

where the coefficients list is

$$\begin{aligned} A &= \xi_1 \xi_2, \\ B &= \xi_1 (\mu + \gamma_1 - \xi_2) + \mu \xi_2, \\ C &= \mu (\mu + \gamma_1 - \xi_1). \end{aligned} \tag{3.12}$$

For simplicity, we divide the right hand side of B and C by $(\mu + \gamma_1)$ and we rearrange the system by using the following notations:

$$R_0 = \frac{\xi_1}{\mu + \gamma_1} \quad \text{and} \quad R_1 = \frac{\xi_2}{\mu + \gamma_1}.$$

Then the coefficients list (3.12) takes the following form:

$$\begin{aligned} A &= \xi_1 \xi_2, \\ B &= \xi_1 (\mu + \gamma_1)(1 - R_1) + \mu \xi_2, \\ C &= \mu (\mu + \gamma_1)(1 - R_0). \end{aligned} \tag{3.13}$$

It is easy to determine that the roots of (3.11) as follows:

$$\begin{aligned} s_1^* &= \frac{-B + \sqrt{B^2 - 4AC}}{2A} \\ s_2^* &= \frac{-B - \sqrt{B^2 - 4AC}}{2A} \end{aligned} \tag{3.14}$$

We can observe that A is always positive from (3.13). We should examine three cases which are $R_0 < 1$, $R_0 = 1$ and $R_0 > 1$.

Before continuing to examine these cases, we recall that $\xi_1 \geq \xi_2$ from (3.5). This assumption in our model, yields that

$$R_0 \geq R_1.$$

We now examine the three cases:

- If $R_0 < 1$ then $C > 0$ and $B > 0$. In this case, since $s_1^* < 0$ and $s_2^* < 0$ there is no positive root. Thus, these roots are not biologically significant.

- If $R_0 = 1$, which satisfies $C = 0$, it is clear that

$$s_1^* = 0 \quad \text{and} \quad s_2^* = -\frac{B}{A}$$

from (3.13). Since $B > 0$ when $R_1 \leq 1$ then s_2^* is negative. So, we can say that s_2^* is not sensible.

On the other hand, if we consider

$$s_1^* = 0 \quad \text{together with} \quad e_1^* = 0$$

then we get

$$q_1^* = 0$$

which yields the smoking-free equilibrium $E_0 = (0,0,0)$.

- If $R_0 > 1$, then $C < 0$, and there are two distinct real roots since

$$B^2 - 4AC > 0.$$

And these roots have always opposite signs.

Now, we will analyze the sign of s_1^* and s_2^*. Considering

$$\sqrt{B^2 - 4AC} > \sqrt{B^2} = \mid B \mid \tag{3.15}$$

as $C < 0$. Taking (3.15) into consideration, we can easily see that

$$\begin{aligned} s_1^* &= \frac{-B + \sqrt{B^2 - 4AC}}{2A} > \frac{-B + \mid B \mid}{2A} \geq 0 \\ s_2^* &= \frac{-B - \sqrt{B^2 - 4AC}}{2A} < \frac{-B - \mid B \mid}{2A} \leq 0 \end{aligned} \tag{3.16}$$

Consequently, $s_1^* > 0$ and $s_2^* < 0$ when $R_0 > 1$. Here, we do not need to calculate s_2^* because s_2^* is not biologically meaningful.

Additionally, we require that

$$0 \leq \frac{-B + \sqrt{B^2 - 4AC}}{2A} \leq 1.$$

since

$$0 \leq s_1^* \leq 1.$$

If we arrange this inequality as given below

$$\begin{aligned} -B+\sqrt{B^2-4AC} &\leq 2A \\ \sqrt{B^2-4AC} &\leq 2A+B \\ B^2-4AC &\leq 4A^2+4AB+B^2 \\ -C &< A+B \end{aligned}$$

then we get a relation between A, B, and C as follows:

$$0 < A+B+C$$

Namely, if this relation satisfied between A, B, and C then s_1^* satisfies $0 \leq s_1^* \leq 1$.

Putting all the values together in the coefficients list (3.13), we find that

$$\begin{aligned} A+B+C &= \xi_1\xi_2+\xi_1\mu+\xi_1\gamma_1-\xi_1\xi_2+\mu\xi_2+\mu^2+\mu\gamma_1-\mu\xi_1 \\ &= \xi_1\gamma_1+\mu\xi_2+\mu^2+\mu\gamma_1 \\ &> 0 \end{aligned}$$

So, the sum of the values of A, B, and C are always positive. Then s_1^* always exists when $R_0 > 1$. Existence of s_1^* guarantees that the positivity of q_1^* from (3.8). In addition, the following condition must hold:

$$s_1^*+q_1^* \leq 1.$$

Under the conditions mentioned above, we conclude that the e-cigarette smoking-free equilibrium always exists when $R_0 > 1$.

Lastly, after some algebraic computations we get

$$s_1^* = \frac{-\xi_1(\mu+\gamma_1-\xi_2)-\mu\xi_2+\sqrt{-4\mu(\mu+\gamma_1-\xi_1)\xi_1\xi_2+(\xi_1(\mu+\gamma_1-\xi_2)+\mu\xi_2)^2}}{2\xi_1\xi_2}.$$

For simplicity, we use the following notations:

$$\mu+\gamma_1-\xi_1 = d$$

and

$$\mu+\gamma_1-\xi_2 = \tilde{d}$$

Then we take

$$-4\mu d\xi_1\xi_2+\left(\xi_1\tilde{d}+\mu\xi_2\right)^2 = \Delta.$$

Therefore, s_1^* can be written as

$$s_1^* = \frac{-\tilde{d}\xi_1 - \mu\xi_2 + \sqrt{\Delta}}{2\xi_1\xi_2}. \tag{3.17}$$

By substituting (3.17) in (3.8), we obtain

$$q_1^* = \frac{\gamma_1\left(\tilde{d}\xi_1 + \mu\xi_2 - \sqrt{\Delta}\right)}{\xi_2\left(\xi_1(\tilde{d} - 2\mu) + \mu\xi_2 - \sqrt{\Delta}\right)}. \tag{3.18}$$

Hence, the **e-cigarette smoking-free equilibrium** is represented by

$$E_1 = (s_1^*, 0, q_1^*).$$

3.3.3 Endemic equilibrium point

In addition to the equilibrium points, which are represented above by E_0 and E_1, there is another equilibrium point, which is the last one, since we have (3.7).

If we substitute (3.7) into the third equation of (3.6) and solve it together with the first equation of (3.6) then we have

$$q_2^* = \frac{\alpha(\gamma_1(\theta\mu + \alpha\xi_1) - \gamma_2(\theta\mu + (\alpha - \theta)\xi_1))}{(\theta\mu + \alpha\xi_1)(\theta(\mu + \gamma_2) + \alpha\xi_2)}$$

and

$$e_2^* = -\frac{\alpha(\theta\gamma_1(\theta\mu + \alpha\xi_1) + (\theta\mu + (\alpha - \theta)\xi_1)(\theta\mu + \alpha\xi_2))}{\theta(\theta\mu + \alpha\xi_1)(\theta(\mu + \gamma_2) + \alpha\xi_2)}$$

where

$$\alpha = c + \mu + \gamma_2.$$

Then the **endemic equilibrium** is represented by

$$E_2 = \left(\frac{\alpha}{\theta}, e_2^*, q_2^*\right).$$

3.4 Local Stability of the Equilibrium Points

3.4.1 Stability of the smoking-free equilibrium

Let us note that

$$R_0 = \frac{\xi_1}{\mu + \gamma_1}.$$

Lemma 3.4.1 *The system* (3.6) *always has the smoking-free equilibrium* $E_0 = (0,0,0)$. *It is locally asymptotically stable if* $R_0 < 1$. *The equilibrium bifurcates at* $R_0 = 1$. *When* $R_0 > 1$, *the smoking-free equilibrium is a saddle.*

Proof From the system (3.4), the smoking-free equilibrium can be indicated by

$$E_0 = (0,0,0).$$

The Jacobian matrix corresponding to this system is as follows:

$$J = \begin{pmatrix} \xi_1(1-s-e-q) - \xi_1 s - \mu - \gamma_1 + \xi_2 q - \theta e & -\xi_1 s - \theta s + c & -\xi_1 s + \xi_2 s \\ \theta e & \theta s - c - \mu - \gamma_2 & 0 \\ \gamma_1 - \xi_2 q & \gamma_2 & -\xi_2 s - \mu \end{pmatrix} \tag{3.19}$$

The Jacobian matrix at E_0 is calculated as given below:

$$J\,|_{E_0} = \begin{pmatrix} \xi_1 - \mu - \gamma_1 & c & 0 \\ 0 & -c - \mu - \gamma_2 & 0 \\ \gamma_1 & \gamma_2 & -\mu \end{pmatrix} \tag{3.20}$$

We get the eigenvalues from characteristic polynomial, which is equivalent to

$$det(J\,|_{E_0} - \lambda I) = 0,$$

yields

$$\lambda_1 = -\mu, \quad \lambda_2 = -c - \mu - \gamma_2 \quad and \quad \lambda_3 = -\mu - \gamma_1 + \xi_1$$

It can be easily seen that λ_1 and λ_2 are always negative since μ, c and γ_2 are always positive. If $\mu + \gamma_1 > \xi_1$ then λ_3 is negative.

In other words,

- if $R_0 < 1$ then all eigenvalues become negative real numbers and it shows that the smoking-free equilibrium is **stable**.
- if $R_0 = 1$ then the system (3.6) has a non-hyperbolic equilibrium. Thus, there is a **bifurcation** for the system (3.6).
- if $R_0 > 1$, the smoking-free equilibrium is a **saddle** since λ_3 becomes positive real number as λ_1 and λ_2 are negative real numbers.

Next generation matrix method

The corresponding basic reproduction number of the smoking-free equilibrium can be found in another way which is called next generation matrix method [44]. In this method, the aim is to determine the spectral radius of the next generation matrix mentioned in [45]. The dynamics are composed by a system of non-linear ordinary

differential equations that represents the change with time for all sub-population [44, 46] . To get R_0, we only take into consideration the cases that apply to infected people. For this reason, we will construct a linearized infected subsystem and we already know that any linear system of ordinary differential equations can be represented by a matrix [47, 48]. Let $x = (s,e,q)^T$, and then the system (3.4) can be broken down as follows:

$$\frac{dx}{dt} = \mathscr{F}(x) - \mathscr{V}(x), \tag{3.21}$$

where $\mathscr{F}(x)$ is the **transmission part** and $\mathscr{V}(x)$ is the **transition part** [45]. In other words, $\mathscr{F}(x)$ denotes the new infection rates and $\mathscr{V}(x)$ denotes all the other rates. Thus, we construct (3.21) by $\mathscr{F}(x)$ and $\mathscr{V}(x)$, which are given below:

$$\begin{aligned} \mathscr{F}(x) &= \begin{pmatrix} \mathscr{F}_1(x) \\ \mathscr{F}_2(x) \\ \mathscr{F}_3(x) \end{pmatrix} = \begin{pmatrix} \xi_1(1-s-e-q)s + \xi_2 sq \\ 0 \\ 0 \end{pmatrix} \\ \mathscr{V}(x) &= \begin{pmatrix} \mathscr{V}_1(x) \\ \mathscr{V}_2(x) \\ \mathscr{V}_3(x) \end{pmatrix} = \begin{pmatrix} \mu s + \gamma_1 s + \theta se - ce \\ -\theta se + ce + \mu e + \gamma_2 e \\ -\gamma_1 s + \xi_2 sq + \mu q - \gamma_2 e \end{pmatrix} \end{aligned} \tag{3.22}$$

