ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY

GENERIC SUBMERSIONS

Ph.D. THESIS

Cem SAYAR

Department of Mathematical Engineering

Mathematical Engineering Programme

AUGUST 2019

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KAPSAMLI SUBMERSİYONLAR

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To a new beginning

FOREWORD

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ABBREVIATIONS

AH	**:** Almost Hermitian manifold
ASK	**:** Almost semi-Kaehler manifold
QK	**:** Quasi Kaehler manifold
NK	**:** Nearly Kaehler manifold
AK	**:** Almost Kaehler manifold
SK	**:** Semi-Kaehler manifold
H	**:** Hermitian manifold
K	**:** Kaehler manifold
l.p.R.	**:** Locally Product Riemannian manifold

SYMBOLS

$\mathbb{R}$	**:** Real Numbers
$\mathbb{R}^n$	**:** n-dimensional Real Space
$\mathbb{C}$	**:** Complex Numbers
$\mathscr{F}(M)$	**:** The set of differentiable functions on M
$grad f$	**:** The Gradient of a function f
$\Gamma(M)$	**:** The set of the vector fields on M
$\dim(M)$	**:** The dimension of M
$i = \overline{1,k}$	**:** i takes the value from 1 to k

GENERIC SUBMERSIONS

SUMMARY

The theory of Riemannian submersion is an area in differential geometry that gives a chance to compare the geometries of two manifolds with a smooth map between them. In this sense, many kinds of Riemannian submersions are defined and studied. In this thesis, we establish a generalization of the theory of submersion step by step.

In the first chapter, the purpose of the thesis, the literature of the theory of submersion and the hypothesis of the thesis are given. Some studies in this area are given. The aim of the study is mentioned.

In the second chapter, the fundamental definitions, equations and theorems on Riemannian geometry are introduced. A brief information on Riemannian manifolds is considered. The concept of distribution is discussed. We give some basic definitions and theorems on distributions. Finally in this chapter, we give some essential knowledge about almost product structure and classification of the manifolds with respect to almost product structure, that is, almost product Riemannian manifolds and locally product Riemannian manifolds.

In the third chapter, the complex manifolds are summarized. After giving the definition of an almost complex structure, the classification of the almost complex manifolds with respect to the almost complex structure are mentioned. Some classes of the almost complex manifolds are introduced. Also, the inclusion relations between complex manifolds are given.

In the fourth chapter, which is the main part of the thesis, Riemannian submersion concept, which is defined by O'Neill, is mentioned. The notion of a fiber, which is a crucial point in the theory of submersion, is introduced. To study and understand the geometry of the fibers, O'Neill tensors A and T also their some properties are given. Furthermore, some fundamental definitions, equations and theorems are introduced about the theory of Riemannian submersion.

After giving the concepts about Riemannian submersion, first we study on anti-invariant submersion and Lagrangian submersion, which is a particular case of an anti-invariant submersion, by taking the total manifold as a locally product Riemannian manifold. In this case, we prove that for a Lagrangian submersion the fibers are always totally geodesic. Moreover, we define the first variational formula of an anti-invariant submersion. By means of that form, we give a new approach to investigate whether the fibers of the submersion are harmonic or not.

Next, we study on semi-invariant submersion by taking the total manifold as a locally product Riemannian. In the present case, an example is given for semi-invariant submersion. Also, we prove some decomposition lemmas. The integrability conditions of the distributions for a semi-invariant submersion are investigated. Moreover, we

investigate the geometry of the fibers of a semi-invariant submersion and we study the totally geodesicness of the fibers. After, we consider the fibers totally umbilical and obtain some results. The canonical structures are considered parallel and we get certain results about relation between canonical structures and the geometry of the fibers. Furthermore, we define the first variational formula of a semi-invariant submersion. By the virtue of that formula, we give a new idea to investigate whether the fibers of the submersion are harmonic or not.

Later, we define a new type of submersion, which is called pointwise semi-slant submersion by considering the total manifold as a locally product Riemannian. We give an example for a pointwise semi-slant submersion. Some decomposition theorems are obtained. Integrability of the distributions are investigated that mentioned in the definition of the pointwise semi-slant submersion. The geometry of the fibers are examined and some results are obtained. The canonical structures and the fibers of the pointwise semi-slant submersion are considered parallel and totally umbilical, respectively, and some consequences are found. Moreover, the first variational formula of a pointwise semi-slant submersion is defined and it is given that a new view to understanding in which conditions the fibers are harmonic.

Finally, the generic submersion (in the sense of Ronsse) is defined, which is the generalization of the all kind of submersions. We study generic submersion by taking the total manifold Kaehler and give some examples for a generic submersion. Also, we give some decomposition theorems and some equations, which have same meaning with Gauss and Weingarten equations in the theory of submanifold, to use in the proofs. The integrability and the totally geodesicness of the distributions, which are mentioned in the definition of the generic submersion, are investigated. By taking the fibers as totally umbilical, we give some results and get a corollary for the minimality of the fibers. We think the canonical structures parallel and obtain some outstanding results.

