ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE

EXACT SOLITON TYPE SOLUTIONS OF HIGHER ORDER DISPERSIVE-CUBIC-QUINTIC NONLINEAR SCHRÖDINGER EQUATION WITH A $\mathscr{PT}$-SYMMETRIC POTENTIAL

M.Sc. THESIS

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Mathematical Engineering Department

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Thesis Advisor: Assoc. Prof. Dr. İlkay BAKIRTAŞ AKAR

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$\mathscr{PT}$ -SİMETRİK BİR POTANSİYEL İÇEREN YÜKSEK MERTEBEDEN DİSPERSİF KÜBİK-KUİNTİK NONLİNEER SCHRÖDINGER DENKLEMİNİN SOLİTON TİPİ KESİN ÇÖZÜMLERİ

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FOREWORD

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TABLE OF CONTENTS

Page

FOREWORD .... **vii**
TABLE OF CONTENTS .... **ix**
ABBREVIATIONS .... **xi**
LIST OF FIGURES .... **xiii**
SUMMARY .... **xv**
ÖZET .... **xvii**
1. INTRODUCTION **1**
1.1 Purpose of Thesis 2
1.2 Literature Review 2
2. EXACT SOLUTION OF CQNLS EQUATION WITH THIRD ORDER DISPERSION AND A $\mathscr{PT}$-SYMMETRIC POTENTIAL **7**
3. NUMERICAL METHODS **13**
3.1 Spectral Renormalization Method 13
4. STABILITY ANALYSIS OF CQNLS EQUATION WITH THIRD ORDER DISPERSION AND A $\mathscr{PT}$-SYMMETRIC POTENTIAL **19**
4.1 Split-Step Fourier Method 19
4.2 Nonlinear Stability Analysis 19
4.3 Linear Stability 22
4.3.1 Linear spectrum 23
5. CONCLUSION **29**
REFERENCES **31**
APPENDICES **35**
APPENDIX A.1 37
CURRICULUM VITAE **39**

ABBREVIATIONS

NLS	:	Nonlinear Schrödinger
CNLS	:	Cubic Nonlinear Schrödinger
CQNLS	:	Cubic Quintic Nonlinear Schrödinger
3OD	:	Third Order Dispersion
4OD	:	Fourth Order Dispersion
PDE	:	Partial Differantial Equation
ODE	:	Ordinary Differantial Equation
$\mathscr{PT}$	:	Parity - Time
SR	:	Spectral Renormalization
KdV	:	Korteweg-de Vries
Eq.	:	Equation

LIST OF FIGURES

Page

Figure 3.1 : Analytically and numerically obtained soliton for $W_1 = 1.8$, $V_2 = 1.2$ and $\beta = 0.1$ 16
Figure 3.2 : Numerically obtained soliton, real and imaginary parts of the $\mathscr{PT}$-symmetric potential........ 16
Figure 3.3 : Real and imaginary parts of the potential for $\mu = 1$, $\alpha = 1$, $W_1 = 1.8$, $V_2 = 1.2$ and various values of β: (a) $\beta = -0.1$, (b) $\beta = 0$, (c) $\beta = 0.1$ 17
Figure 3.4 : Numerically obtained solitons of CQNLS equation with 3OD for various potential depths of $\mathscr{PT}$-symmetric potential for (a) $\beta = -0.1$, (b) $\beta = 0$, (c) $\beta = 0.1$........ 17
Figure 4.1 : Nonlinearly stable (marked as green) and nonlinearly unstable (marked as red) solitons of (a) CQNLS equation with 3OD for $\beta = -0.1$; (b) CQNLS equation without 3OD; (c) CQNLS equation with 3OD for $\beta = 0.1$ for varying potential depths of the $\mathscr{PT}$-symmetric potential........ 20
Figure 4.2 : Nonlinear stability of soliton for $\alpha = 1, \mu = 1$, $\beta = 0.1$, $W_1 = 0.6$ and $V_2 = 2$ with a $\mathscr{PT}$-symmetric potential; (a) Numerically produced soliton (blue dashes) on top of the solution after the evolution (red solid), (b) Nonlinear evolution of the soliton, (c)The view from top and (d) Maximum amplitude as a function of the propagation distance z........ 21
Figure 4.3 : Nonlinear instability of soliton for $\alpha = 1, \mu = 1$, $\beta = 0.1$, $W_1 = 1.8$ and $V_2 = 2.3$ with a $\mathscr{PT}$-symmetric potential; (a) Numerically produced soliton (blue dashes) on top of the solution after the evolution (red solid), (b) Nonlinear evolution of the soliton, (c)The view from top and (d) Maximum amplitude as a function of the propagation distance z. 21
Figure 4.4 : Nonlinear stability of soliton for $\alpha = 1, \mu = 1$, $\beta = -0.1$, $W_1 = 0.2$ and $V_2 = 3.3$ with a $\mathscr{PT}$-symmetric potential; (a) Numerically produced soliton (blue dashes) on top of the solution after the evolution (red solid), (b) Nonlinear evolution of the soliton, (c)The view from top and (d) Maximum amplitude as a function of the propagation distance z. 22
Figure 4.5 : Nonlinear instability of soliton for $\alpha = 1, \mu = 1$, $\beta = -0.1$, $W_1 = 1.7$ and $V_2 = 2$ with a $\mathscr{PT}$-symmetric potential; (a) Numerically produced soliton (blue dashes) on top of the solution after the evolution (red solid), (b) Nonlinear evolution of the soliton, (c)The view from top and (d) Maximum amplitude as a function of the propagation distance z........ 22

Figure 4.6 : Linear spectrum of CQNLS equation (a) with 3OD for $\beta = -0.1$; (b) without 3OD; (c) with 3OD for $\beta = 0.1$ for $W_1 = 4$ and $V_2 = 3.7$ 26
Figure 4.7 : Linear spectrum of numerically obtained solitons of the CQNLS equation with $\mathscr{P}\mathscr{T}$-symmetric potential and 3OD for $\beta = 0.1$, $W_1 = 4$ and $V_2 = 3.7$ in the (a) cubic case (a=1, b=0) and cubic-quintic cases: (b) a=1, b=0.2, (c) a=1, b=0.4, (d) a=1, b=0.6, (e) a=1, b=0.8, (f) a=1, b=1. 27

EXACT SOLITON TYPE SOLUTIONS OF HIGHER ORDER DISPERSIVE-CUBIC-QUINTIC NONLINEAR SCHRÖDINGER EQUATION WITH A $\mathscr{PT}$-SYMMETRIC POTENTIAL

SUMMARY

Nowadays, nonlinear equations are used to model many different problems. Nonlinear wave propagations shine as the most important developments in nonlinear scientific researches. Generally, starting with the solution of the partial differential equation (PDE), a nonlinear system should be investigated as an approach to the related experimental system in fluid mechanics, optics, plasma physics and biology. Types of the solutions of these kinds of equations are nonlinear wave types, and some of these wave types called soliton. Solitons are the propagating localized waves without changing their speeds and shapes. They are resistant to collision and can preserve their properties after collision. For many years, PDEs such as Korteweg-de Vries (KdV), sine-Gordon and nonlinear Schrödinger (NLS) have been used in the modeling of nonlinear waves and soliton type solutions of these equations are widely investigated in the literature.

In literature, the analytical and numerical solutions of the NLS equation with cubic and / or cubic-quintic nonlinearity and second order dispersion are investigated in detail. In inter-continent data transmission problems, it is important to investigate the contribution of third-order dispersion terms to the problem as well as the second order dispersion. Another factor which affects the shape and stability of optical pulses is the external optical potentials.

In this thesis, the numerical existence and stability analysis of the soliton solutions of the cubic-quintic nonlinear Schrödinger (CQNLS) equation with the third-order dispersion, an external potential with parity time ($\mathscr{PT}$)-symmetry properties are analyzed and the results are shown with various graphics.

In chapter 1, the historical development of the solitons has been shortly discussed and then the structure and the application areas of the CQNLS equation which includes a $\mathscr{PT}$-symmetric potential have been explained. Afterwards the aim of the thesis and the necessary literature review have been given.

In chapter 2, an ansatz solution has been proposed in order to produce the analytical solution under a $\mathscr{PT}$-symmetric potential and then the exact solution has been obtained via necessary substitutions. In addition, the structure of the $\mathscr{PT}$-symmetric potential was determined.

In chapter 3, a numerical method which is called spectral renormalization (SR) that has been previously used in solutions of numerous PDEs is explained. This method is originally developed by Ablowitz and Musslimani and in this chapter, we have modified the method for our problem. Using this method, numerical solutions of the CQNLS equation with the third-order dispersion containing an external potential were obtained. Then, the numerical solutions obtained by SR algorithm and the exact

solutions obtained in chapter 2 have been observed to be consistent; thus, the suitability of the numerical method and the accuracy of applied algorithm have been tested and found to be in good agreement. The effects of obtained external potential coefficients on the soliton solution have been analyzed and the results have been shown with adequate number of graphics. Furthermore, the effect of third order dispersion term on the existence of soliton solution is analyzed and the results have been graphically expressed.

In chapter 4, the split-step Fourier method is used to analyze the nonlinear stability of previously obtained solitons. The split-step Fourier method has been modified to apply the CQNLS equation with an external potential and third order dispersion term. Then the effect of different potential depths on stability properties has been depicted in graphics. Additionally, the effect of third order dispersion on the stability of soliton was investigated.

Not only nonlinear stability but also linear stability of the obtained solitons has been investigated. A brief information of the linear spectrum has been given and its application to the CQNLS equation has been explained, and finally, the effect of third order dispersion on linear stability has been shown by graphics.

$\mathscr{PT}$ -SİMETRİK BİR POTANSİYEL İÇEREN YÜKSEK MERTEBEDEN DİSPERSİF KÜBİK-KUİNTİK NONLİNEER SCHRÖDINGER DENKLEMİNİN SOLİTON TİPİ KESİN ÇÖZÜMLERİ

ÖZET

Günümüzde, birçok farklı problemin modellenmesinde doğrusal olmayan (nonlineer) denklem sistemlerinden yararlanılmaktadır. Nonlineer bilimsel araştırma alanlarında en büyük gelişmeler nonlineer dalga yayılımı problemleri konusunda öne çıkmaktadır. Genellikle, nonlineer bir sistemin araştırılmasına; akışkanlar mekaniği, optik, plazma fiziği ve biyolojideki ilişkili deneysel sistemin bir yaklaşımı olarak, kısmi türevli diferansiyel denklemin çözümünün elde edilmesiyle başlanmaktadır. Bu tipteki denklemlerin çözümleri nonlineer dalga tipindedir ve bunların bir kısmı soliton olarak adlandırılır. Solitonlar; yayılırken yayılma hızı ve formunu koruyarak ilerleyen lokalize dalgalardır. Karşılıklı çarpışmaya dayanıklıdırlar (elastik çarpışma) ve sahip oldukları özellikleri çarpışma sonrasında koruyabilirler. KdV sine-Gordon ve NLS gibi kısmi türevli diferansiyel denklemler, nonlineer dalga yayılımının modellenmesinde kullanılmakta olup bu denklemlerin soliton tipi çözümleri literatürde geniş yer almaktadır.