The related Jacobian matrices of $\mathscr{F}(x)$ and $\mathscr{V}(x)$ are easily calculated,

$$\begin{aligned} D\mathscr{F}(x) &= \begin{pmatrix} -\xi_1(-1+e+q+2s) + q\xi_2 & -s\xi_1 & s(-\xi_1+\xi_2) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ D\mathscr{V}(x) &= \begin{pmatrix} e\theta + \mu + \gamma_1 & -c + s\theta & 0 \\ -e\theta & c + \mu + \gamma_2 - s\theta & 0 \\ -\gamma_1 + q\xi_2 & -\gamma_2 & \mu + s\xi_2 \end{pmatrix} \end{aligned} \tag{3.23}$$

Thus, the Jacobian matrices evaluated at the smoking-free equilibrium point $E_0 = (0,0,0)$ can be shown by $\mathscr{F}(x)$ and $\mathscr{V}(x)$, respectively

$$\begin{aligned} F &= \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ V &= \begin{pmatrix} \mu + \gamma_1 & -c & 0 \\ 0 & c + \mu + \gamma_2 & 0 \\ -\gamma_1 & -\gamma_2 & \mu \end{pmatrix} \end{aligned} \tag{3.24}$$

To construct the next generation matrix FV^{-1}, we firstly find the inverse of V. Then V^{-1} can be easily calculated as

$$V^{-1} = \begin{pmatrix} \frac{1}{\mu+\gamma_1} & \frac{c}{(\mu+\gamma_1)(c+\mu+\gamma_2)} & 0 \\ 0 & \frac{1}{c+\mu+\gamma_2} & 0 \\ \frac{\gamma_1}{\mu(\mu+\gamma_1)} & \frac{\mu\gamma_2+\gamma_1(c+\gamma_2)}{\mu(\mu+\gamma_1)(c+\mu+\gamma_2)} & \frac{1}{\mu} \end{pmatrix}, \tag{3.25}$$

We easily get the next generation matrix by some computations, which is

$$FV^{-1} = \begin{pmatrix} \frac{\xi_1}{\mu+\gamma_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.26}$$

Last step is determining the **spectral radius** (ρ) of the next generation matrix [49]. More clearly, we calculate the maximum eigenvalue of FV^{-1}. It can be clearly seen that the maximum eigenvalue equals to $\frac{\xi_1}{\mu+\gamma_1}$ from (3.26). Thus, we have

$$R_0 = \rho(FV^{-1}) = \frac{\xi_1}{\mu+\gamma_1}. \tag{3.27}$$

3.4.2 Stability of the e-cigarette smoking-free equilibrium

In Section (3.3.2), we defined the following parameters:

$$\begin{aligned} R_0 &= \frac{\xi_1}{\mu+\gamma_1}, \\ \lambda_1 &= \theta s_1^* - c - \mu - \gamma_2, \\ \tilde{B} &= (\mu + s_1^*\xi_2)(\gamma_1(\mu + s_1^*\xi_1) + (\mu + s_1^*\xi_2)(2\mu + (2s_1^* - 1)\xi_1 + s_1^*\xi_2)) \\ \tilde{C} &= (\mu + s_1^*\xi_2)((\mu + (2s_1^* - 1)\xi_1)(\mu + s_1^*\xi_2)^2 + \gamma_1(\mu^2 + s_1^*\xi_1(2\mu + s_1^*\xi_2))). \end{aligned}$$

Lemma 3.4.2 *If $R_0 > 1$ and $s_1^* + q_1^* \leq 1$ then the e-cigarette smoking free equilibrium $E_1 = (s_1^*, 0, q_1^*)$ exists. Moreover, E_1 is stable if and only if $\lambda_1 < 0$ together with $\tilde{B} > 0$ and $\tilde{C} > 0$.*

Proof We do not substitute the values of s_1^* and q_1^* explicitly to simplify the notations. For that reason, we use $e_1^* = 0$ and get the Jacobian matrix as

$$J = \begin{pmatrix} \xi_1(1 - s_1^* - q_1^*) - \xi_1 s_1^* - \mu - \gamma_1 + \xi_2 q_1^* & -\xi_1 s_1^* - \theta s_1^* + c & -\xi_1 s_1^* + \xi_2 s_1^* \\ 0 & \theta s_1^* - c - \mu - \gamma_2 & 0 \\ \gamma_1 - \xi_2 q_1^* & \gamma_2 & -\xi_2 s_1^* - \mu \end{pmatrix} \tag{3.28}$$

We have constructed the characteristic polynomial using the relationship between q^* and s^*, which is obtained in (3.8), as follows:

$$(-c + \theta s_1^* - \lambda - \mu - \gamma_2)(\tilde{A}\lambda^2 + \tilde{B}\lambda + \tilde{C}) = 0 \tag{3.29}$$

where the coefficients list is

$$\begin{aligned} \tilde{A} &= (\mu + s_1^*\xi_2)^2 \\ \tilde{B} &= (\mu + s_1^*\xi_2)(\gamma_1(\mu + s_1^*\xi_1) + (\mu + s_1^*\xi_2)(2\mu + (2s_1^* - 1)\xi_1 + s_1^*\xi_2)) \\ \tilde{C} &= (\mu + s_1^*\xi_2)((\mu + (2s_1^* - 1)\xi_1)(\mu + s_1^*\xi_2)^2 + \gamma_1(\mu^2 + s_1^*\xi_1(2\mu + s_1^*\xi_2))) \end{aligned} \tag{3.30}$$

It can be clearly seen that the first eigenvalue of $J\,|_{E_1}$ is found as

$$\lambda_1 = (-c + \theta s_1^* - \mu - \gamma_2).$$

Since we have a quadratic equation as

$$\tilde{A}\lambda^2 + \tilde{B}\lambda + \tilde{C} = 0 \tag{3.31}$$

in (3.29), it is easily seen that we have two other eigenvalues as stated below:

$$\begin{aligned} \lambda_2 = & \quad \frac{-\tilde{B} + \sqrt{\tilde{B}^2 - 4\tilde{A}\tilde{C}}}{2\tilde{A}} \\ \lambda_3 = & \quad \frac{-\tilde{B} - \sqrt{\tilde{B}^2 - 4\tilde{A}\tilde{C}}}{2\tilde{A}} \end{aligned} \tag{3.32}$$

Further, we already know

$$\begin{aligned} \lambda_2 + \lambda_3 &= -\frac{\tilde{B}}{2\tilde{A}} \\ \lambda_2\lambda_3 &= \frac{\tilde{C}}{\tilde{A}} \end{aligned} \tag{3.33}$$

To interpret the signs of the eigenvalues, we should examine the signs of $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$. We take notice that $\tilde{A}$ is always positive since $\mu > 0$, $\xi_2 > 0$ and s_1^*. Thus, we should study the signs of $\tilde{B}$ and $\tilde{C}$. We have two cases here as given below:

- If $\tilde{C} < 0$, it is clearly seen that

  $$\tilde{B}^2 - 4\tilde{A}\tilde{C} > 0$$

  and it yields

  $$\sqrt{\tilde{B}^2 - 4\tilde{A}\tilde{C}} > 0. \tag{3.34}$$

  It shows that λ_2 and λ_3 are real roots. And we also obtain

  $$\lambda_2\lambda_3 < 0 \tag{3.35}$$

  in (3.33). It can be clearly seen that λ_2 and λ_3 always have opposite signs. Thus, if $\tilde{C} < 0$ then the quadratic equation always has two real roots and these roots always have opposite signs.

  Consequently, the e-cigarette smoking free equilibrium is **saddle**, irrespective of the sign of λ_1.

- If $\tilde{C} > 0$, the information we have is inadequate to say

$$\sqrt{\tilde{B}^2 - 4\tilde{A}\tilde{C}} > 0$$

or

$$\sqrt{\tilde{B}^2 - 4\tilde{A}\tilde{C}} < 0.$$

For this reason, we can not determine whether these eigenvalues are real or not. However, we can analyze the signs of the real parts of the eigenvalues.

In this case, we always have

$$\lambda_2 \lambda_3 > 0 \tag{3.36}$$

since $\tilde{C} > 0$. So, we should examine the sign of $\tilde{B}$ as follows:

  - If $\tilde{B} < 0$, we observe

$$\lambda_2 + \lambda_3 > 0. \tag{3.37}$$

Putting (3.36) and (3.37) together yields

$$Re(\lambda_2) > 0 \quad \text{and} \quad Re(\lambda_3) > 0.$$

Thus, the e-cigarette smoking-free equilibrium is **unstable**. More specifically,

    * If $\lambda_1 < 0$ then the e-cigarette smoking free equilibrium is **saddle**.
    * If $\lambda_1 > 0$ then the e-cigarette smoking free equilibrium is **unstable focus** or **unstable node**.

  - If $\tilde{B} > 0$, we easily observe

$$\lambda_2 + \lambda_3 < 0. \tag{3.38}$$

If we consider (3.36) with (3.38), then we have

$$Re(\lambda_2) < 0 \quad \text{and} \quad Re(\lambda_3) < 0.$$

Moreover,

    * If $\lambda_1 > 0$, then the e-cigarette smoking free equilibrium is **saddle**.
    * If $\lambda_1 < 0$, then the e-cigarette smoking free equilibrium is **stable focus** or **stable node**.

3.4.3 Stability of the endemic equilibrium

Let us note that

$$\hat{B} = \gamma_1 + \mu - \xi_1 + 2\xi_1 s_2^* + e_2^*(\theta + \xi_1).$$

In the next Lemma, we assume that $\xi_1 = \xi_2$ since otherwise we do not get analytical expressions. We will construct the Jacobian matrix at $E_2 = (\frac{\alpha}{\theta}, e_2^*, q_2^*)$ and then we will assume that $\xi_1 = \xi_2$. In (3.45), we will rearrange the Jacobian matrix using $\xi_1 = \xi_2$.

Lemma 3.4.3 *Suppose that $\xi_1 = \xi_2$. If*

$$\frac{\theta\xi_1(\theta\mu + \alpha\xi_2)}{(\theta\mu + \alpha\xi_1)(\theta\gamma_1 + \theta\mu + \alpha\xi_2)} \geq 1 \quad \textit{and} \quad \frac{\theta\gamma_2\xi_1 + \gamma_1(\theta\mu + \alpha\xi_1)}{\gamma_2(\theta\mu + \alpha\xi_1)} \geq 1$$

with

$$s_2^* + e_2^* + q_2^* < 1$$

then the endemic equilibrium $E_2 = (s_2^*, e_2^*, q_2^*)$ exists. Moreover, E_2 is stable if and only if $\hat{B} > 0$.

Proof We have proved that the system (3.6) has the endemic equilibrium, yet this equilibrium is not always biologically meaningful. Before continuing to stability analysis we firstly find some restrictions to have admissible equilibrium solutions. To get meaningful results, e_2^* and q_2^* must respectively satisfy

$$-\alpha\theta\gamma_1(\theta\mu + \alpha\xi_1) - \alpha\theta\mu(\theta\mu + \alpha\xi_2) - \alpha^2\xi_1(\theta\mu + \alpha\xi_2) + \alpha\theta\xi_1(\theta\mu + \alpha\xi_2) \geq 0 \tag{3.39}$$

and

$$\alpha\gamma_1(\theta\mu + \alpha\xi_1) - \alpha\gamma_2\theta\mu - \alpha\gamma_2(\alpha - \theta)\xi_1 \geq 0 \tag{3.40}$$

These conditions are respectively equivalent to

$$\frac{\theta\xi_1(\theta\mu + \alpha\xi_2)}{(\theta\mu + \alpha\xi_1)(\theta\gamma_1 + \theta\mu + \alpha\xi_2)} \geq 1 \tag{3.41}$$

and

$$\frac{\theta\gamma_2\xi_1 + \gamma_1(\theta\mu + \alpha\xi_1)}{\gamma_2(\theta\mu + \alpha\xi_1)} \geq 1 \tag{3.42}$$

in (3.39) and (3.40).

Besides, we take into account

$$s_2^* + e_2^* + q_2^* < 1. \tag{3.43}$$

We can now study on stability analysis of the endemic equilibrium. The Jacobian matrix at $E_2 = (\frac{\alpha}{\theta}, e_2^*, q_2^*)$ is found as follows:

$$J = \begin{pmatrix} \xi_1(1-\frac{\alpha}{\theta}-e_2^*-q_2^*) - \xi_1\frac{\alpha}{\theta} - \mu - \gamma_1 + \xi_2 q_2^* - \theta e_2^* & -\xi_1\frac{\alpha}{\theta} - \alpha + c & -\xi_1\frac{\alpha}{\theta} + \xi_2\frac{\alpha}{\theta} \\ \theta e_2^* & 0 & 0 \\ \gamma_1 - \xi_2 q_2^* & \gamma_2 & -\xi_2\frac{\alpha}{\theta} - \mu \end{pmatrix} \tag{3.44}$$

Under the assumption $\xi_1 = \xi_2$, the Jacobian matrix becomes

$$J = \begin{pmatrix} \xi_1(1-s_2^*-e_2^*-q_2^*) - \xi_1 s_2^* - \mu - \gamma_1 + \xi_1 q_2^* - \theta e_2^* & -\xi_1 s_2^* - \alpha + c & 0 \\ \theta e_2^* & 0 & 0 \\ \gamma_1 - \xi_1 q_2^* & \gamma_2 & -\xi_1 s_2^* - \mu \end{pmatrix} \tag{3.45}$$

For simplicity, if we consider

$$\begin{aligned} \hat{A} &= 1, \\ \hat{B} &= \gamma_1 + \mu - \xi_1 + 2\xi_1 s_2^* + e_2^*(\theta + \xi_1), \\ \hat{C} &= e_2^*\theta(\alpha - c + \xi_1 s_2^*) \end{aligned} \tag{3.46}$$

we obtain the eigenvalues of (3.45) as listed below:

$$\begin{aligned} \lambda_1 &= -\mu - \xi_1 s_2^* \\ \lambda_2 &= \frac{1}{2}\left(-\hat{B} - \sqrt{\hat{B}^2 - 4\hat{A}\hat{C}}\right) \\ \lambda_3 &= \frac{1}{2}\left(-\hat{B} + \sqrt{\hat{B}^2 - 4\hat{A}\hat{C}}\right). \end{aligned} \tag{3.47}$$

If we consider (3.47) with $s_2^* > 0$, $\xi_1 > 0$ and $\mu > 0$ we can clearly see that $\lambda_1 < 0$.

To interpret the stability of the endemic equilibrium point, we firstly construct the quadratic equation whose roots correspond to λ_2 and λ_3. Then we have the following quadratic equation:

$$\hat{A}\lambda^2 + \hat{B}\lambda + \hat{C} = 0 \tag{3.48}$$

the coefficients of which are given in (3.46). We should examine the sign of $\hat{C}$.

We have an observation which indicates that $\hat{C}$ is always positive since the equilibrium is always positive. We can easily see that $\hat{C}$ can not be negative in the following equation:

$$\begin{aligned} \alpha - c &= (c + \mu + \gamma_2) - c \\ &= \mu + \gamma_2. \end{aligned} \tag{3.49}$$

If we combine (3.49) with $\hat{C} = e_2^* \theta(\alpha - c + \xi_1 s_2^*)$, we get

$$\hat{C} = e_2^* \theta(\mu + \gamma_2 + \xi_1 s_2^*) \quad \Longrightarrow \quad \hat{C} > 0 \tag{3.50}$$

since all the parameters are positive. So, we should examine the sign of $\hat{B}$ since $\hat{C}$ is always positive. Now, we have two cases which are given below:

- If $\hat{B} > 0$, then we have

$$\begin{aligned} \gamma_1 + \mu + 2\xi_1 s_2^* + e_2^*(\theta + \xi_1) > \xi_1 \quad \Longrightarrow \quad & e_2^*(\theta + \xi_1) > \xi_1 - 2\xi_1 s_2^* - \mu - \gamma_1 \\ & e_2^* > \frac{\xi_1(1 - 2s_2^*) - \mu - \gamma_1}{\theta + \xi_1}. \end{aligned} \tag{3.51}$$

  If we consider (3.48), we always achieve

$$\begin{aligned} \lambda_2 + \lambda_3 &= -\frac{\hat{B}}{2\hat{A}} \\ \lambda_2 \lambda_3 &= \frac{\hat{C}}{\hat{A}} \end{aligned} \tag{3.52}$$

  Putting (3.50) and (3.52) together yields

$$Re(\lambda_2) < 0 \quad \text{and} \quad Re(\lambda_3) < 0 \tag{3.53}$$

  since

$$\lambda_2 + \lambda_3 < 0 \quad \text{and} \quad \lambda_2 \lambda_3 > 0.$$

  Thus, the endemic equilibrium is **stable focus** or **stable node** since λ_1, λ_2, $\lambda_3 < 0$.

- If $\hat{B} < 0$, we obtain

$$e_2^* < \frac{\xi_1(1 - 2s_2^*) - \mu - \gamma_1}{\theta + \xi_1}.$$

  In the same technique where $\hat{B} > 0$, we get

$$Re(\lambda_2) > 0 \quad \text{and} \quad Re(\lambda_3) > 0. \tag{3.54}$$

  Consequently, the endemic equilibrium is **saddle** since $\lambda_1 < 0$ while λ_2 and λ_3 are positive.

To achieve better interpretations, this analysis will be done with numerical solutions in the next chapters.

3.5 Global Stability

3.5.1 Global Stability of the smoking-free equilibrium point

Let us recall

$$R_0 = \frac{\xi_1}{\mu + \gamma_1}.$$

Lemma 3.5.1 *If $R_0 < 1$ and $\frac{\mu\gamma_2}{\gamma_1} < \mu + \frac{\alpha\xi_1}{\theta} + \frac{\xi_1\gamma_2}{\gamma_1}$ are satisfied then the smoking-free equilibrium is globally asymptotically stable.*

Remark 2 *When $\frac{\gamma_2}{\gamma_1} < 1$ or $\mu < \xi_1$ is satisfied, we obtain $\frac{\mu\gamma_2}{\gamma_1} < \mu + \frac{\alpha\xi_1}{\theta} + \frac{\xi_1\gamma_2}{\gamma_1}$.*

Proof We construct a Lyapunov function to examine the global stability of E_0. We consider the following Lyapunov function:

$$V(s,e,q) = s + he + kq. \tag{3.55}$$

with $h,k > 0$. It is easy to see that $V(E_0) = 0$ and $V(s,e,q) > 0$ for all $(s,e,q) \neq (0,0,0)$. Then we need to find the derivative of the Lyapunov function and we will choose h and k accordingly. Let us define h and k as

$$k = \frac{\mu + \gamma_1 - \xi_1}{\gamma_1} \quad \textit{and} \quad h = 1 + \frac{\xi_1}{\theta} \tag{3.56}$$

Then we find the derivative of the Lyapunov function:

$$\begin{aligned} \dot{V} = &-\xi_1 s^2 - k\mu q + s(\xi_1 - \mu - \gamma_1 + k\gamma_1) + \\ &e(c - hc - h\mu - h\gamma_2 + k\gamma_2) + sq(-\xi_1 + \xi_2 - k\xi_2) + se(-\xi_1 - \theta + h\theta) \end{aligned} \tag{3.57}$$

E_0 is locally asymptotically stable if and only if $R_0 < 1$. This yields

$$\xi_1 < \mu + \gamma_1 \tag{3.58}$$

and this also guarantees that $k > 0$. However, this is not enough to obtain $\dot{V} < 0$. We now clearly require $\dot{V} < 0$. It is easy to see that h is always positive. Using the positiveness of h together with $c - hc - h\mu - h\gamma_2 + k\gamma_2 < 0$ we get

$$\frac{\mu\gamma_2}{\gamma_1} < \mu + \frac{\alpha\xi_1}{\theta} + \frac{\xi_1\gamma_2}{\gamma_1}. \tag{3.59}$$

where $\alpha = c + \mu + \gamma_2$. By (3.56), it is easy to see that $\dot{V} < 0$ since $\xi_1 \geq \xi_2$. We proved that when (3.58) and (3.59) are satisfied together then E_0 is **globally asymptotically stable** in line with Theorem (1.2.2).

3.5.2 The unsuccessful search for a Lyapunov function for the global stability of the e-cigarette smoking-free and endemic equilibrium points

To examine the stability of the e-cigarette smoking-free and endemic equilibrium points we studied on different forms of Lyapunov function [50, 51]. For instance, we considered the functions

$$V = x_1(s-s^*)^2 + x_2(e-e^*)^2 + x_3(q-q^*)^2 \tag{3.60}$$

and

$$V = x_1\left(s-s^*-s^*\ln\left(\frac{s}{s^*}\right)\right) + x_2\left(e-e^*-e^*\ln\left(\frac{e}{e^*}\right)\right) + \left(q-q^*-q^*\ln\left(\frac{q}{q^*}\right)\right). \tag{3.61}$$

However, we could not find conditions on x_1, x_2 and x_3 for which the above choices of V are Lyapunov Functions. Then we considered another Lyapunov function candidate for e-cigarette smoking-free equilibrium which is given below:

$$V = x_1\left(s-s^*-s^*\ln\left(\frac{s}{s^*}\right)\right) + x_2\left(e-e^*-e^*\ln\left(\frac{e}{e^*}\right)\right) + x_3\left(q-q^*-q^*\ln\left(\frac{q}{q^*}\right)\right) \tag{3.62}$$

Using (3.62) gives us more useful calculations to use Theorem (1.2.2). Firstly, let us find the derivative of (3.62):

$$\begin{aligned}\dot{V} &= x_1\dot{s}\left(1-\frac{s^*}{s}\right) + x_2\dot{e}\left(1-\frac{e^*}{e}\right) + x_3\dot{q}\left(1-\frac{q^*}{q}\right)\\ &= x_1\left[\xi_1(1-s-e-q) - \mu - \gamma_1 + \xi_2 q - \theta e + \frac{ce}{s}\right](s-s^*)\\ &+ x_2\left[\theta s - c - \mu - \gamma_2\right](e-e^*) + x_3\left[\gamma_1\frac{s}{q} - \xi_2 s - \mu + \gamma_2\frac{e}{q}\right](q-q^*)\end{aligned} \tag{3.63}$$

If we substitute

$$\begin{aligned}\theta s^* &= c + \mu + \gamma_2,\\ \mu + \gamma_1 &= \xi_1(1-s^*-e^*-q^*) + \xi_2 q^* - \theta e^* + \frac{ce^*}{s^*},\\ \mu &= \gamma_1\frac{s^*}{q^*} - \xi_2 s^* + \gamma_2\frac{e^*}{q^*}\end{aligned} \tag{3.64}$$

into (3.63), we obtain

$$\begin{aligned}\dot{V} &= x_1\left[\xi_1\left[(s^*-s)+(e^*-e)+(q^*-q)\right] + \xi_2(q-q^*) - \theta(e-e^*) + c(\frac{e}{s}-\frac{e^*}{s^*})\right](s-s^*)\\ &+ x_2\theta(s-s^*)(e-e^*) + x_3\left[\gamma_1\left(\frac{s}{q}-\frac{s^*}{q^*}\right) - \xi_2(s-s^*) + \gamma_2\left(\frac{e}{q}-\frac{e^*}{q^*}\right)\right](q-q^*)\\ &= -x_1\xi_1(s-s^*)^2 + (s-s^*)(e-e^*)\left[-x_1\xi_1 - x_1\theta + x_2\theta\right] + (q-q^*)(s-s^*)\left[x_1\xi_2 - x_1\xi_1 - x_3\xi_2\right]\\ &+ cx_1\left(\frac{e}{s}-\frac{e^*}{s^*}\right)(s-s^*) + \gamma_2 x_3\left(\frac{e}{q}-\frac{e^*}{q^*}\right)(q-q^*) + \gamma_1 x_3\left(\frac{s}{q}-\frac{s^*}{q^*}\right)(q-q^*).\end{aligned} \tag{3.65}$$

We can not still yet show that $\dot{V} < 0$. Thus, the above V does not satisfy the condition of the theorem (1.2.2), either. As a result, the existence of Lyapunov functions for the model or not is still an open problem. Future research could be focused on finding appropriate Lyapunov function(s) for the problem.