Bir optik dalganın (atım) çeşitli optik malzemeler içinde yayılımı NLS denklemi ile temsil edilir. NLS denklemi, genellikle bir piko saniyelik zaman ölçeğinde optik atımların, doğrusal olmayan yayılımlarını modellemekte yaygın olarak kullanılmaktadır. Bu denklem, Erwin Schrödinger tarafından 1927'de geliştirilmiş ve yaygın olarak kullanılmıştır. Literatürde kübik ve/veya kübik-kuintik nonlineerite ve yalnızca ikinci mertebe dispersiyon içeren NLS denkleminin analitik ve sayısal çözümleri etraflıca incelenmiştir. Kıtalar arası veri iletimi problemlerinde, ikinci mertebe dispersiyonun yanı sıra üçüncü dereceden dispersiyon terimlerinin probleme katkılarının araştırılması önem kazanmaktadır. Optik atımların biçimlerine ve kararlılıklarına etki eden bir başka faktör de dış optik potansiyellerdir. Bir dış potansiyelin eklendiği NLS denklemi, literatürde Gross-Pitaevskii denklemi olarak adlandırılır. Son yıllarda, kuantum mekaniği problemlerinde, parite-zaman simetrisine sahip potansiyeller ilgi çekmektedir.

Bu tezde,

$$V_{\mathscr{PT}} = V(x) + iW(x). \quad (1)$$

şeklinde kompleks yapıda bir dış potansiyel kullanılmıştır ve bu potansiyel $\mathscr{PT}$-simetrik özelliğindedir; yani potansiyelin reel kısmı olan V(x), çift fonksiyon olma özelliğine sahipken; imaginer kısmı olan W(x), tek fonksiyon olma özelliğine sahiptir. Böylece, $V(-x) = V(x)$ ve $W(-x) = -W(x)$ ilişkisi sağlanır.

Bu çalışmada,

$$iu_z + \alpha u_{xx} + i\beta u_{xxx} + |u|^2 u + |u|^4 u + V_{PT} u = 0. \quad (2)$$

olarak verilen, üçüncü mertebe dispersiyon terimi ve $\mathscr{PT}$-simetri özelliğine sahip bir dış potansiyel içeren, CQNLS denkleminin soliton çözümlerinin sayısal varlığı ve kararlılık analizleri incelenmiş; sonuçlar çeşitli grafikler ile gösterilmiştir.

Yukarıda verilen denklemde, u kompleks değerli türevlenebilir fonksiyonu, u_{xx} kırılımı (dispersiyon) modelleyen terimi, α ikinci dereceden dispersiyon teriminin katsayısını, β üçüncü mertebe dispersiyon teriminin katsayısını ve $V_{\mathscr{PT}}$ $\mathscr{PT}$-simetri özelliği olan potansiyeli temsil eder. Bu tezin amacı, $\mathscr{PT}$-simetri özelliği olan bir dış potansiyelin ve üçüncü mertebe dispersiyon teriminin soliton çözümünde ve çözümlerin kararlılığında yaptığı etkiyi incelemektir.

Bölüm 1'de, solitonların tarihsel gelişiminden kısaca söz edilmiş, $\mathscr{PT}$ simetrik potansiyel içeren, doğrusal olmayan CQNLS denkleminin yapısı ve uygulama alanları anlatılmıştır. Sonrasında tezin amacı ve gerekli literatür taraması verilmiştir.

Bölüm 2'de, aşağıda verilen, $V+iW$ kompleks yapılı potansiyel içeren CQNLS denkleminin analitik çözümlerini üretebilmek için $u(x,z)=f(x)e^{i(\mu z+g(x))}$ çözüm önerisi yapılmıştır.

$$iu_z+\alpha u_{xx}+i\beta u_{xxx}+|u|^2u+|u|^4u+V_{\mathscr{PT}}u=0.$$

Burada $f(x)$ ve $g(x)$, yapısı henüz belli olmayan reel değerli fonksiyonlardır. Önerilen bu çözüm, denklemde yerine konularak kesin çözümler elde edilmiştir. Ayrıca, kompleks potansiyelin reel kısmı

$$V(x)=V_0+V_1\sec h(x)+V_2\text{sech}^2(x)+V_3\text{sech}^3(x)+V_4\text{sech}^4(x). \tag{3}$$

şeklinde bir çift fonksiyon, imaginer kısmı ise,

$$W(x)=W_0\text{sech}^2(x)\tanh(x)+W_1\text{sech}(x)\tanh(x)+W_2\tanh(x). \tag{4}$$

şeklinde bir tek fonksiyon olarak elde edilmiştir. Böylece, bulunan potansiyel $\mathscr{PT}$-simetrik yapıda olup aşağıdaki gibi belirlenmiştir:

$$\begin{aligned} V_{\mathscr{PT}} &= [V_0+V_1\text{sech}(x)+V_2\text{sech}^2(x)+V_3\text{sech}^3(x)+V_4\text{sech}^4(x)] \\ &\quad +i[W_0\text{sech}^2(x)\tanh(x)+W_1\text{sech}(x)\tanh(x)+W_2\tanh(x)]. \end{aligned} \tag{5}$$

Bölüm 3'te, Ablowitz ve Musslimani'nin geliştirdiği, çeşitli alanlarda kullanılan bir sayısal yöntem olan SR yönteminden bahsedilmiştir. SR yöntemiyle soliton çözüm elde etmek için,

$$w_0=e^{-x^2}. \tag{6}$$

Gaussian başlangıç koşulu kullanılmış ve yakınsama koşulu 10^{-12} olarak alınmıştır. Bu sayısal yöntem kullanılarak, bir dış potansiyel içeren üçüncü dereceden dispersiyon terimi bulunan CQNLS denkleminin sayısal çözümleri elde edilmiştir. Daha sonra, SR algoritması MATLAB bilgisayar programına aktarılarak elde edilen sayısal çözümler ile Bölüm 2'de elde edilen kesin çözümlerin üst üste düştüğü gözlemlenmiş ve böylece kullanılan sayısal yöntemin uygunluğu ve uygulanan algoritmanın doğruluğu test edilmiştir.

MATLAB programı kullanılarak; belirli bir potansiyel derinliğinde, 3OD teriminin katsayısındaki değişimin potansiyelin yapısına etkisi grafiksel olarak incelenmiştir. Elde edilen dış potansiyelin katsayılarının soliton çözüme etkileri sayısal olarak incelenmiş ve sonuçlar grafik ile gösterilmiştir. Ayrıca, üçüncü mertebe dispersiyon teriminin, elde edilen potansiyel altında soliton çözüme etkisi grafiksel olarak ifade edilmiştir.

Bölüm 4'te, elde edilen solitonların kararlılık analizini yapmak için kullanılan ayrık adımlı Fourier metodundan bahsedilmiştir. Daha sonra nonlineer stabilite (kararlılık) analizi için ayrık adımlı Fourier metodu; $\mathscr{P}\mathscr{T}$ dış potansiyeli ve üçüncü mertebe dispersiyon içeren CQNLS denklemine uygulanmıştır. Üç farklı 3OD terimi katsayısı için 0 ile 4 arasında değişen potansiyel derinliklerinde nonlineer stabilite bölgeleri çizilmiş ve 3OD teriminin solitonun kararlılığı üzerindeki etkisi araştırılmıştır. Nonlineer olarak stabil ve stabil olmayan soliton örnekleri çeşitli grafiklerle gösterilmiştir.
Nonlineer stabilitenin yanı sıra, solitonların lineer stabilitesi de incelenmiştir. Lineer spektrum hakkında kısa bir bilgi verilerek, CQNLS denklemine uygulanışı anlatılmıştır. Belirli bir potansiyel derinliğinde, üç farklı β katsayısı için, elde edilen solitonların lineer spektrumları bulunarak 3OD teriminin lineer stabiliteye etkisi incelenmiştir. Ayrıca, a; kübik nonlineerite teriminin, b; kuintik nonlineerite teriminin katsayıları olmak üzere,

$$iu_z + \alpha u_{xx} + i\beta u_{xxx} + a|u|^2 u + b|u|^4 u + V_{\mathscr{P}\mathscr{T}} u = 0. \tag{7}$$

denklemi ele alınarak, 0'dan 1'e; 0.2 artımla değişen b katsayısı için solitonların lineer spektrumları çizilmiş ve kuintic nonlineeritenin, solitonun lineer stabilitesine etkisi üzerinde çalışılmıştır.

1. INTRODUCTION

Solitons arises in many fields of nonlinear science like nonlinear optics, Bose-Einstein condensates, plasma physics, biology, fluid mechanics [1], [2]. Thus, in recent years, the importance of optical solitons has increased rapidly. NLS equation which was discovered by Erwin Schrödinger in 1927 is used for modeling nonlinear propagation of optical pulses on a picosecond time scale. In order to obtain the soliton type solutions of NLS equation, diverse analytical and numerical methods are used. He et al. use the similarity transformation method under certain parametric conditions in order to investigate exact bright, dark and gray analytical nonautonomous soliton solutions of generalized CQNLS equation with spatially inhomogeneous group velocity dispersion and amplification or attention in [3]. Exact solutions of quintic NLS equations with time and space modulated nonlinearities and potentials are obtained through similarity transformations by Belmonte-Beitia et al [4]. They also develop nontrivial explicit solutions such as periodic, quasi periodic, bright and dark solitons. Novel bright solitons of the NLS equation with third order dispersion (3OD) in some complex $\mathscr{PT}$-symmetric potentials (e.g. physically relevant $\mathscr{PT}$-symmetric Scarf-II like and harmonic-Gaussian potentials) are demonstrated by Chen et al. [5]. Their conclusion concludes the dynamical phoneme of soliton equation in the presence of 3OD and $\mathscr{PT}$-symmetric potentials arising in nonlinear fiber optics and other physically relevant fields. Using the extended hyperbolic auxiliary equation method, Zhu derives new exact travelling solutions of the high-order NLS equation in [6]. In general, NLS equation is defined as

$$iu_z + u_{xx} + \alpha|u|^2 u = 0. \tag{1.1}$$

Here, u represents the differentiable complex valued; u_{xx} represents the diffraction; z is a scaled propagation distance; α is the coefficient of cubic nonlinearities.

Moreover, higher order dispersion terms are needed because the dynamics of pulses smaller than 1 picosecond cannot be managed with cubic nonlinear Schrödinger (CNLS) equation. The contribution of 3OD is important for performance enhancement in data transmission across continents. Furthermore, another structure that affects the shape and stability of optical pulses is the external optical potential added to the system.

In this thesis, CQNLS equation with 3OD term and $\mathscr{P}\mathscr{T}$-symmetric optical potential given as

$$iu_z + \alpha u_{xx} + i\beta u_{xxx} + |u|^2 u + |u|^4 u + V_{\mathscr{P}\mathscr{T}} u = 0. \tag{1.2}$$

is examined. Here, β is a coefficient of 3OD term and $\mathscr{P}\mathscr{T}$ is a symmetric external potential. Then, exact and numerical solutions of the Eq. (1.2) are obtained and stabilities of soliton solutions are examined.

A flexible novel numerical scheme with which to compute self-localized states of nonlinear wave guides and also be applied to many nonlinear systems is improved by Ablowitz and Musslimani. They also represent how to solve lattice solutions by means of SR method in [7]. The SR method presents numerically solutions of existence and stability properties of the (1+1)D CQNLS equation with a $\mathscr{P}\mathscr{T}$-symmetric potential for diverse potential depths and in different self-focusing/defocusing cubic-quintic media. In addition to numerically and analytically obtained solutions, their linear and nonlinear stability properties are examined through linear spectrum analysis and by direct simulations in [8].

1.1 Purpose of Thesis

In this thesis, examining of the existence of soliton solutions and their stability analysis of CQNLS equation with 3OD and $\mathscr{P}\mathscr{T}$-symmetric potential is aimed.

1.2 Literature Review

Exact solutions of one/two dimensional NLS equations (knowns as the Gross-Pitaevski equation in Bose-Einstein condensates) with several complex $\mathscr{P}\mathscr{T}$-symmetric nonlinear wave equation starting from both $\mathscr{P}\mathscr{T}$-symmetric (e.g. the Kdv equation) and non $\mathscr{P}\mathscr{T}$-symmetric (e.g. the Burgers equation) nonlinear wave equations are found by some complex $\mathscr{P}\mathscr{T}$-symmetric extension principle in [9]. Many novel solutions which fundamentally differentiate from the others of CQNLS equation are obtained by means of the complete discrimination system method in [10]. The existence and the stability of lattice solitons in $\mathscr{PT}$-symmetric mixed linear-nonlinear optical lattice in Kerr media are represented in [11]. They also find that the combination of $\mathscr{PT}$-symmetric linear and nonlinear lattices can stabilize lattice solitons and can improve unique soliton properties. Göksel et al.[10] study the numerical existence and nonlinear stability of fundamental solitons in saturable media with crystal and certain type of quasi crystal lattices by means of computational methods. They also examined the effect of lattice depth to the gap width and in a certain parameter regime of the lattice depth and the propagation constant, the first nonlinear band-gap structures are obtained in [12]. The effect of the competing nonlinearity and the gain-loss coefficient on the existence and stability of the both 2D fundamental solution and vortex solitons are investigated in [13]. The existence and stability regions are also obtained for fundamental and vortex solitons. In addition, the coefficients of nonlinear terms and the propagation constants of solitons determine the whole nonlinearity. The existence and stability of solitons forming in $\mathscr{PT}$-symmetric optical lattice with spatially periodic modulation of the local strength of nonlinear media are investigated in [14]. They also present that the spatial modulation of the nonlinearity significantly effects the stability of solitons in $\mathscr{PT}$–symmetric optical lattices.

Spatial localized mode solution of a (2+1) dimensional NLS equation with constant diffraction and cubic-quintic nonlinearity in $\mathscr{PT}$-symmetric potential is investigated in [15]. They also examined the linear stability of these solutions. Exact spatial localized mode solutions in a cubic-quintic medium with harmonic and $\mathscr{PT}$-symmetric potentials are obtained according to their results. The detailed analysis of conditions for the stable propagation of (1+1)D spatial solutions in media exhibiting nonlinearities up to the seventh order is studied in [16]. Yang et al. [17] numerically investigated stability of soliton families in 1D NLS equations with non-$\mathscr{PT}$-symmetric complex potentials. Eigenvalues of linear-stability operators of solitons appear in quartest $(\lambda, -\lambda, \lambda^*, -\lambda^*)$, such as conservative systems and $\mathscr{PT}$-symmetric systems are the most significant numerical findings which are obtained in [17]. Results of solitons in models of waveguides with focusing or defocusing saturable nonlinearity and $\mathscr{PT}$-symmetric complex valued external potential of the Scarf-II type are searched by Li et al. in [18]. They also indicate that the instability of the stationary solutions can be mitigated or completely suppressed according to their results. Khare et al. [19] study the consequences of competing nonlinearities on beam dynamics in $\mathscr{PT}$-symmetric potentials. The effect of nonlinearity on beam dynamics in $\mathscr{PT}$-symmetric potentials are studied by Musslimani in [20]. Musslimani particularly represents the main features of Floquet-Bloch (FB) mode in $\mathscr{PT}$-symmetric optical lattice in [21].

In [22], in order to obtain the solution of NLS type equation, fast numerical methods are developed in fiber optics. While for solving a stiff system of ordinary differential equations (ODEs) Split-Step method is utilized, implicit Runge Kutta formulas of Gauss type are handled for more complex nonlinearities. In [23], exploiting weak external localized potentials which can convey or reflect the soliton related to the initial speed, coactions of soliton are examined. Li J. et al. [24] reproduce two species of Gauss-type solitons in cubic-quintic-septimal nonlinear media and also, form properties of two species of Gauss-type solitons are compared. Then, analysis of linear stability of solutions is studied by means of the method of eigenvalue. Furthermore, in order to inspect Gauss-type solitons' stability in diverse nonlinear media, numerical simulation based on the split-step Fourier method is used in [24]. In defocusing $\mathscr{PT}$-symmetric nonlocal nonlinear media, the existence, stability and inner coactions of two dimensional multipole solitons are studied in [25]. In addition, easier stabilizing of dipole solitons with intermediate nonlocality is presented.

In [26], group of periodic solutions of NLS equation with periodically modulated 3OD and complex valued potential is examined not only analytically but also numerically. With the usage of plane wave expansion method, Liu B. et al. [26] have observed periodic solutions band structure of the stability problem together with periodic complex potential. Generic complex hyperbolic refractive index distribution and fourth order diffraction (FOD) featured $\mathscr{PT}$-symmetric optical media, at the same time existence and the stability of solitons are researched in [27]. In the linear situation, it is numerically presented that the $\mathscr{PT}$ breaking points can be changed by the FOD parameter. Xu B. et al. [28] have studied the (1+1)-dimensional higher-order NLS equation with $\mathscr{PT}$-symmetric potentials. According to [28], propagation constant of soliton is identified by the factors of the fourth-order and the second-order diffraction, while phase of soliton is detected by the factors of the gain or loss distribution. For the generalized nonlinear Schrödinger (DNLS) equation the system of a Lagrangian and Hamiltonian are acquired and using the method of amplitude ansatz, equations' bright, dark and bright-dark solitary wave solutions are reproduced in [29]. Besides, existence of solitons is provided by exploiting some conditions. Moreover, with the aid of method of the standard linear stability analysis, stability analysis is examined.

2. EXACT SOLUTION OF CQNLS EQUATION WITH THIRD ORDER DISPERSION AND A $\mathscr{PT}$-SYMMETRIC POTENTIAL

Exact solutions provide one to understand the structure of the complex nonlinear physical phenomena which is related to wave propagation in a higher-order CQNLS equation with $\mathscr{PT}$-symmetric potential.

General form of CQNLS equation with a complex potential in the form $V+iW$ is given below:

$$iu_z+\alpha u_{xx}+i\beta u_{xxx}+|u|^2u+|u|^4u+(V+iW)u=0. \tag{2.1}$$

Obviously $u=0$ is the trivial solution of Eq. (2.1). To obtain non-zero solutions, we assume that $u\neq 0$. When Eq. (2.1) is divided by u, then we have the following equation:

$$i\frac{u_z}{u}+\alpha\frac{u_{xx}}{u}+i\beta\frac{u_{xxx}}{u}+|u|^2+|u|^4+V+iW=0. \tag{2.2}$$

The following ansatz is used to get non-zero stationary solutions:

$$u(x,z)=f(x)e^{i(\mu z+g(x))}. \tag{2.3}$$

where $f(x)$ and $g(x)$ are real-valued functions different than zero, u is a function of x and z to be determined and μ is the propagation constant. Taking derivatives of Eq. (2.3) with respect to z and x, results in following equations

$$u_z=f(x)i\mu e^{i(\mu z+g(x))}=i\mu u. \tag{2.4}$$

$$u_{xx}=e^{i(\mu z+g(x))}[f''(x)+2if'(x)g'(x)+if(x)g''(x)-f(x)(g'(x))^2]. \tag{2.5}$$

$$\begin{aligned}u_{xxx}=e^{i(\mu z+g(x))}[&f'''(x)+3if''(x)g'(x)+3if'(x)g''(x)-3f'(x)(g'(x))^2\\&-3f(x)g'(x)g''(x)+if(x)g'''(x)-if(g'(x))^3].\end{aligned} \tag{2.6}$$

$$|u|^2=f(x)e^{i(\mu z+g(x))}f(x)e^{-i(\mu z+g(x))}=(f(x))^2. \tag{2.7}$$

$$|u|^4=(f(x))^4. \tag{2.8}$$

Substituting Eq. (2.4)-Eq. (2.8) into Eq. (2.2) yields

$$[-\mu+\alpha\frac{f''(x)}{f(x)}-\alpha(g'(x))^2-3\beta\frac{f''(x)g'(x)}{f(x)}+\beta(g'(x))^3-3\beta\frac{f'(x)g''(x)}{f(x)}$$
$$-\beta g'''(x)+(f(x))^2+(f(x))^4+V(x)]+i[2\alpha\frac{f'(x)g'(x)}{f(x)}+\alpha g''(x) \tag{2.9}$$
$$-3\beta\frac{f'(x)(g'(x))^2}{f(x)}-3\beta g'(x)g'''(x)+\beta\frac{f'''(x)}{f(x)}+W(x)]=0.$$

To obtain soliton solutions, we used the following ansatz

$$f(x)=f_0\sec h^p(x),\qquad g'(x)=g_0\sec h^q(x). \tag{2.10}$$

where f_0 and g_0 are non-zero real constants and $p,q\in N$. In order to simplify Eq. (2.9), we need to calculate the derivatives of the functions f and g. By using Eq. (2.10) we obtain

$$f'(x)=-f_0p\tanh(x)\mathrm{sech}^p(x). \tag{2.11}$$

$$f''(x)=-f_0p(1+p)\sec h^{p+2}(x)+f_0p^2\sec h^p(x). \tag{2.12}$$

$$f'''(x)=f_0[-p^3+(p^3+3p^2+2p)\sec h^2(x)]\sec h^p(x)\tanh(x). \tag{2.13}$$

$$g'(x)=g_0\mathrm{sech}^q(x). \tag{2.14}$$

$$g''(x)=g_0q\sec h^q(x)\tanh(x). \tag{2.15}$$

$$g'''(x)=g_0q^2\sec h^q(x)-g_0(q^2+q)\sec h^{q+2}(x). \tag{2.16}$$

Substituting Eq. (2.10)-Eq. (2.16) into Eq. (2.9) we obtain

$$-\mu+\alpha p^2+[-\alpha(p^2+p)]\sec h^2(x)+[-\beta g_0(3p^2+3pq+q^2)]\sec h^q(x)$$
$$+[-\alpha g_0{}^2]\sec h^{2q}(x)+[\beta g_0(3p^2+3p+3pq+q^2+q)]\sec h^{q+2}(x)$$
$$+[\beta g_0{}^3]\sec h^{3q}(x)+[f_0{}^2]\sec h^{2p}(x)+[f_0{}^4]\sec h^{4p}(x)+V(x) \tag{2.17}$$
$$+i[[3\beta g_0{}^2(p+q)]\sec h^{2q}(x)\tanh(x)+[-\alpha g_0(2p+q)]\sec h^q(x)\tanh(x)$$
$$+[\beta(p^3+3p^2+2p)]\sec h^2(x)\tanh(x)+[-\beta p^3]\tanh(x)+W(x)]=0.$$