3.6 Numerical Results

In this section, we present some numerical simulations using the parameters given in Table (3.1), which was used in [7], for the system (3.4). In line with the information obtained from [7], we assumed that the mortality rate, μ, is estimated by the inverse of life expectancy at birth for the total population in the United States [52]. The data used in this study were obtained from people who smoked for 50 years according to National Health Interview Survey(NHIS) [53]. Taking this data into consideration, we changed our parameters to some extent in order to get more meaningful result for our model.

Table 3.1 : Description and estimation of parameters.

Parameter	**Description**	**Value** $(year^{-1})$	**Reference**
μ	Birth and death rates	$1/79.8$	Estimated in [52]
ξ_1	Transmission rate from potential smoker compartment to smoker compartment due to peer pressure	0.1961	Estimated in [54]
ξ_2	Rate of relapse due to peer pressure	0.0101	Estimated in [54]
γ_1	Treatment rate of people who quit smoking by their own will	0.0772	Estimated in [54]
γ_2	The cessation rate by using e-cigarette	0.1008	Estimated in [7]
c	The return rate to smoker class, after using e-cigarettes, by their own will	0.0822	Estimated in [7]
θ	The transformation rate from smokers to e-cigarettes smokers due to peer pressure	0.1245	Estimated in [7]

3.6.1 Numerical verification of stability of the smoking-free equilibrium

The parameters are selected in compliance with the Table (3.1). We investigate the simulations of the smoking-free equilibrium in four cases.

In the **first case**, we choose

$$\gamma_1 = 0.35 \tag{3.66}$$

instead of the one given in Table (3.1). The corresponding R_0 value is

$$R_0 = 0.5409 < 1.$$

If we consider the condition which is mentioned in (3.59) and rearrange it as

$$\Re = \mu + \frac{\alpha\xi_1}{\theta} + \frac{\xi_1\gamma_2}{\gamma_1} - \frac{\mu\gamma_2}{\gamma_1}$$

where $\alpha = c + \mu + \gamma_2$.

Then we find

$$\Re = 0.3733 > 0.$$

We give the simulations where $R_0 < 1$ as $\Re > 0$ with different initial conditions in Figures (3.2) and (3.3). The figures verify that the smoking-free equilibrium

$$E_0 = (s_0^*, e_0^*, q_0^*) = (0,0,0)$$

is **globally asymptotically stable**.

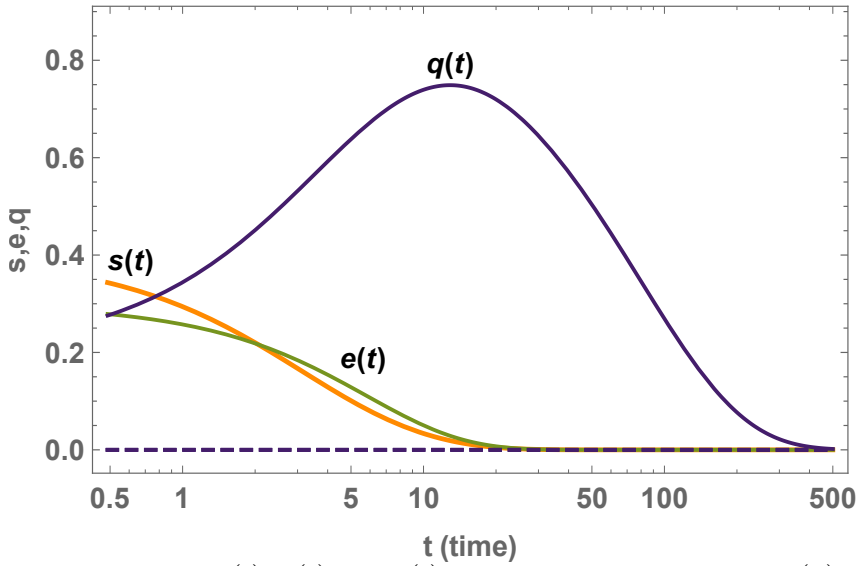


Figure 3.2 : Graph of $s(t)$, $e(t)$ and $q(t)$ with the initial conditions $s(0) = 0.4$, $e(0) = 0.3$ and $q(0) = 0.2$. $R_0 = 0.5409$ and $\Re = 0.3733$.

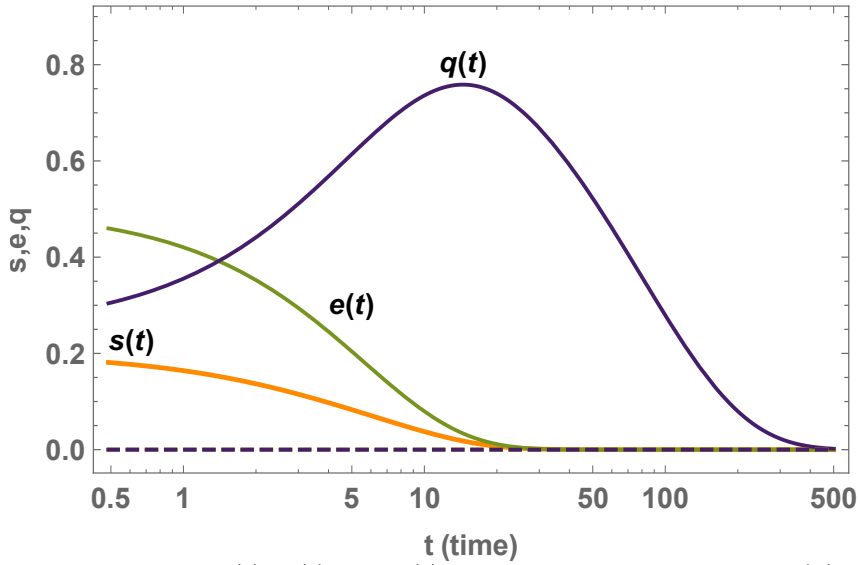


Figure 3.3 : Graph of $s(t)$, $e(t)$ and $q(t)$ with the initial conditions $s(0) = 0.2$, $e(0) = 0.5$ and $q(0) = 0.25$. $R_0 = 0.5409$ and $\Re = 0.3733$.

In the **second case**, we investigate the case which satisfies $R_0 < 1$ with $\Re < 0$. Choosing

$$\mu = 0.14, \quad \xi_1 = 0.1, \quad \gamma_1 = 0.015 \quad \text{and} \quad \theta = 1$$

yields

$$R_0 = 0.6451 \quad \text{and} \quad \Re = -0.0965.$$

Figures (3.4) and (3.5) are given for different initial conditions which are

$$s(0) = 0.4, \quad e(0) = 0.3, \quad q(0) = 0.2$$

and

$$s(0) = 0.2, \quad e(0) = 0.5, \quad q(0) = 0.25$$

respectively.

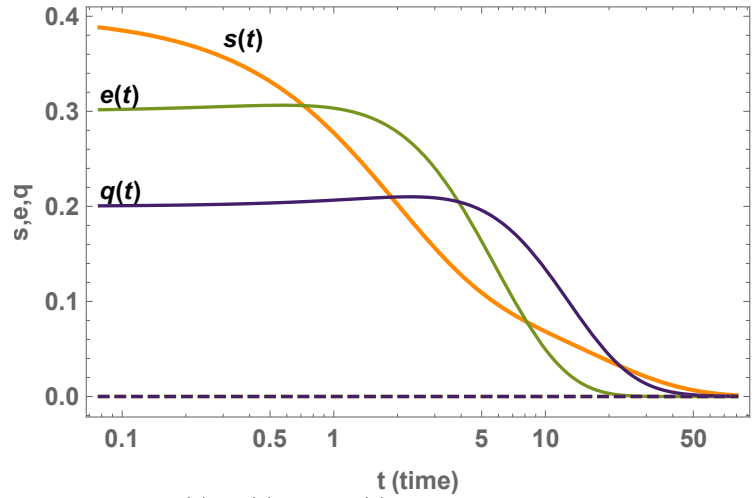


Figure 3.4 : Graph of $s(t)$, $e(t)$ and $q(t)$ with the initial conditions $s(0) = 0.4$, $e(0) = 0.3$ and $q(0) = 0.2$. Parameter values $\mu = 0.14$, $\xi_1 = 0.1$, $\gamma_1 = 0.015$ and $\theta = 1$. $R_0 = 0.6451$ and $\Re = -0.0965$.

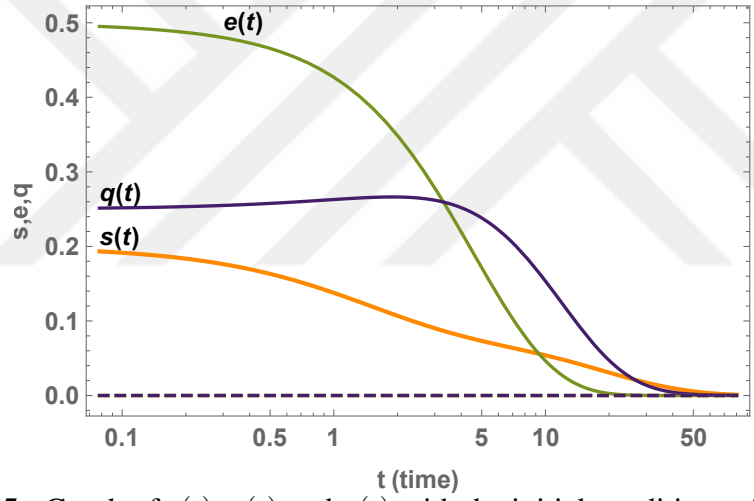


Figure 3.5 : Graph of $s(t)$, $e(t)$ and $q(t)$ with the initial conditions $s(0) = 0.2$, $e(0) = 0.5$ and $q(0) = 0.25$. Parameter values Parameter values $\mu = 0.14$, $\xi_1 = 0.1$, $\gamma_1 = 0.015$ and $\theta = 1$. $R_0 = 0.6451$ and $\Re = -0.0965$.

The figures suggest that E_0 is **globally asymptotically stable** for $R_0 < 1$ and $\Re < 0$. If we compare the first case with the second case, we see that the positiveness of $\Re$ is insignificant when $R_0 < 1$. Thus, the smoking-free equilibrium is **globally asymptotically stable** when $R_0 < 1$. Consequently, we see that $\Re > 0$ is necessary for Lemma (3.5.1) but actually it is not necessary to interpret the stability of the system when $R_0 < 1$.