When we split Eq. (2.17) into real and imaginary parts, we get the expressions for the real and imaginary parts of the complex potential as we can see below:

Real Part

The real part of the Eq. (2.17) can be written as,

$$[-\mu+\alpha p^2]+[-\alpha(p^2+p)]\sec h^2(x)+[-\beta g_0(3p^2+3pq+q^2)]\sec h^q(x)$$
$$+[-\alpha g_0{}^2]\sec h^{2q}(x)+[\beta g_0(3p^2+3p+3pq+q^2+q)]\sec h^{q+2}(x) \tag{2.18}$$
$$+[\beta g_0{}^3]\sec h^{3q}(x)+[f_0{}^2]\sec h^{2p}(x)+[f_0{}^4]\sec h^{4p}(x)+V(x)=0.$$

The real part of the complex potential is found as

$$\begin{aligned}V(x)=V_0+V_1\text{sech}^2(x)+V_2\text{sech}^q(x)+V_3\text{sech}^{2q}(x)+V_4\text{sech}^{q+2}(x)\\ +V_5\text{sech}^{3q}(x)+V_6\text{sech}^{2p}(x)+V_7\text{sech}^{4p}(x).\end{aligned} \tag{2.19}$$

where

$$V_0=\mu-\alpha p^2. \tag{2.20}$$

$$V_1=\alpha(p^2+p). \tag{2.21}$$

$$V_2=\beta g_0(3p^2+3pq+q^2). \tag{2.22}$$

$$V_3=\alpha g_0{}^2. \tag{2.23}$$

$$V_4=-\beta g_0(3p^2+3p+3pq+q^2+q). \tag{2.24}$$

$$V_5=-\beta g_0{}^3. \tag{2.25}$$

$$V_6=-f_0{}^2. \tag{2.26}$$

$$V_7=-f_0{}^4. \tag{2.27}$$

We can see in the following form that $V(x)$ is indeed an even function

$$\begin{aligned}V(-x)&=V_0+V_1\text{sech}^2(-x)+V_2\text{sech}^q(-x)+V_3\text{sech}^{2q}(-x)+V_4\text{sech}^{q+2}(-x)\\ &\quad+V_5\text{sech}^{3q}(-x)+V_6\text{sech}^{2p}(-x)+V_7\text{sech}^{4p}(-x)\\ &=V_0+V_1\text{sech}^2(x)+V_2\text{sech}^q(x)+V_3\text{sech}^{2q}(x)+V_4\text{sech}^{q+2}(x)\\ &\quad+V_5\text{sech}^{3q}(x)+V_6\text{sech}^{2p}(x)+V_7\text{sech}^{4p}(x)\\ &=V(x).\end{aligned} \tag{2.28}$$

Now, $V(x)$ can be simplified by equating the powers of $sech(x)$. Considering the case of $p=q=1$, then Eq. (2.19) can be rewritten as following form,

$$\begin{aligned}V(x)=[\mu-\alpha]+[7\beta g_0]\sec h(x)+[2\alpha+\alpha g_0{}^2-f_0{}^2]\text{sech}^2(x)\\ +[-\beta g_0(11+g_0{}^2)]\text{sech}^3(x)+[-f_0{}^4]\text{sech}^4(x).\end{aligned} \tag{2.29}$$

Then, the real part of the complex potential is found as

$$V(x) = V_0 + V_1 \sec h(x) + V_2 \text{sech}^2(x) + V_3 \text{sech}^3(x) + V_4 \text{sech}^4(x). \tag{2.30}$$

where

$$V_0 = \mu - \alpha. \tag{2.31}$$

$$V_1 = 7\beta g_0. \tag{2.32}$$

$$V_2 = 2\alpha + \alpha {g_0}^2 - {f_0}^2. \tag{2.33}$$

$$V_3 = -\beta g_0(11 + {g_0}^2). \tag{2.34}$$

$$V_4 = -{f_0}^4. \tag{2.35}$$

Imaginary Part

The complex part of the Eq. (2.17) can be written as

$$\begin{aligned} &[3\beta {g_0}^2(p+q)]\text{sech}^{2q}(x)\tanh(x) + [-\alpha g_0(2p+q)]\text{sech}^q(x)\tanh(x) \\ &+[\beta(p^3+3p^2+2p)]\text{sech}^2(x)\tanh(x) + [-\beta p^3]\tanh(x) + W(x) = 0. \end{aligned} \tag{2.36}$$

Then the imaginary part of the complex potential is obtained as

$$W(x) = W_0 \text{sech}^{2q}(x)\tanh(x) + W_1 \text{sech}^q(x)\tanh(x) + W_2 \sec h^2(x)\tanh(x) + W_3\tanh(x). \tag{2.37}$$

where

$$W_0 = -3\beta {g_0}^2(p+q). \tag{2.38}$$

$$W_1 = \alpha g_0(2p+q). \tag{2.39}$$

$$W_2 = -\beta(p^3+3p^2+2p). \tag{2.40}$$

$$W_3 = \beta p^3. \tag{2.41}$$

We can see in the following form that $W(x)$ is indeed an odd function.

$$\begin{aligned} W(-x) &= W_0\text{sech}^{2q}(-x)\tanh(-x) + W_1\text{sech}^q(-x)\tanh(-x) \\ &\quad +W_2\text{sech}^2(-x)\tanh(-x) + W_3\tanh(-x) \\ &= W_0\text{sech}^{2q}(x)(-\tanh(x)) + W_1\text{sech}^q(x)(-\tanh(x)) \\ &\quad +W_2\text{sech}^2(x)(-\tanh(x)) + W_3(-\tanh(x)) \\ &= -W(x). \end{aligned} \tag{2.42}$$

Considering the case of $p = q = 1$, then we can rewritten Eq. (2.37) as following form,

$$W(x) = -6\beta({g_0}^2+1)\sec h^2(x)\tanh(x) + 3\alpha g_0 \sec h(x)\tanh(x) + \beta\tanh(x). \tag{2.43}$$

Then the imaginary part of the complex potential is obtained as

$$W(x) = W_0 \text{sech}^2(x)\tanh(x) + W_1 \text{sech}(x)\tanh(x) + W_2 \tanh(x). \tag{2.44}$$

where

$$W_0 = -6\beta({g_0}^2 + 1). \tag{2.45}$$

$$W_1 = 3\alpha g_0. \tag{2.46}$$

$$W_2 = \beta. \tag{2.47}$$

Attention should be paid in case of $p = q = 1$, by considering Eq. (2.29) and Eq. (2.43) the analytical solution of the problem can begin with

$$u(x,z) = f_0 \sec h(x) e^{i[\mu z + g_0 \arctan h(x) \sinh(x)]}. \tag{2.48}$$

Consequently, Eq. (1) corresponding to $\mathscr{PT}$-symmetric potential with the real and imaginary parts in Eq. (2.30) and Eq. (2.44) can be given as

$$\begin{aligned} V_{PT} = {} & [V_0 + V_1 \text{sech}(x) + V_2 \text{sech}^2(x) + V_3 \text{sech}^3(x) + V_4 \text{sech}^4(x)] \\ & + i[W_0 \text{sech}^2(x)\tanh(x) + W_1 \text{sech}(x)\tanh(x) + W_2 \tanh(x)]. \end{aligned} \tag{2.49}$$

3. NUMERICAL METHODS

3.1 Spectral Renormalization Method

In nonlinear optics, localized wave solutions namely solitons establish a significant solution class. To compute localized solitons in nonlinear waveguides, a numerical scheme based on a Fourier iteration method was proposed by Petviashvili [30]. Then, this method was improved by Musslimani and Ablowitz [7]. Fourier iteration is the basis of this method. Like a NLS-type equation, fundamental equation leading the soliton is transformed into Fourier space through this method. The main purpose of this method is to specify a nonlinear nonlocal integral equation linked to an algebraic equation. Numerical scheme does not diverge through the coupling.

In nonlinear optics and related fields such as Bose–Einstein condensation and fluid mechanics, this method have comprehensive utilization. Subsequently, solitons in diverse self-focusing/self-defocusing cubic-quintic media are obtained using SR method by Göksel [8].

In this chapter, numerical solutions of the 3OD CQNLS equation with $\mathscr{P}\mathscr{T}$-symmetric potential in Eq. (2) will be achieved by using the SR method as follows:

$$iu_z + \alpha u_{xx} + i\beta u_{xxx} + |u|^2 u + |u|^4 u + V_{\mathscr{P}\mathscr{T}} u = 0. \tag{3.1}$$

The following equations are obtained by using $u(x,z) = f(x)e^{i\mu z}$ formula where μ is the propagation constant (or eigenvalue) and $f(x)$ is a complex-valued function:

$$\begin{aligned}
u_z &= i\mu f e^{i\mu z}. \\
u_{xx} &= f_{xx} e^{i\mu z}. \\
u_{xxx} &= f_{xxx} e^{i\mu z}. \\
u^* &= f e^{-i\mu z}. \\
|u|^2 &= |f|^2. \\
|u|^4 &= |f|^4.
\end{aligned} \tag{3.2}$$

The following nonlinear equation for f is acquired by putting Eq. (3.2) into Eq. (3.1)

$$-\mu f e^{i\mu z} + \alpha f_{xx} e^{i\mu z} + i\beta f_{xxx} e^{i\mu z} + |f|^2 f e^{i\mu z} + |f|^4 f e^{i\mu z} + V_{\mathscr{P}\mathscr{T}} f e^{i\mu z} = 0. \tag{3.3}$$

After simplifying these equations we get

$$-\mu f + \alpha f_{xx} + i\beta f_{xxx} + |f|^2 f + |f|^4 f + V_{\mathscr{P}\mathscr{T}} f = 0. \tag{3.4}$$

Taking the Fourier transformation of Eq. (3.4) gives us

$$\mathscr{F}\{-\mu f\} + \mathscr{F}\{\alpha f_{xx}\} + \mathscr{F}\{i\beta f_{xxx}\} + \mathscr{F}\{|f|^2 f\} + \mathscr{F}\{|f|^4 f\} + \mathscr{F}\{V_{\mathscr{P}\mathscr{T}} f\} = \mathscr{F}\{0\}. \tag{3.5}$$