3.6.1.1 Bifurcation at $R_0 = 1$

In the **third case**, we examine the dynamics of system (3.4) which satisfies $R_0 = 1$. The first observation of this case shows that $\mathfrak{R}$ is always positive when $R_0 = 1$. It is easy to see this from the following notation:

$$\mathfrak{R} = \mu + \frac{\alpha\xi_1}{\theta} + \frac{\gamma_2}{\gamma_1}(\xi_1 - \mu). \tag{3.67}$$

since

$$R_0 = 1 \quad \Longrightarrow \quad \xi_1 = \mu + \gamma_1 \tag{3.68}$$

Substituting (3.68) in (3.67) yields

$$\begin{aligned} \mathfrak{R} &= \mu + \frac{\alpha\xi_1}{\theta} + \frac{\gamma_2}{\gamma_1}(\mu + \gamma_1 - \mu) \\ &= \mu + \frac{\alpha\xi_1}{\theta} + \gamma_2. \end{aligned} \tag{3.69}$$

It is obvious that the value of $\mathfrak{R}$ is always positive when $R_0 = 1$. It is also easy to observe that $\mathfrak{R}$ is always positive when $R_0 \geq 1$.

To get the illustrations of this case, we choose

$$\mu = 0.0125 \quad \text{and} \quad \xi_1 = 0.0897$$

and we get

$$R_0 = 1 \quad \text{and} \quad \mathfrak{R} = 0.2541.$$

Then we obtain the following graphics for different initial values and these graphics suggest that the smoking-free equilibrium E_0 is **globally asymptotically stable** when $R_0 = 1$. In other words, the number of smokers, e-cigarette smokers and quitters eventually goes down to zero when $R_0 = 1$. However, this convergence occurs in a very long time as seen in Figures (3.6) and (3.7).

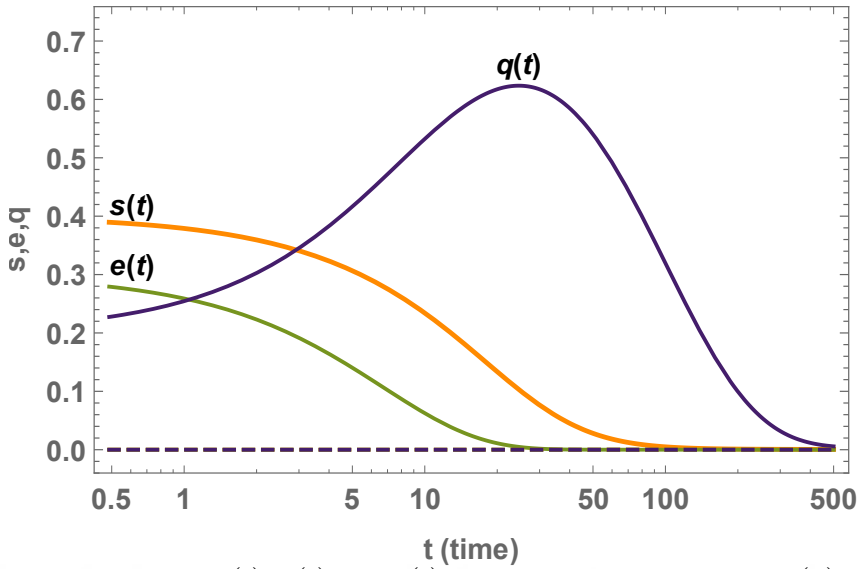


Figure 3.6 : Graph of $s(t)$, $e(t)$ and $q(t)$ with the initial conditions $s(0) = 0.4$, $e(0) = 0.3$ and $q(0) = 0.2$. Parameter values $\mu = 0.0125$ and $\xi_1 = 0.0897$. $R_0 = 1$ and $\Re = 0.2541$.

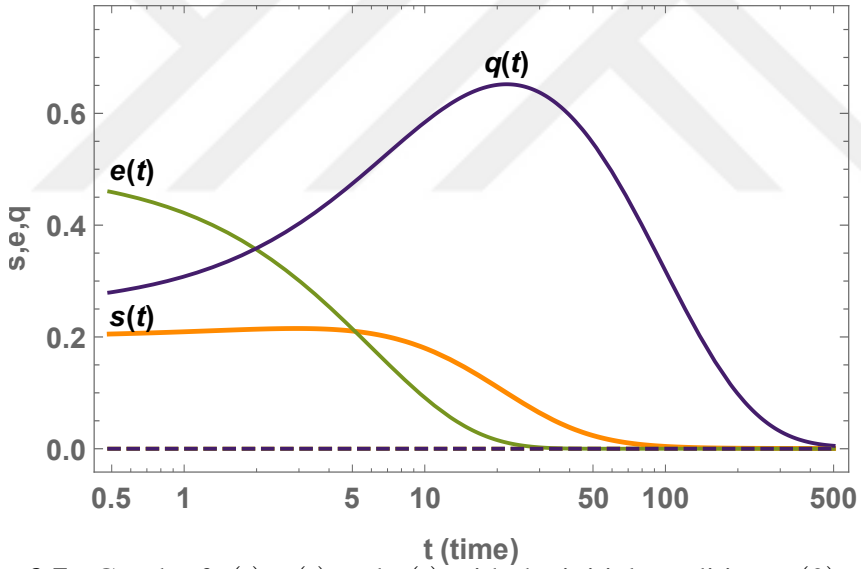


Figure 3.7 : Graph of $s(t)$, $e(t)$ and $q(t)$ with the initial conditions $s(0) = 0.2$, $e(0) = 0.5$ and $q(0) = 0.25$. Parameter values $\mu = 0.0125$ and $\xi_1 = 0.0897$. $R_0 = 1$ and $\Re = 0.2541$.

As discussed before, there is a bifurcation at $R_0 = 1$. That is for $R_0 > 1$ the smoking-free equilibrium loses stability. To see the bifurcation, we take

$$\mu = 0.0125 \quad \text{and} \quad \xi_1 = 0.098$$

and we obtain

$$R_0 = 1.0925 \quad \text{and} \quad \Re = 0.2780. \tag{3.70}$$

The corresponding figure is shown by Figure (3.8) with the initial conditions

$$s(0) = 0.4, \quad e(0) = 0.3 \quad \text{and} \quad q(0) = 0.2. \tag{3.71}$$

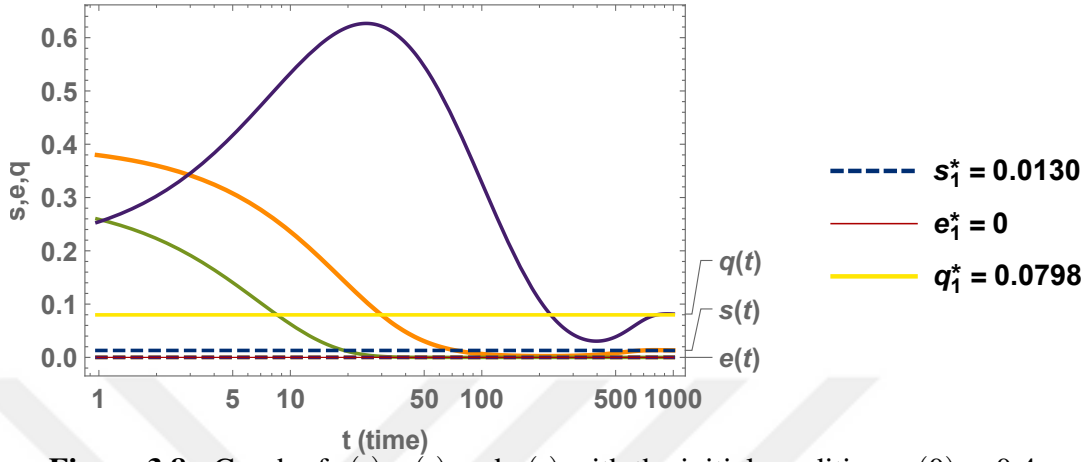


Figure 3.8 : Graph of $s(t)$, $e(t)$ and $q(t)$ with the initial conditions $s(0) = 0.4$, $e(0) = 0.3$ and $q(0) = 0.2$. Parameter values $\mu = 0.0125$ and $\xi_1 = 0.098$. $R_0 = 1.0925$ and $\Re = 0.2780$.

We also obtain

$$R_0 = 1.1148 \quad \text{and} \quad \Re = 0.2837 \tag{3.72}$$

and

$$R_0 = 1.2151 \quad \text{and} \quad \Re = 0.3096 \tag{3.73}$$

by taking

$$\mu = 0.0125 \quad \text{and} \quad \xi_1 = 0.1$$

and

$$\mu = 0.0125 \quad \text{and} \quad \xi_1 = 0.109$$

respectively. Figures (3.9) and (3.10) are given with the same initial conditions mentioned in (3.71).

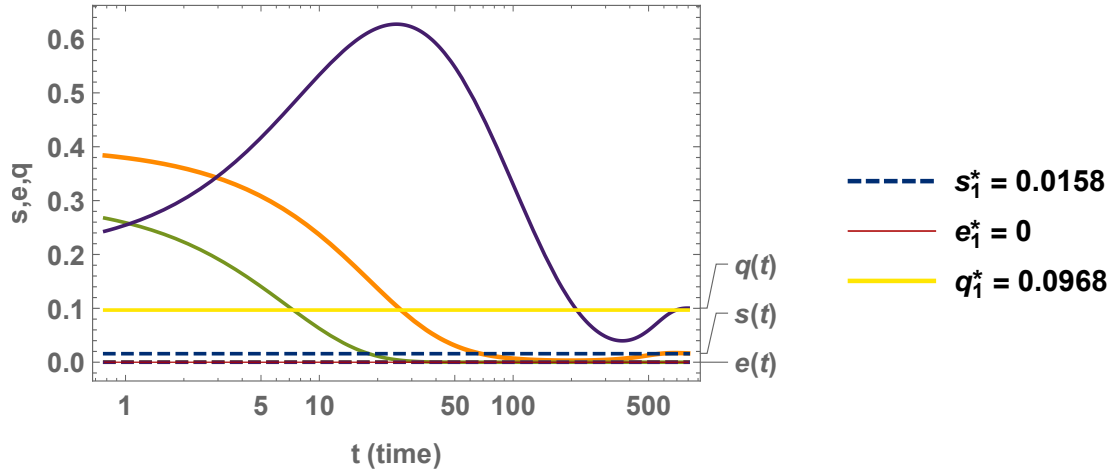


Figure 3.9 : Graph of $s(t)$, $e(t)$ and $q(t)$ with the initial conditions $s(0) = 0.4$, $e(0) = 0.3$ and $q(0) = 0.2$. Parameter values $\mu = 0.0125$ and $\xi_1 = 0.1$. $R_0 = 1.1148$ and $\mathfrak{R} = 0.2837$.