Here, Fourier transformation is represented by $\mathscr{F}$ and the following equation is obtained with utilizing this transformation's qualities:

$$-\mu \hat{f} + \alpha (ik_x)^2 \hat{f} + i\beta (ik_x)^3 \hat{f} + \mathscr{F}\{|f|^2 f\} + \mathscr{F}\{|f|^4 f\} + \mathscr{F}\{(V+iW) f\} = 0. \tag{3.6}$$

where $\mathscr{F}(f) = \hat{f}$ and k_x are Fourier variables. When we solve the Eq. (3.6) for the $\hat{f}$, we see

$$\hat{f} = \frac{\mathscr{F}\{|f|^2 f\} + \mathscr{F}\{|f|^4 f\} + \mathscr{F}\{(V+iW) f\}}{[\mu + \alpha {k_x}^2 - \beta {k_x}^3]}. \tag{3.7}$$

In order to find $f(x)$, this equation could be indexed and utilized but the scheme does not converge. At this point, we should make acquainted with a new field variable $f(x) = \lambda w(x)$ with $\lambda \in R^+$ where λ is a parameter to be determined. So, putting $f(x) = \lambda w(x)$ into Eq. (3.7) yields

$$\lambda \hat{w} = \frac{\mathscr{F}\{|w|^2 |\lambda|^2 w\lambda\} + \mathscr{F}\{|w|^4 |\lambda|^4 w\lambda\} + \mathscr{F}\{(V+iW)\lambda w\}}{\mu + \alpha {k_x}^2 - \beta {k_x}^3}. \tag{3.8}$$

Therefore, $\hat{w}$ satisfies

$$\hat{w} = \frac{\mathscr{F}\{|w|^2 |\lambda|^2 w\} + \mathscr{F}\{|w|^4 |\lambda|^4 w\} + \mathscr{F}\{(V+iW) w\}}{\mu + \alpha {k_x}^2 - \beta {k_x}^3}. \tag{3.9}$$

For finding out w, Eq. (3.9) can be utilized in an iterative method. In order to succeed this, we can calculate $\hat{w}$ using the following iteration approach:

$$\hat{w}_{n+1} = \frac{|\lambda|^2 \mathscr{F}\{|w_n|^2 w_n\} + |\lambda|^4 \mathscr{F}\{|w_n|^4 w_n\} + \mathscr{F}\{(V+iW) w_n\}}{\mu + \alpha {k_x}^2 - \beta {k_x}^3}, \quad n \in N. \tag{3.10}$$

with the initial condition taken as a Gaussian type function

$$w_0 = e^{-x^2}. \tag{3.11}$$

where our convergence criterions are $|w_{n+1} - w_n| < 10^{-12}$. Multiplying both sides of Eq. (3.9) by $(\mu + \alpha {k_x}^2 - \beta {k_x}^3)$ and we obtain

$$(\mu + \alpha {k_x}^2 - \beta {k_x}^3)\hat{w} = |\lambda|^2 \mathscr{F}\{|w|^2 w\} + |\lambda|^4 \mathscr{F}\{|w|^4 w\} + \mathscr{F}\{(V + iW)w\}. \quad (3.12)$$

When we take all terms of Eq. (3.12) to the left side, we lead to following equation

$$(\mu + \alpha {k_x}^2 - \beta {k_x}^3)\hat{w} - |\lambda|^2 \mathscr{F}\{|w|^2 w\} - |\lambda|^4 \mathscr{F}\{|w|^4 w\} - \mathscr{F}\{(V + iW)w\} = 0. \quad (3.13)$$

After multiplying Eq. (3.13) by $\hat{w}^*$ which is the conjugate of $\hat{w}$, we get

$$(\mu + \alpha {k_x}^2 - \beta {k_x}^3)|w|^2 - |\lambda|^2 \mathscr{F}\{|w|^2 w\}\hat{w}^* - |\lambda|^4 \mathscr{F}\{|w|^4 w\}\hat{w}^* - \mathscr{F}\{(V + iW)w\}\hat{w}^* = 0. \quad (3.14)$$

Moreover, the following equation is obtained when we take integral of Eq. (3.14):

$$\begin{aligned} &\int_{-\infty}^{\infty} (\mu + \alpha {k_x}^2 - \beta {k_x}^3)|w|^2 dk - |\lambda|^2 \int_{-\infty}^{\infty} \mathscr{F}\{|w|^2 w\}\hat{w}^* dk \\ -|\lambda|^4 &\int_{-\infty}^{\infty} \mathscr{F}\{|w|^4 w\}\hat{w}^* dk - \int_{-\infty}^{\infty} \mathscr{F}\{(V + iW)w\}\hat{w}^* dk = 0. \end{aligned} \quad (3.15)$$

or in a more compact form

$$\begin{aligned} &- \int_{-\infty}^{\infty} \left[-\mathscr{F}\{(V + iW)w\}\hat{w}^* + (\mu + \alpha {k_x}^2 - \beta {k_x}^3)|w|^2 \right] dk \\ &+ |\lambda|^2 \int_{-\infty}^{\infty} \mathscr{F}\{|w|^2 w\}\hat{w}^* dk + |\lambda|^4 \int_{-\infty}^{\infty} \mathscr{F}\{|w|^4 w\}\hat{w}^* dk = 0. \end{aligned} \quad (3.16)$$

Actually, Eq. (3.16) is a fourth order polynomial of λ which has the form $P(\lambda) = a\lambda^4 + b\lambda^2 + c$; thus, roots of this polynomial i.e. λ is worked out exploiting the following formula:

$$\lambda_{1;2} = \pm \sqrt{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}. \quad (3.17)$$

where

$$a = \int_{-\infty}^{\infty} \mathscr{F}\{|w|^4 w\}\hat{w}^* dk. \quad (3.18)$$

$$b = \int_{-\infty}^{\infty} \mathscr{F}\{|w|^2 w\}\hat{w}^* dk. \quad (3.19)$$

$$c = - \int_{-\infty}^{\infty} [-\mathscr{F}\{(V + iW)w\}\hat{w}^* + (\mu + \alpha {k_x}^2 - \beta {k_x}^3)|w|^2] dk. \quad (3.20)$$

The required soliton will be $f(x) = \lambda(wx) = \lambda \mathscr{F}^{-1}(\hat{w})$ when the iteration converges. Thus, the soliton is obtained from a convergent iterative scheme.

In Fig. 3.1, the soliton numerically obtained by the SR method which is defined above is plotted with dashed green solid line while analytically obtained soliton which is explained in Chapter 1 is plotted with red solid line. It is seen from the figure that, two solitons overlap and obtained numerical solution satisfies Eq. (3.1) with absolute error is 10^{-9}. Therefore, it shows that SR method used in this chapter is suitable for getting solitons.

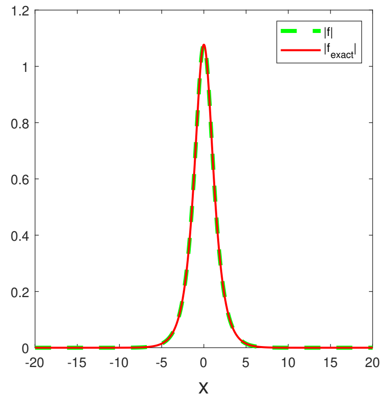

Figure 3.1 : Analytically and numerically obtained soliton for $W_1 = 1.8$, $V_2 = 1.2$ and $\beta = 0.1$

In Fig. 3.2, numerically obtained solution, its real and imaginary parts of the specific $\mathscr{PT}$-symmetric potential which is depicted in Chapter 1 are plotted. In Fig. 3.3, for given potential depths and a constant value of μ and α, the effect of 3OD term β to potential that is given in Chapter 2 is viewed.

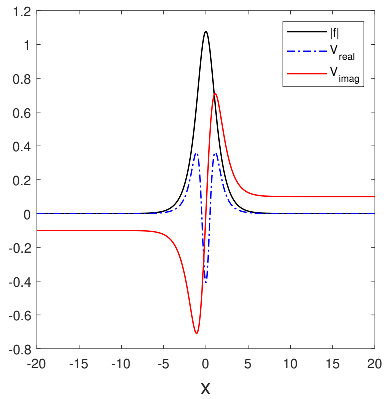

Figure 3.2 : Numerically obtained soliton, real and imaginary parts of the $\mathscr{PT}$-symmetric potential.

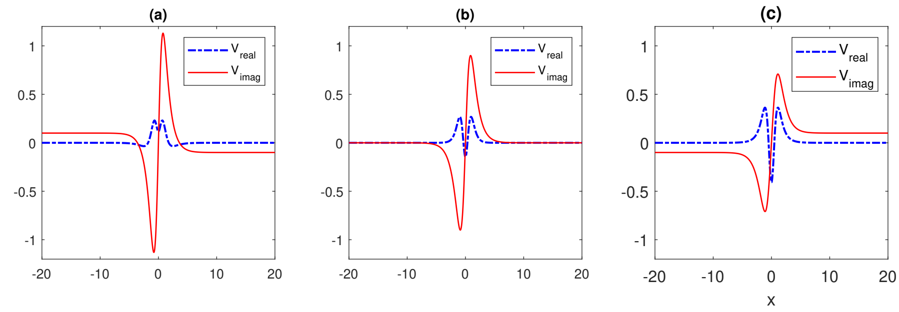

Figure 3.3 : Real and imaginary parts of the potential for $\mu = 1$, $\alpha = 1$, $W_1 = 1.8$, $V_2 = 1.2$ and various values of β: (a) $\beta = -0.1$, (b) $\beta = 0$, (c) $\beta = 0.1$.

In Fig. 3.4, existence regions of numerical solitons of CQNLS equation with 3OD for various values of β are plotted. Therefore, it can be seen from this figure, 3OD has positive effect for obtaining soliton solutions.

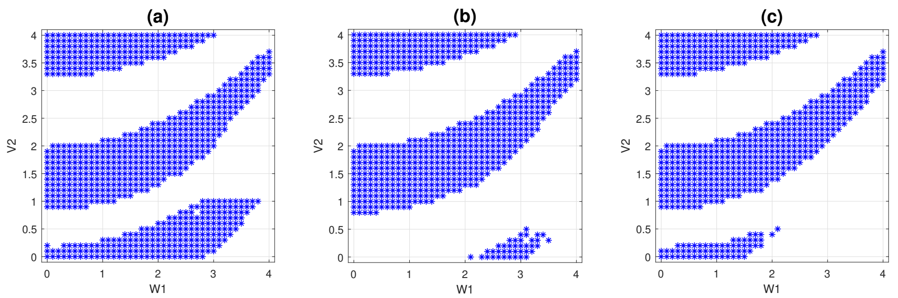

Figure 3.4 : Numerically obtained solitons of CQNLS equation with 3OD for various potential depths of $\mathscr{PT}$-symmetric potential for (a) $\beta = -0.1$, (b) $\beta = 0$, (c) $\beta = 0.1$.

4. STABILITY ANALYSIS OF CQNLS EQUATION WITH THIRD ORDER DISPERSION AND A $\mathscr{PT}$-SYMMETRIC POTENTIAL

4.1 Split-Step Fourier Method

The split-step method is one of the evolution method that the evolution equation is split into several pieces. The thought of the split step method occurred long time ago. This method was greatly improved in later years and high order schemes were forged for linear equations in two decades. After starting to be famous, this method was often made use of physicist.

4.2 Nonlinear Stability Analysis

During direct simulations, solitons are defined nonlinearly unstable if they preserve their shape, maximum amplitude and position. To investigate the nonlinear stability of solitons, they are derived throughout long distance. To be able to perform this, Split-Step Fourier Method is used to advance in z.

Consider the form of nonlinear PDE for $u(x,z)$ which can be written as

$$u_z = (M+N)u. \tag{4.1}$$

where M and N are operators independent of z. If Eq. (2.1) is rewritten, we have

$$u_z = i(\alpha\partial_{xx} + i\beta\partial_{xxx})u + i(|u|^2 + |u|^4 + V_{\mathscr{PT}})u. \tag{4.2}$$

and thus, it can be split as in Eq. (4.1) with the operator $M = i(\alpha\partial_{xx} + i\beta\partial_{xxx})$ and the operator $N = i(|u|^2 + |u|^4 + V_{\mathscr{PT}})$.

$u_z = Mu$ which is solved by means of Fourier transform. After applying the Fourier transform both sides of equation

$$u_z = i\alpha u_{xx} - \beta u_{xxx}. \tag{4.3}$$

then the below equation is obtained.

$$\hat{u}_z = \left(i\alpha(ik_x)^2 - \beta(ik_x)^3\right)\hat{u} = -i(\alpha {k_x}^2 - \beta {k_x}^3)\hat{u}\ . \tag{4.4}$$

Therefore, this equation is actually an ordinary differential equation (ODE) of $\hat{u}$ and its exact solution can be obtained as in Eq. (4.5). Moreover, by taking the inverse Fourier transform of $\hat{u}$, u is found as below.

$$\hat{u} = C_1 e^{-i(\alpha {k_x}^2 - \beta {k_x}^3)z} \quad \Rightarrow \quad u = \mathscr{F}^{-1}\left(C_1 e^{-i(\alpha {k_x}^2 - \beta {k_x}^3)z}\right) . \tag{4.5}$$

The second step $u_z = Nu$, i.e.

$$u_z = i(|u|^2 + |u|^4 + V_{\mathscr{P}\mathscr{T}})u. \tag{4.6}$$

has the exact solution

$$u = C_2 e^{i(|u|^2 + |u|^4 + V_{\mathscr{P}\mathscr{T}})z} . \tag{4.7}$$

Having found solutions to both parts, the split-step Fourier method can now be employed for the CQNLS equation by using any splitting scheme.

Using split-step Fourier method, nonlinearly stability of obtained solitons is investigated. In Fig. (4.1), stability regions of solitons are plotted for a three different values of beta and for potential depths varying from 0 to 4. It can be seen from this figure that positive values of beta (here $\beta = 0.1$) improves the nonlinear stability while negative beta (here $\beta = -0.1$) causes a decline in nonlinear stability.

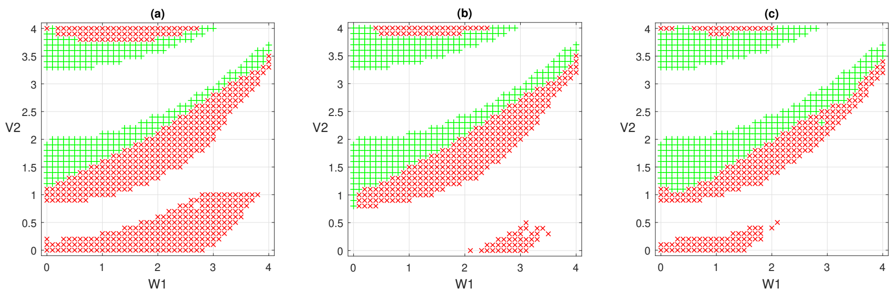

Figure 4.1 : Nonlinearly stable (marked as green) and nonlinearly unstable (marked as red) solitons of (a) CQNLS equation with 3OD for $\beta = -0.1$; (b) CQNLS equation without 3OD; (c) CQNLS equation with 3OD for $\beta = 0.1$ for varying potential depths of the $\mathscr{P}\mathscr{T}$-symmetric potential.

In Fig. (4.2) and Fig. (4.4), nonlinearly stable solitons are shown. It is seen from these figures that the solitons conserve their shapes and maximum amplitudes during the evolution. In Fig. (4.3) and Fig. (4.5), the nonlinear evolutions of solitons are depicted and it can be seen from these figures that as the propagation distance increases, the maximum amplitude of the soliton becomes oscillatory and after $z = 50$, the shape of

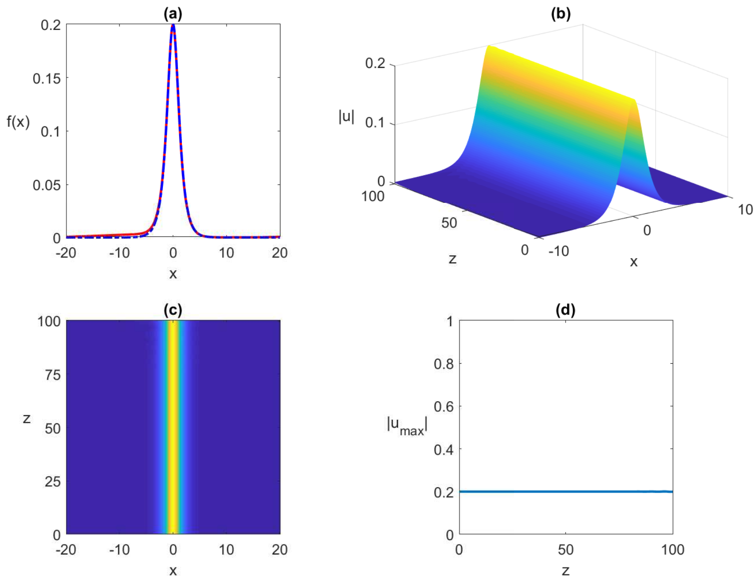

Figure 4.2 : Nonlinear stability of soliton for $\alpha = 1, \mu = 1$, $\beta = 0.1$, $W_1 = 0.6$ and $V_2 = 2$ with a $\mathscr{PT}$-symmetric potential; (a) Numerically produced soliton (blue dashes) on top of the solution after the evolution (red solid), (b) Nonlinear evolution of the soliton, (c)The view from top and (d) Maximum amplitude as a function of the propagation distance z.

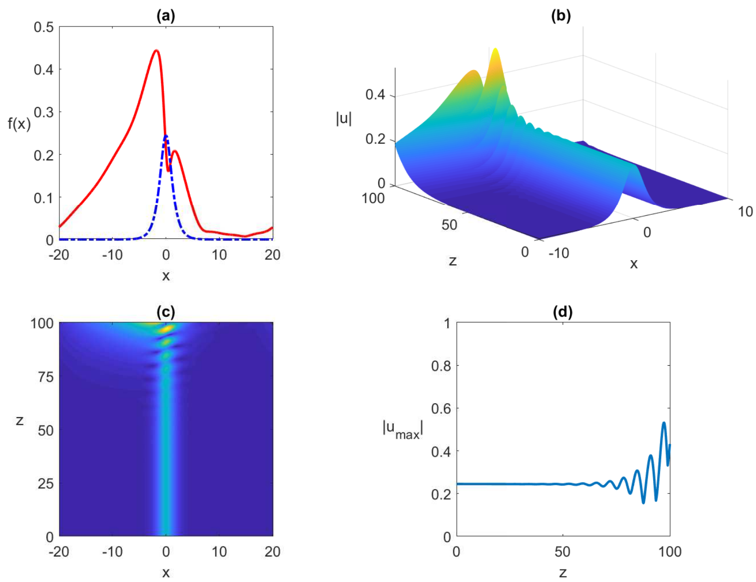

Figure 4.3 : Nonlinear instability of soliton for $\alpha = 1, \mu = 1$, $\beta = 0.1$, $W_1 = 1.8$ and $V_2 = 2.3$ with a $\mathscr{PT}$-symmetric potential; (a) Numerically produced soliton (blue dashes) on top of the solution after the evolution (red solid), (b) Nonlinear evolution of the soliton, (c)The view from top and (d) Maximum amplitude as a function of the propagation distance z.

the soliton is deteriorated which leads to nonlinear instability for the chosen potential depths.

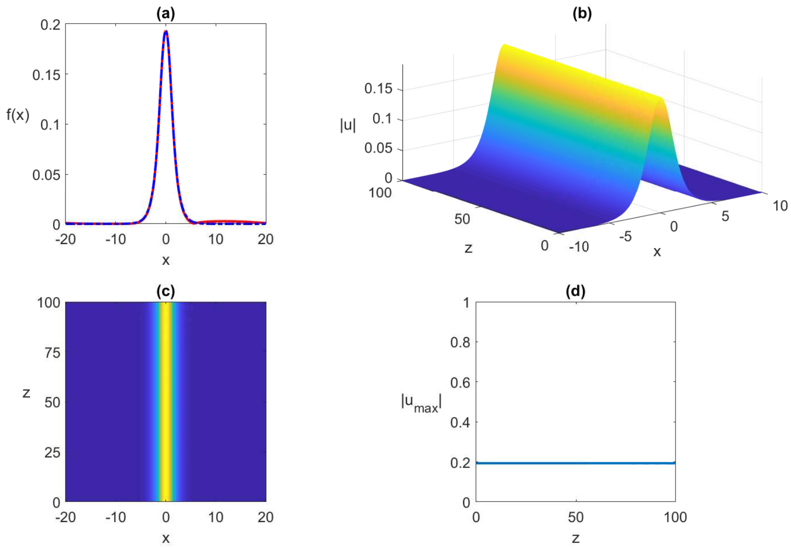

Figure 4.4 : Nonlinear stability of soliton for $\alpha = 1, \mu = 1, \beta = -0.1, W_1 = 0.2$ and $V_2 = 3.3$ with a $\mathscr{PT}$-symmetric potential; (a) Numerically produced soliton (blue dashes) on top of the solution after the evolution (red solid), (b) Nonlinear evolution of the soliton, (c)The view from top and (d) Maximum amplitude as a function of the propagation distance z.