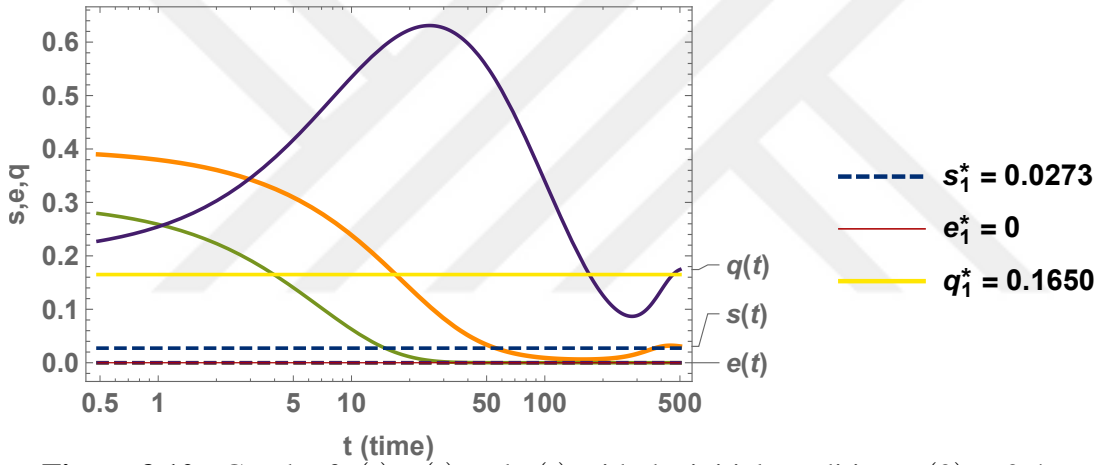


Figure 3.10 : Graph of $s(t)$, $e(t)$ and $q(t)$ with the initial conditions $s(0) = 0.4$, $e(0) = 0.3$ and $q(0) = 0.2$. Parameter values $\mu = 0.0125$ and $\xi_1 = 0.109$. $R_0 = 1.2151$ and $\mathfrak{R} = 0.3096$.

The Figures (3.8), (3.9) and (3.10) show that the smoking-free equilibrium exhibits a transition, which is called a bifurcation, when we increase the value of R_0 with a very small perturbation from the value of 1. Then the smoking-free equilibrium becomes **unstable**.

In the **fourth case**, to see the solution behaviour of the system (3.4) we consider a case where $R_0 > 1$. From now on, we fix the value of the basic reproduction number as

$$R_0 = 2.1854 > 1$$

by parameters given in Table (3.1). Furthermore, we assume the initial conditions as

$$s(0) = 0.4, \quad e(0) = 0.3 \quad \text{and} \quad q(0) = 0.2$$

for all the simulations in the second case. Additionally, let us recall that $\mathfrak{R}$ is always positive when $R_0 \geq 1$ from (3.67), (3.68) and (3.69).

The parameters as

$$\xi_2 = 0.2, \quad \gamma_2 = 0.01 \quad \text{and} \quad \theta = 0.0124$$

and we find

$$\mathfrak{R} = 1.6925. \tag{3.74}$$

and the corresponding figure is given in Figure (3.11).

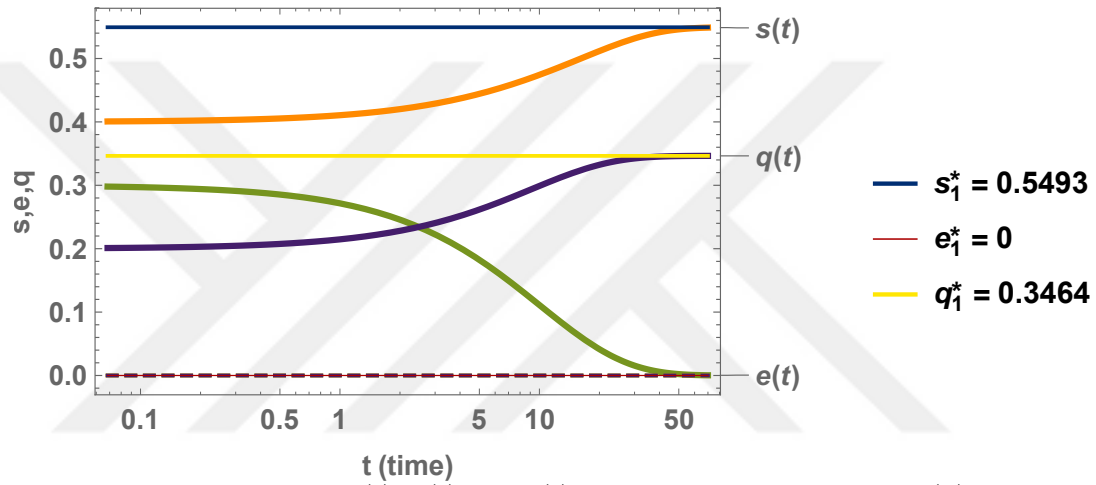


Figure 3.11 : Graph of $s(t)$, $e(t)$ and $q(t)$ with the initial conditions $s(0) = 0.4$, $e(0) = 0.3$ and $q(0) = 0.2$. Parameter values $\xi_2 = 0.2$, $\gamma_2 = 0.01$ and $\theta = 0.0124$. $R_0 = 2.1854$ and $\mathfrak{R} = 1.6925$.

We give the second figure by using all the parameters as given in Table (3.1). Then we find

$$\mathfrak{R} = 0.5601 \tag{3.75}$$

and we obtain the corresponding simulation as seen in Figure (3.12).

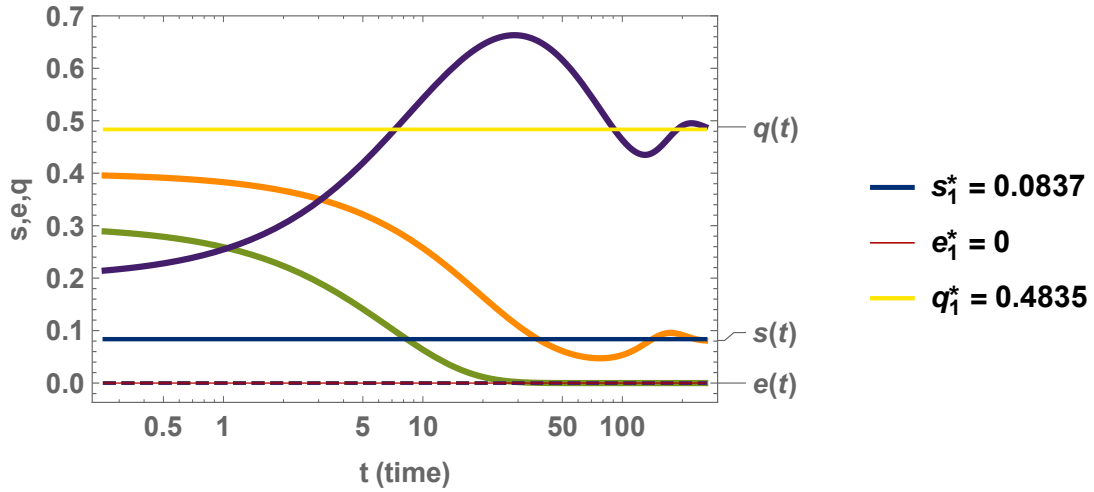


Figure 3.12 : Graph of $s(t)$, $e(t)$ and $q(t)$ with the initial conditions $s(0) = 0.4$, $e(0) = 0.3$ and $q(0) = 0.2$. $R_0 = 2.1854$ and $\mathfrak{R} = 0.5601$.

For the fifth and sixth figures in the second case we obtain

$$\mathfrak{R} = 0.0469 \tag{3.76}$$

and

$$\mathfrak{R} = 0.0433 \tag{3.77}$$

by choosing

$$\gamma_2 = 0.0108, \quad c = 0.008, \quad \theta = 0.7$$

and

$$\gamma_2 = 0.0108, \quad c = 0.0001, \quad \theta = 0.9$$

respectively.

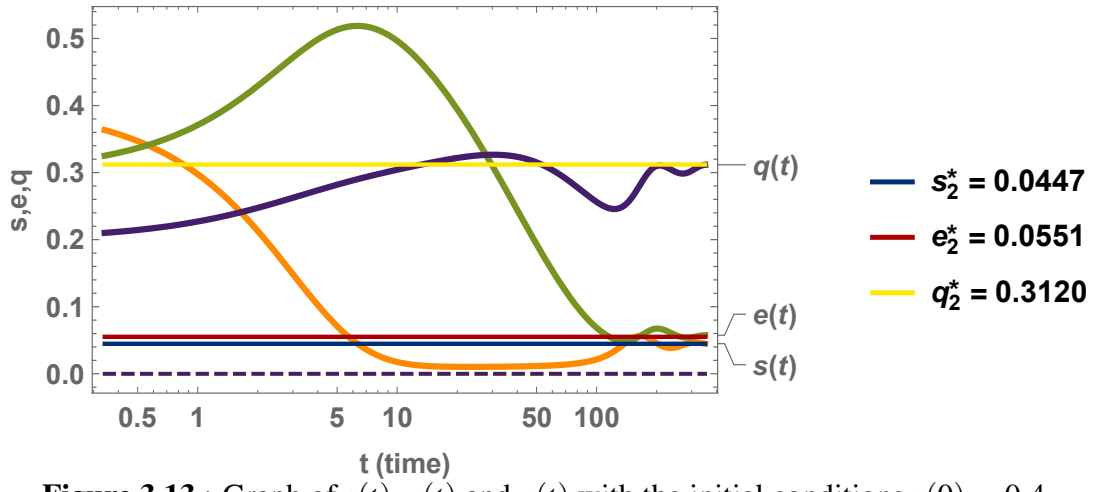


Figure 3.13 : Graph of $s(t)$, $e(t)$ and $q(t)$ with the initial conditions $s(0) = 0.4$, $e(0) = 0.3$ and $q(0) = 0.2$. Parameter values $\gamma_2 = 0.0108$, $c = 0.008$ and $\theta = 0.7$. $R_0 = 2.1854$ and $\mathfrak{R} = 0.0469$.

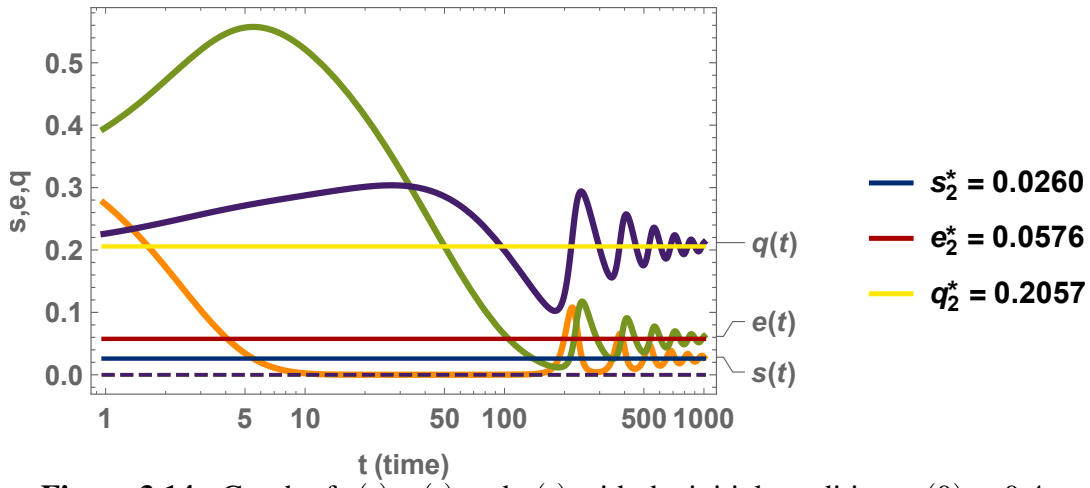


Figure 3.14 : Graph of $s(t)$, $e(t)$ and $q(t)$ with the initial conditions $s(0) = 0.4$, $e(0) = 0.3$ and $q(0) = 0.2$. Parameter values $\gamma_2 = 0.0108$, $c = 0.0001$ and $\theta = 0.9$. $R_0 = 2.1854$ and $\Re = 0.0433$.

As seen in Figures (3.11) and (3.12), the e-cigarette smoking-free equilibrium is stable while the smoking-free equilibrium is unstable. Observing Figures (3.13) and (3.14) shows that the smoking-free equilibrium is unstable while the endemic equilibrium is stable. In addition to above all, putting (3.74), (3.75), (3.76) together with (3.77) shows that if we decrease the value of $\Re$ the solutions switch from e-cigarette smoking-free equilibrium and then approach the endemic equilibrium when R_0 is sufficiently larger than unity.