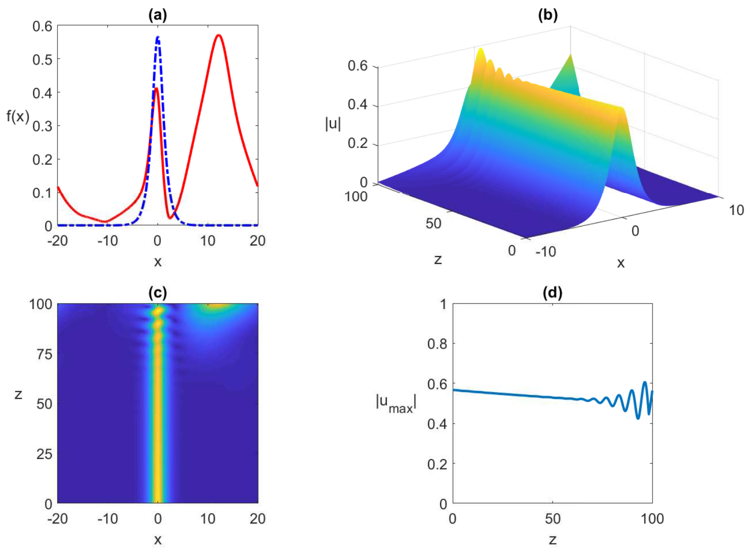

Figure 4.5 : Nonlinear instability of soliton for $\alpha = 1, \mu = 1, \beta = -0.1, W_1 = 1.7$ and $V_2 = 2$ with a $\mathscr{PT}$-symmetric potential; (a) Numerically produced soliton (blue dashes) on top of the solution after the evolution (red solid), (b) Nonlinear evolution of the soliton, (c)The view from top and (d) Maximum amplitude as a function of the propagation distance z.

4.3 Linear Stability

Linear stability will be investigated by acquiring and analyzing the linear spectrum and/or by evolving the linearized solitons.

4.3.1 Linear spectrum

Linear stability spectrum or short, linear spectrum are the eigenvalues of the linear stability operator of a soliton. These eigenvalues give information about the linear stability of a soliton.

Eq. (2.1) can be written as the following (1+1)D NLS equation having general type of nonlinearities:

$$iu_z(x) + \alpha u_{xx}(x) + i\beta u_{xxx}(x,z) + F(|u(x)|^2)u(x) + V_{\mathscr{PT}}(x)u(x) = 0. \tag{4.8}$$

where $F(\cdot) \in \mathbb{R}$ and $F(0) = 0$. As explained before, Eq. (4.8) admits soliton solutions of the form $u(x) = f(x)\, e^{i\mu z}$. Substituting

$$\begin{aligned} u_z &= i\mu f e^{i\mu z}. \\ u_{xx} &= f_{xx}\, e^{i\mu z}. \\ u_{xxx} &= f_{xxx}\, e^{i\mu z}. \\ |u|^2 &= uu^* = f e^{i\mu z} f^* e^{-i\mu z} = ff^* = |f|^2. \end{aligned} \tag{4.9}$$

in Eq. (4.8) and multiplying by $e^{-i\mu z}$ gives

$$-\mu f + \alpha f_{xx} + i\beta f_{xxx} + F(|f|^2)f + V_{\mathscr{PT}} f = 0. \tag{4.10}$$

For investigating the linear stability, the soliton solution is perturbed as follows

$$u(x) = \left[f(x) + g(x)e^{\lambda z} + h^*(x)e^{\lambda^* z}\right] e^{i\mu z}. \tag{4.11}$$

where g and h are perturbation eigenfunctions and λ is the eigenvalue.

$$\begin{aligned} u_z &= \left(\lambda g e^{\lambda z} + \lambda^* h^* e^{\lambda^* z} + i\mu f + i\mu g e^{\lambda z} + i\mu h^* e^{\lambda^* z}\right) e^{i\mu z}. \\ u_{xx} &= \left(f_{xx} + g_{xx} e^{\lambda z} + h^*_{xx} e^{\lambda^* z}\right) e^{i\mu z}. \\ u_{xxx} &= \left(f_{xxx} + g_{xxx} e^{\lambda z} + h^*_{xxx} e^{\lambda^* z}\right) e^{i\mu z}. \end{aligned} \tag{4.12}$$

$$\begin{aligned} |u|^2 = uu^* &= \left(f + g e^{\lambda z} + h^* e^{\lambda^* z}\right) e^{i\mu z} \left(f^* + g^* e^{\lambda^* z} + h e^{\lambda z}\right) e^{-i\mu z} \\ &= ff^* + fg^* e^{\lambda^* z} + fh e^{\lambda z} + f^* g e^{\lambda z} + gg^* e^{(\lambda+\lambda^*)z} \\ &\quad + gh e^{2\lambda z} + f^* h^* e^{\lambda^* z} + g^* h^* e^{2\lambda^* z} + hh^* e^{(\lambda+\lambda^*)z} \\ &\simeq |f|^2 + \left(g^* e^{\lambda^* z} + h e^{\lambda z}\right) f + \left(g e^{\lambda z} + h^* e^{\lambda^* z}\right) f^*. \end{aligned} \tag{4.13}$$

Using linear Taylor expansion $F(x+h) = F(x) + hF'(x) + O(h^2)$,

$$\begin{aligned} F(|u|^2) &= F\left(|f|^2 + \left[\left(g^* e^{\lambda^* z} + h e^{\lambda z}\right) f + \left(g e^{\lambda z} + h^* e^{\lambda^* z}\right) f^*\right]\right) \\ &\simeq F(|f|^2) + \left[\left(g^* e^{\lambda^* z} + h e^{\lambda z}\right) f + \left(g e^{\lambda z} + h^* e^{\lambda^* z}\right) f^*\right] F'(|f|^2)\,. \end{aligned} \tag{4.14}$$

Hence,

$$\begin{aligned} & F(|u|^2) u e^{-i\mu z} \\ &= F(|f|^2) f + \left[\left(g^* e^{\lambda^* z} + h e^{\lambda z}\right) f^2 + \left(g e^{\lambda z} + h^* e^{\lambda^* z}\right) |f|^2\right] F'(|f|^2) \\ &\quad + F(|f|^2) g e^{\lambda z} \\ &\quad + \left[\left(g g^* e^{(\lambda+\lambda^*)z} + g h e^{2\lambda z}\right) f + \left(g^2 e^{2\lambda z} + g h^* e^{(\lambda+\lambda^*)z}\right) f^*\right] F'(|f|^2) \\ &\quad + F(|f|^2) h^* e^{\lambda^* z} \\ &\quad + \left[\left(g^* h^* e^{2\lambda^* z} + |h|^2 e^{(\lambda+\lambda^*)z}\right) f + \left(g h^* e^{(\lambda+\lambda^*)z} + (h^*)^2 e^{2\lambda^* z}\right) f^*\right] F'(|f|^2) \\ &\simeq F(|f|^2)\left[f + g e^{\lambda z} + h^* e^{\lambda^* z}\right] \\ &\quad + F'(|f|^2)\left[\left(f^2 h + |f|^2 g\right) e^{\lambda z} + \left(f^2 g^* + |f|^2 h^*\right) e^{\lambda^* z}\right]\,. \end{aligned} \tag{4.15}$$

Substituting Eq.(4.11), (4.12) and (4.15) into Eq.(4.8) gives

$$\begin{aligned} & i\left(\lambda g e^{\lambda z} + \lambda^* h^* e^{\lambda^* z} + i\mu f + i\mu g e^{\lambda z} + i\mu h^* e^{\lambda^* z}\right) e^{i\mu z} \\ &+ \alpha\left(f_{xx} + g_{xx} e^{\lambda z} + h^*_{xx} e^{\lambda^* z}\right) e^{i\mu z} \\ &+ i\beta\left(f_{xxx} + g_{xxx} e^{\alpha z} + h^*_{xxx} e^{\lambda^* z}\right) e^{i\mu z} \\ &+ \left\{\begin{array}{l} F(|f|^2)\left[f + g e^{\lambda z} + h^* e^{\lambda^* z}\right] \\ + F'(|f|^2)\left[\left(f^2 h + |f|^2 g\right) e^{\lambda z} + \left(f^2 g^* + |f|^2 h^*\right) e^{\lambda^* z}\right] \end{array}\right\} e^{i\mu z} \\ &+ V_{\mathscr{PT}}\left(f + g e^{\lambda z} + h^* e^{\lambda^* z}\right) e^{i\mu z} = 0\,. \end{aligned} \tag{4.16}$$

Grouping the terms and multiplying by $e^{-i\mu z}$ yields

$$\begin{aligned} & \left[-\mu f + \alpha f_{xx} + i\beta f_{xxx} + F(|f|^2) f + V_{\mathscr{PT}} f\right] \\ &+ \left\{\begin{array}{l} i\lambda g - \mu g + \alpha g_{xx} + i\beta g_{xxx} + F(|f|^2) g \\ + \left(f^2 h + |f|^2 g\right) F'(|f|^2) + V_{\mathscr{PT}} g \end{array}\right\} e^{\lambda z} \\ &+ \left\{\begin{array}{l} i\lambda^* h^* - \mu h^* + \alpha h^*_{xx} + i\beta h^*_{xxx} + F(|f|^2) h^* \\ + \left(f^2 g^* + |f|^2 h^*\right) F'(|f|^2) + V_{\mathscr{PT}} h^* \end{array}\right\} e^{\lambda^* z} \\ &= 0\,. \end{aligned} \tag{4.17}$$

Since f is a solution seen in Eq.(4.10), the first bracket in Eq.(4.17) is equal to zero. For Eq.(4.17) to hold true, the factors of the exponentials must be zero simultaneously.

Hence, one has on one hand

$$i\lambda g-\mu g+\alpha g_{xx}+i\beta g_{xxx}+F(|f|^2)g+\left(f^2h+|f|^2g\right)F'(|f|^2)+V_{\mathscr{PT}}g=0. \quad (4.18)$$

which can be rewritten as

$$\alpha g_{xx}+i\beta g_{xxx}+\left[F(|f|^2)+F'(|f|^2)|f|^2-\mu+V_{\mathscr{PT}}\right]g+F'(|f|^2)f^2h=-i\lambda g. \quad (4.19)$$

and moreover, by Eq.(4.17), second exponential factor is equal to zero.

$$i\lambda^*h^*-\mu h^*+\alpha h^*_{xx}+i\beta h^*_{xxx}+F(|f|^2)h^*+\left(f^2g^*+|f|^2h^*\right)F'(|f|^2)+V_{\mathscr{PT}}h^*=0. \quad (4.20)$$

which can be rewritten as

$$\alpha h^*_{xx}+i\beta h^*_{xxx}+\left[F(|f|^2)+F'(|f|^2)|f|^2-\mu+V_{\mathscr{PT}}\right]h^*+F'(|f|^2)f^2g^*=-i\lambda^*h^*. \quad (4.21)$$

Taking the conjugate of Eq.(4.21) gives

$$\alpha h_{xx}+i\beta h_{xxx}+\left[F(|f|^2)+F'(|f|^2)|f|^2-\mu+V^*_{\mathscr{PT}}\right]h+F'(|f|^2)\left(f^2\right)^*g=i\lambda h. \quad (4.22)$$

Multiplying Eq.(4.22) by -1 gives

$$-\alpha h_{xx}-i\beta h_{xxx}-\left[F(|f|^2)+F'(|f|^2)|f|^2-\mu+V^*_{\mathscr{PT}}\right]h-F'(|f|^2)\left(f^2\right)^*g=-i\lambda h. \quad (4.23)$$

Writing Eq.(4.19) and (4.23) in matrix form yields

$$i\begin{bmatrix} L_1 & L_2 \\ -L_2{}^* & -L_1{}^* \end{bmatrix}\begin{bmatrix} g \\ h \end{bmatrix}=\lambda\begin{bmatrix} g \\ h \end{bmatrix} \quad (4.24)$$

where

$$\begin{aligned} L_1 &= \alpha\partial_{xx}+i\beta\partial_{xxx}+F(|f|^2)+F'(|f|^2)|f|^2-\mu+V_{\mathscr{PT}}. \\ L_2 &= F'(|f|^2)f^2 \ . \end{aligned} \quad (4.25)$$

For the cubic-quintic nonlinearity,

$$\begin{aligned} F(x) &= ax+bx^2. \\ F'(x) &= a+2bx \ . \end{aligned} \quad (4.26)$$

Using Eq.(4.26) in Eq.(4.25) yields

$$\begin{aligned} L_1 &= \alpha\partial_{xx}+i\beta\partial_{xxx}+2a|f|^2+3b|f|^4-\mu+V_{\mathscr{PT}}. \\ L_2 &= af^2+2bf^3f^* \ . \end{aligned} \quad (4.27)$$

If the soliton and potential are real, i.e. $f,\ V_{\mathscr{P}\mathscr{T}} \in \mathbb{R}$, Eq.(4.24) becomes

$$i\begin{bmatrix} L_1 & L_2 \\ -L_2 & -L_1 \end{bmatrix}\begin{bmatrix} g \\ h \end{bmatrix} = \lambda \begin{bmatrix} g \\ h \end{bmatrix} \tag{4.28}$$

where

$$\begin{aligned} L_1 &= \alpha\partial_{xx} + i\beta\partial_{xxx} + 2af^2 + 3bf^4 - \mu + V_{\mathscr{P}\mathscr{T}}. \\ L_2 &= af^2 + 2bf^4\ . \end{aligned} \tag{4.29}$$

Linear spectrum of numerically obtained solitons are found by taking $\beta = -0.1$, $\beta = 0$ and $\beta = 0.1$ in order to examine the impact of the 3OD term to linear stability of solitons of CQNLS equation. It can be seen from Fig. (4.6) that, obtained solitons for $\beta = -0.1$ and $\beta = 0.1$ are found to be linearly stable while soliton obtained for $\beta = 0$ is linearly unstable. This fact reveals that adding 3OD to the problem increases linear stability properties of CQNLS solitons in this specific $\mathscr{P}\mathscr{T}$-symmetric potential.

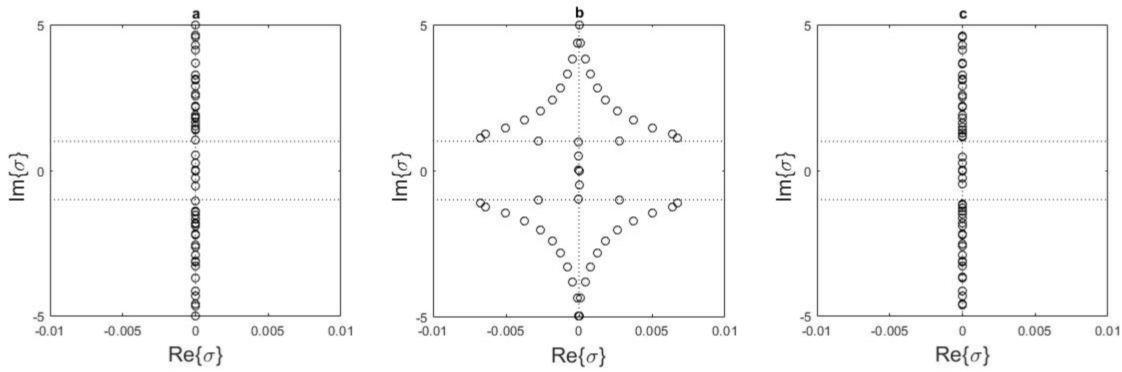

Figure 4.6 : Linear spectrum of CQNLS equation (a) with 3OD for $\beta = -0.1$; (b) without 3OD; (c) with 3OD for $\beta = 0.1$ for $W_1 = 4$ and $V_2 = 3.7$

In order to investigate the effect of quintic nonlinearity on linear stability of soliton solutions, we consider following equation with coefficients of cubic and quintic nonlinearities as *a* and *b* respectively

$$iu_z + \alpha u_{xx} + i\beta u_{xxx} + a|u|^2u + b|u|^4u + V_{\mathscr{P}\mathscr{T}}u = 0. \tag{4.30}$$

Here we fix the cubic nonlinearity ($a = 1$) and assume that b is varying from 0 to 1 by 0.2 steps. To investigate the effect of increasing quintic nonlinearity, linear spectra of solitons are plotted in Fig. (4.7) for progressively increasing b values. It can be concluded from Fig. (4.7), while other parameters are fixed, increasing the quintic nonlinearity has a positive effect on linear stability of a soliton.

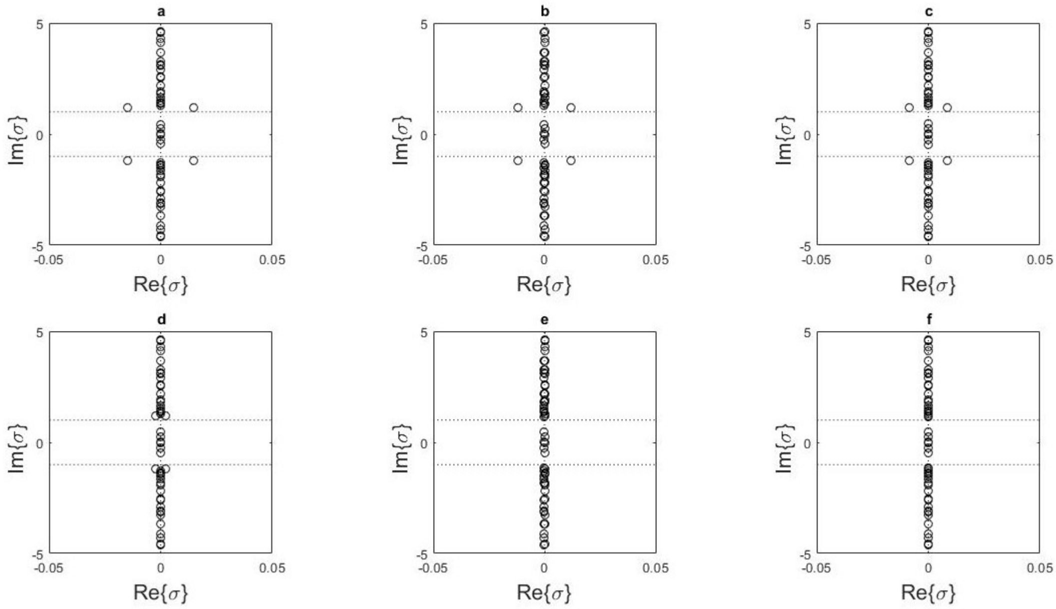

Figure 4.7 : Linear spectrum of numerically obtained solitons of the CQNLS equation with $\mathscr{P}\mathscr{T}$-symmetric potential and 3OD for $\beta = 0.1$, $W_1 = 4$ and $V_2 = 3.7$ in the (a) cubic case (a=1, b=0) and cubic-quintic cases: (b) a=1, b=0.2, (c) a=1, b=0.4, (d) a=1, b=0.6, (e) a=1, b=0.8, (f) a=1, b=1.

5. CONCLUSION

In this thesis, we have investigated the existence and the stability of soliton type solutions of higher order dispersive-cubic quintic nonlinear Schrödinger equation with a $\mathscr{PT}$-symmetric potential. As the $\mathscr{PT}$-symmetric potential, an extension of well known Scarff II type potential is taken into account.

Firstly, exact soliton type solution of CQNLS equation with 3OD and a $\mathscr{PT}$-symmetric potential is obtained and structure of $\mathscr{PT}$-symmetric potential is determined.

In the second section, SR method is explained and modified in order to obtain numerical solutions of the equation. Then, it is shown that the analytical and the numerical soliton solutions overlap; here, the numerical solutions satisfy the Eq. (1.2) with an error less than 10^{-8}. Moreover, effect of 3OD term on existence region of soliton type solutions is studied and shown that adding the 3OD term to the problem enlarges soliton existence region for both positive and negative coefficients.

In the last section of this thesis, using Split-Step Fourier method, nonlinear stability properties of previously obtained solitons are examined and nonlinear stability/instability regions are depicted for varying values of the coefficients of potential depths. It is observed that, for negative 3OD term the nonlinear stability region of the solitons is smaller than that of the region for positive 3OD term. So one can conclude that, considering a positive higher order dispersion may enlarge the nonlinear stability region for CQNLS equation for this specific type of $\mathscr{PT}$-symmetric potential.

Moreover, linear stability analysis based on the effect of 3OD term and quintic term on linear stability are studied. For fixed potential depth values it is observed that 3OD term has a positive effect on the linear stability for both negative and positive coefficients of 3OD. Also, the effect of quintic term on linear stability is investigated by slowly increasing the coefficient of the quintic term from 0 to 1. It is shown that quintic nonlinearity has a positive effect on linear stability of the solitons.

For future studies, the competing nonlinearity can be discussed for a deeper understanding of the effect of cubic and quintic nonlinearities. Also, in order to observe the contribution of fourth order dispersion (4OD) to the problem, one may consider taking both 3OD and 4OD into account and study the competing dispersion effect by comparing the results to the existing literature.

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APPENDICES

APPENDIX A.1 : Fourier Transform

APPENDIX A.1

Fourier Transform

The Fourier transform of f(x) is

$$F(k_x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(k_x)x} dx \tag{A.1}$$

for a continuous, smooth and absolutely integrable function $f(x)$ and conversely, *the inverse Fourier transform of* $F(k_x)$

$$F(k_x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i(k_x)x} \mathrm{d}x \tag{A.2}$$

$\mathscr{F}(f) = \hat{f}$ is called the Fourier transform of f and $\mathscr{F}^{-1}(\hat{f})$ is called the inverse Fourier transform of $\hat{f}$.

One of the classical properties of Fourier transform is given as

$$\mathscr{F}\left[\frac{d}{dx} f(x)\right] = ik_x \mathscr{F}(f(x)) = ik_x \hat{f} \tag{A.3}$$

This property can be generalized to higher order differantiation property of Fourier transform as Eq. (**A.4**).

$$\mathscr{F}\left[\frac{d^n}{(dx)^n} f(x)\right] = (ik_x)^n \mathscr{F}(f(x)) = (ik_x)^n \hat{f}, \qquad n \in N \tag{A.4}$$

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