3.6.2 Numerical verification of stability of the e-cigarette smoking-free equilibrium

To show the solution behaviour of the e-cigarette smoking-free E_1, we conduct some simulations.

In the **first case**, we choose all the parameters as given in Table (3.1). For these values of the parameters we obtain the following steady states:

$$s_1^* = 0.0837, \quad e_1^* = 0 \quad \text{and} \quad q_1^* = 0.4835.$$

Clearly, the e-cigarette smoking-free equilibrium is meaningful, biologically. We remark here that

$$R_0 = 2.1854.$$

We also get

$$\lambda_1 < 0, \quad \tilde{B} > 0 \quad \text{and} \quad \tilde{C} > 0$$

from subsection (3.4.2). We conduct two simulations for this case with different initial conditions:

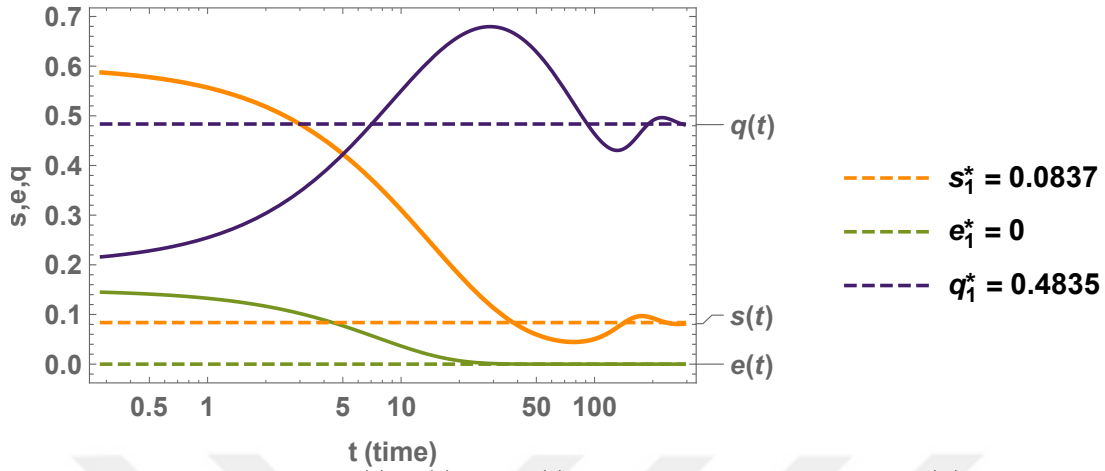


Figure 3.15 : Graph of $s(t)$, $e(t)$ and $q(t)$ with the initial conditions $s(0) = 0.6$, $e(0) = 0.15$ and $q(0) = 0.2$. $R_0 = 2.1854$.

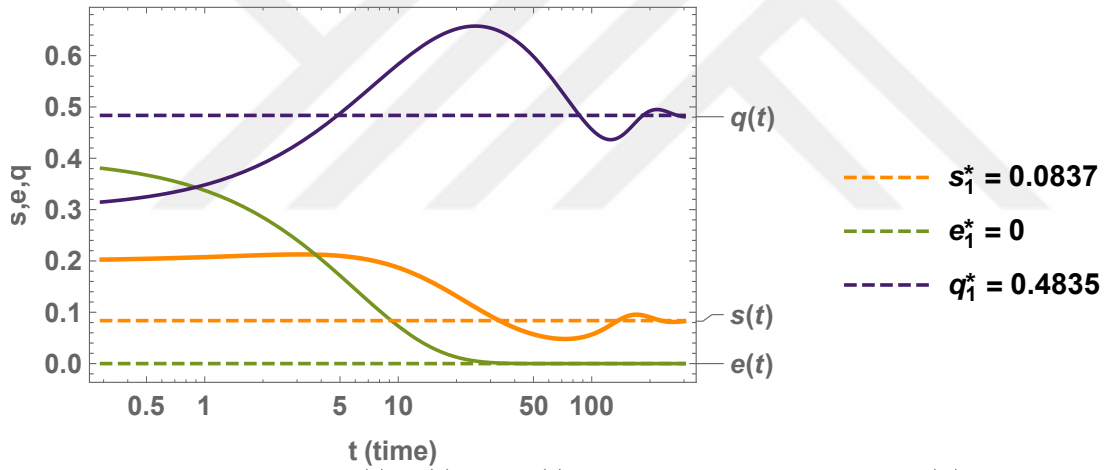


Figure 3.16 : Graph of $s(t)$, $e(t)$ and $q(t)$ with the initial conditions $s(0) = 0.2$, $e(0) = 0.4$ and $q(0) = 0.3$. $R_0 = 2.1854$.

As in figures (3.15) and (3.16), the e-cigarette smoking-free equilibrium is **stable**.

In the **second case**, we change some parameters in compliance with Table (3.1). Taking

$$\gamma_2 = 0.008, \quad c = 0.0002 \quad \text{and} \quad \theta = 0.5.$$

yields

$$R_0 = 2.1854, \quad \lambda_1 > 0, \quad \tilde{B} > 0 \quad \text{and} \quad \tilde{C} > 0.$$

We also have the steady states

$$s_1^* = 0.0837, \quad e_1^* = 0 \quad \text{and} \quad q_1^* = 0.4835.$$

Using the same initial conditions as in figures (3.15) and (3.16) respectively, we obtain the figures which are given below for the second case.

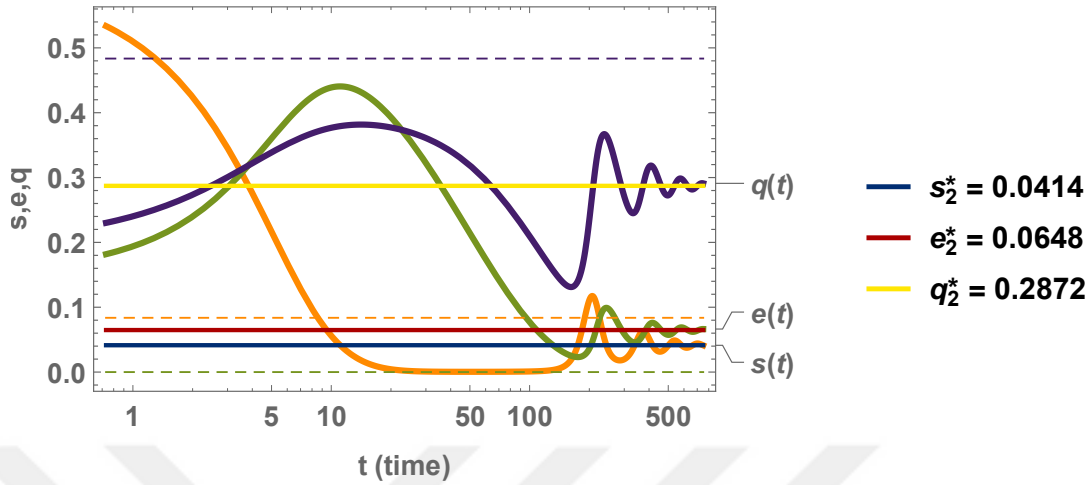


Figure 3.17 : Graph of $s(t)$, $e(t)$ and $q(t)$ with the initial conditions $s(0) = 0.6$, $e(0) = 0.15$ and $q(0) = 0.2$. Parameter values $\gamma_2 = 0.008$, $c = 0.0002$ and $\theta = 0.5$. $R_0 = 2.1854$.

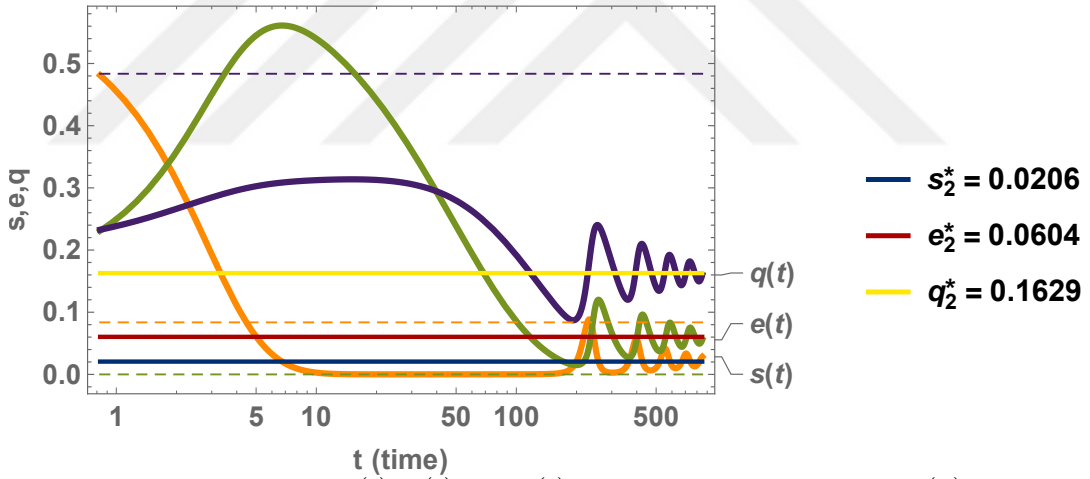


Figure 3.18 : Graph of $s(t)$, $e(t)$ and $q(t)$ with the initial conditions $s(0) = 0.2$, $e(0) = 0.4$ and $q(0) = 0.3$. Parameter values $\gamma_2 = 0.008$, $c = 0.0002$ and $\theta = 0.5$. $R_0 = 2.1854$.

We clearly see that the e-cigarette smoking-free equilibrium is unstable as the endemic equilibrium

$$E_2^* = (0.0414, 0.0648, 0.2872)$$

is stable in Figure (3.17). Additionally, if we observe Figure (3.18), we realize that the solution curves approach

$$E_2^* = (0.0206, 0.0604, 0.1629).$$

Consequently, the e-cigarette smoking-free equilibrium is unstable if $\lambda_1 > 0$ when $\hat{B}$ is positive. It is clear to see that the systems oscillates to the endemic equilibrium point. We note that the e-cigarette smoking-free equilibrium exists when $R_0 > 1$ as seen in all the figures which are given in e-cigarette smoking-free equilibrium simulations.

3.6.3 Numerical verification of stability of the endemic equilibrium

We use all the parameters given in Table (3.1) and the steady states as

$$s_2^* = 1.5705, \quad e_2^* = -1.0724 \quad \text{and} \quad q_2^* = 0.4628.$$

As we know, these steady states are not biologically meaningful. For this reason, we select some parameters as given below:

$$\gamma_1 = 0.05, \quad \gamma_2 = 0.02, \quad c = 0.01 \quad \text{and} \quad \theta = 0.39. \tag{3.78}$$

In the **first case**, we have two figures for different initial values:

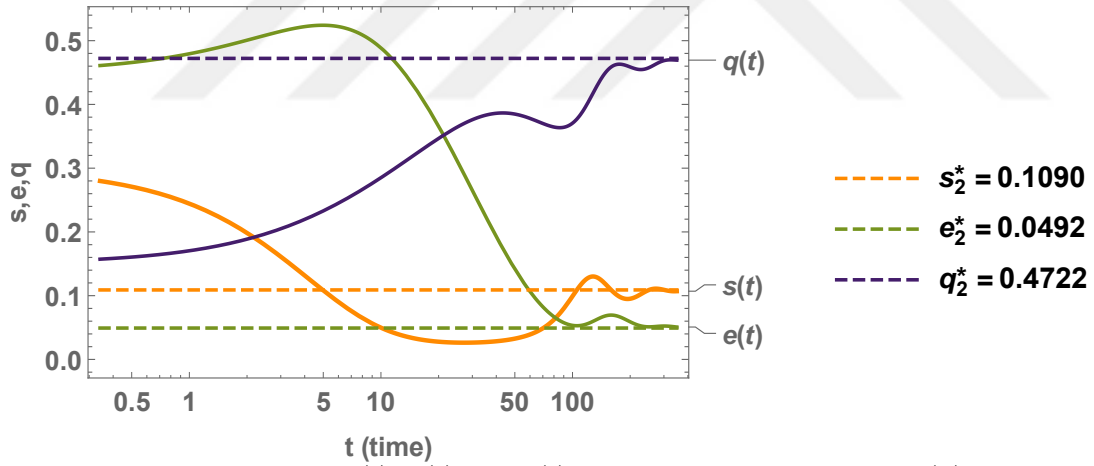


Figure 3.19 : Graph of $s(t)$, $e(t)$ and $q(t)$ with the initial conditions $s(0) = 0.3$, $e(0) = 0.45$ and $q(0) = 0.15$. Parameter values $\gamma_1 = 0.05$, $\gamma_2 = 0.02$, $c = 0.01$ and $\theta = 0.39$.

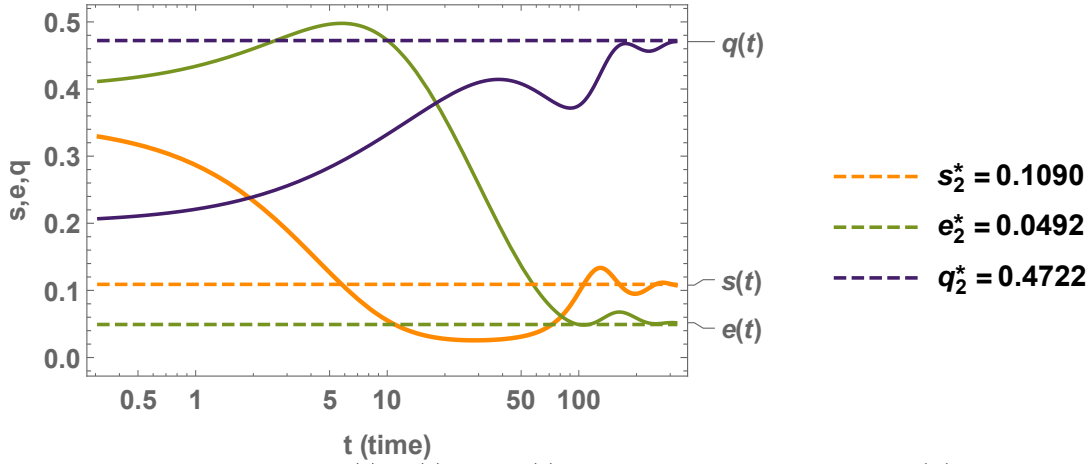


Figure 3.20 : Graph of $s(t)$, $e(t)$ and $q(t)$ with the initial conditions $s(0) = 0.35$, $e(0) = 0.4$ and $q(0) = 0.2$. Parameter values $\gamma_1 = 0.05$, $\gamma_2 = 0.02$, $c = 0.01$ and $\theta = 0.39$.

As seen in Figures (3.19) and (3.20), the endemic equilibrium is **stable**.

In the **second case**, to see the stability of the endemic equilibrium point, we have $\xi_1 = \xi_2$ as chosen in (3.4.3).

We use the parameters as mentioned in (3.78) together with

$$\xi_1 = \xi_2 = 0.1961.$$

Then we have the steady states

$$s_2^* = 0.1090, \quad e_2^* = 0.2269 \quad \text{and} \quad q_2^* = 0.2945.$$

We observe

$$\hat{B} > 0$$

from subsection (3.4.3).

If we use the same initial conditions for Figures (3.19) and (3.20) respectively, then we get the Figures (3.21) and (3.22).

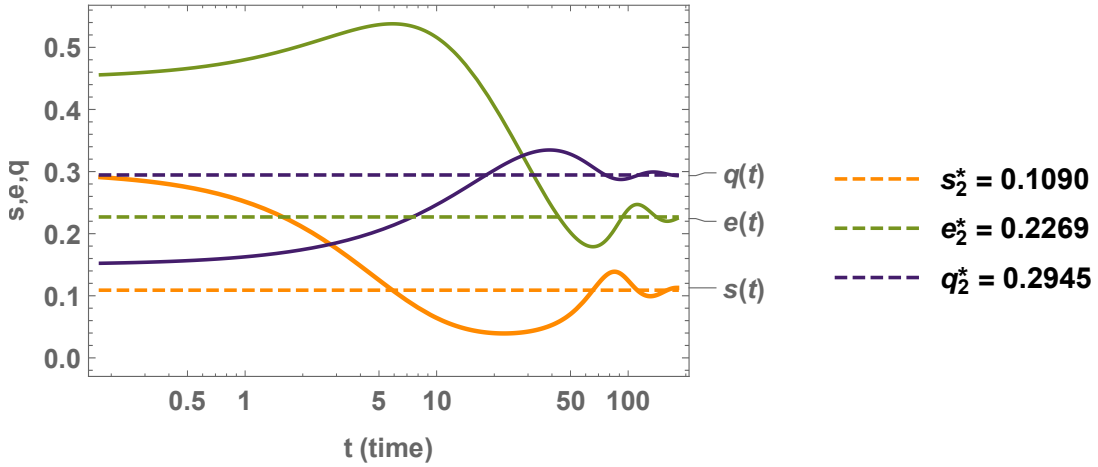


Figure 3.21 : Graph of $s(t)$, $e(t)$ and $q(t)$ with the initial conditions $s(0) = 0.3$, $e(0) = 0.45$ and $q(0) = 0.15$. Parameter values $\xi_1 = \xi_2 = 0.1961$.

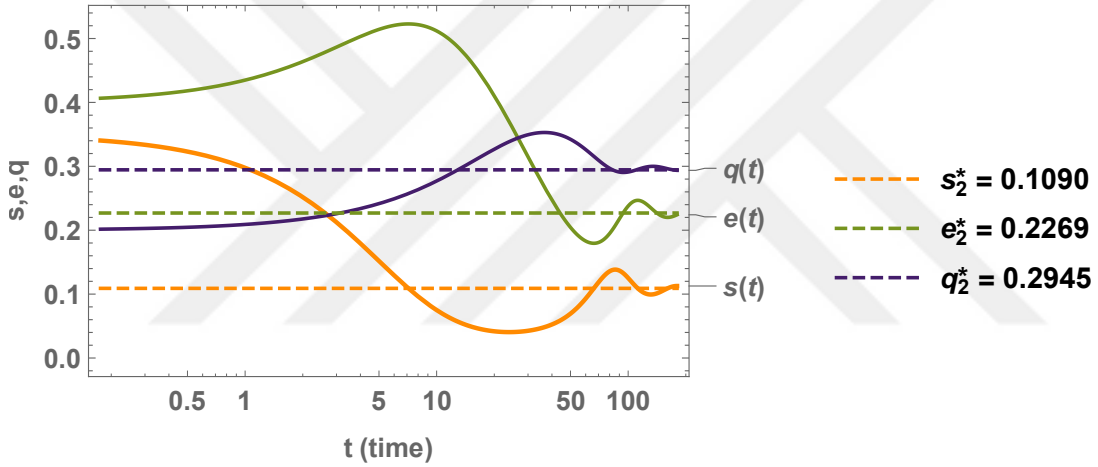


Figure 3.22 : Graph of $s(t)$, $e(t)$ and $q(t)$ with the initial conditions $s(0) = 0.35$, $e(0) = 0.4$ and $q(0) = 0.2$. Parameter values $\xi_1 = \xi_2 = 0.1961$.

We clearly see that the endemic equilibrium is **stable** for $\xi_1 = \xi_2$.

3.7 Conclusions

In this study, our main objective was to investigate the effect of e-cigarettes on smoking cessation. We have constructed a differential equation model to examine the impact of using e-cigarettes on smoking cessation by peer pressure and analyzed their dynamical behaviours.

The model exhibits three equilibrium solutions which are the smoking-free equilibrium, e-cigarette smoking-free equilibrium and endemic equilibrium. We established necessary conditions for the existence of these equilibrium solutions.

We also obtained necessary and sufficient conditions for the local stability of these equilibria.

By the next generation matrix method, we defined the basic reproduction number for the local stability of the smoking-free equilibrium and by constructing a Lyapunov function we assigned the condition for the global stability of the smoking-free equilibrium. To complete the analysis and illustrate the theoretical results achieved in the sections before, we performed numerical simulations using Mathematica. The numerical simulations were performed using the data obtained from people who smoked for 50 years according to National Health Interview Survey(NHIS) [53]. Taking this data into consideration, we changed our parameters to some extent in order to get more meaningful results for our model.

As we already know, the basic reproduction number has great importance in epidemiological models. In this model, we have obtained the basic reproduction number for the smoking-free equilibrium and e-cigarette smoking-free equilibrium. In the numerical figures plotted for smoking-free equilibrium, we observed that when $R_0 < 1$, the smoking-free equilibrium E_0 is globally asymptotically stable. We noticed that the case where $R_0 < 1$ is a sufficient condition to achieve the global asymptotically stable results for the smoking-free equilibrium. In other words, the value of $\mathfrak{R}$ is not necessary to interpret the stability of the system when $R_0 < 1$.

We noted that when $R_0 > 1$, the smoking-free equilibrium E_0 is unstable while the e-cigarette equilibrium or the endemic equilibrium is stable. We have questioned the importance of $\mathfrak{R}$ in this situation. Our results indicate that another non-dimensional parameter $\mathfrak{R}$ controls whether the endemic or the e-cigarette smoking-free equilibrium becomes stable for $R_0 > 1$.

When $R_0 = 1$, the smoking-free equilibrium is globally asymptotically stable. Yet, this convergency is effective in too long time. We also considered the situation which provides a transition when we increased the value of R_0 with very small perturbation from unity. This transition is called a bifurcation. In other words, the equilibrium point changes its behaviour from stable to unstable with very small perturbation of R_0.

The observation of the numerical simulations of the e-cigarette smoking-free equilibrium verifies our theoretical results. We observed that the e-cigarette

smoking-free equilibrium exists when $R_0 > 1$. Additionally, the figures show that if λ_1 becomes positive then the system oscillates to the endemic equilibrium point as the other stability conditions are satisfied.

Moreover, we examined the illustrations related with the endemic equilibrium and we found that the endemic equilibrium does not exist with the data given in the table in [7]. We changed these parameters in a nominal way to get a result where the endemic equilibrium is stable. However, we could not find any condition for the endemic equilibrium is stable when $\xi_1 \neq \xi_2$. But on the other hand, we confirmed the stability conditions when $\xi_1 = \xi_2$. This means that, the endemic equilibrium is stable. We obviously see the importance of e-cigarettes since the number of smokers decrease and the number of quitters increase as the number of e-cigarette smokers increase.

In conclusion, by the established dynamical model we verified the efficacy of e-cigarettes for different situations. In other words using e-cigarettes can properly suppress the desire for tobacco cigarettes and may be a successful way in preventing tobacco smoking. Yet, we should always consider the possibility of relapse for smokers and e-cigarette smokers. Using e-cigarettes is quite effective to decrease the number of tobacco smokers, but our analysis indicates that it does not have a remarkable effect on the number of quitters. We conclude that e-cigarette is not a tool to quit tobacco smoking. We therefore recommend that, the society should be made conscious about the correct methods of quitting smoking and create awareness about the right methods of smoking cessation.

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