In future, it is estimated that the curvature relations between total manifold, base manifold and fibers of a generic submersion can be investigated. Also, for the total manifold of a generic submersion, the following problem can be studied: "in which conditions the total manifold can be Einstein space?". On the other hand, it is known that all these theory of submersion have a relation with Physics. Especially, the following question can be answered: "What is the relation of a generic submersion with Physics?". Finally, the theory of submersion has a relation with statistical machine learning processes, which is popular area in the world. Generic submersion and statistical machine learning process relation can be investigated.

KAPSAMLI SUBMERSİYONLAR

ÖZET

Diferansiyel geometride submersiyon teorisi, iki manifoldun geometrisini, aralarında tanımlanan düzgün bir dönüşüm yardımıyla, karşılaştırma şansı sunan bir alandır. Bu bağlamda literatürde, çok sayıda submersiyon çeşidi tanımlandı ve çalışıldı. Verilen bir submersiyon için lifler kaynak manifoldun alt manifoldu olduğundan, alt manifold teorisindeki yaklaşımların bir çoğundan faydalanarak submersiyon teorisinde ilerlemeler kaydedilmiştir. Biz ise, bu çalışmada, alt manifold teorisinde var olan yaklaşımları da kullanarak, submersiyon teorisi için adım adım bir genelleme inşa ediyoruz.

İlk bölümde, alt manifold teorisi ve submersiyon teorisi arasındaki ilişkiden bahsedilmiş olup, literatürde submersiyon teorisi ile alakalı kronolojik olarak elde edilen gelişmelere yer verilmiştir. Kaynak manifoldun seçimine bağlı olarak (kontakt manifold, hemen hemen kompleks manifold v.b.) tanımlanan submersiyon tiplerinden bahsedilmiş olup, çalışmalar refere edilmiştir. Ayrıca, bu tezin hipotezi ve amacı da anlatılmıştır.

İkinci bölümde, Riemann geometrisindeki temel tanımlar, denklemler ve teoremler tanıtılmıştır. Riemann manifoldları hakkında kısaca bilgiler verilmiştir. Distribüsyon kavramı hakkında tezde kullanılacak bilgilere kısaca değinilmiştir. Distribüsyonlar hakkında bazı temel tanım ve teoremlere yer verilmiştir. Bu bölümde son olarak, hemen hemen çarpım yapısı tanıtılmış olup, bu yapıya göre bazı (hemen hemen çarpım Riemann manifoldları, yerel çarpım Riemann manifoldları v.b.) manifold sınıflarının tanımları verilmiştir.

Üçüncü bölümde, kompleks manifoldlar hakkında temel bilgilere yer verilmiştir. Hemem hemen kompleks yapının tanımı ve bu yapının Riemann metriği ile olan ilişkisi verildikten sonra, bu yapıya göre manifoldların sınıflandırılması üzerinde durulmuştur. Bazı kompleks manifold sınıflarının tanımlarından bahsedilmiş olup, bu sınıflar arasındaki kapsama bağıntılarından söz edilmiştir.

Tezin ana parçasını oluşturan dördüncü bölümde, Riemann submersiyonu kavramı tanıtılmış olup, kaynak ve hedef manifoldları arasındaki vektör alanları ilişkilerinden bahsedilmiştir. Submersiyon teorisinde önemli yer tutan lif kavramı tanıtılmıştır. Liflerin boyutları, teğetleri ve normalleri hakkında bilgiler verilmiştir. Liflerin geometrisini incelememize yarayan ve temelini oluşturan, T ve A O'Neill tensörleri ve bu tensörlerin bazı temel özellikleri verilmiştir. Ayrıca, submersiyon teorisi ile alakalı bazı temel tanım, denklem ve teoremler verilmiştir.

Riemann teorisi ile ilgili gereken temel bilgiler verildikten sonra, ilk olarak, kaynak manifoldunu yerel çarpım Riemann manifoldu alarak, ters-değişmez submersiyonları ve ters-değişmez submersiyonların özel bir hali olan Lagrangian submersiyonları çalıştık. Bu durumda, Lagrangian submersiyonlar için, lifleri incelediğimizde, daima

tamamen jeodezik olduğunu elde ettik. Ayrıca, ters-değişmez submersiyonlar için birinci varyasyonel formülünü tanımladık. Bu formül yardımıyla, ters-değişmez submersiyonların liflerinin harmonik olup olmadığının araştırılması konusunda gerek yeter koşul verdik. Bu yaklaşım ile literatüre farklı bir bakış açısı sunmuş olduk.

Daha sonra, yine kaynak manifoldunu yerel çarpım Riemann manifoldu alarak, yarı-değişmez submersiyonları çalıştık. Yarı-değişmez submersiyon için örnek verildi. Kanonik yapıları ve hemen hemen çarpım yapısını kullanarak bazı ayrışım yardımcı önermeleri kanıtladık. Bu tip submersiyonlardaki distribüsyonlar için integrallenebilme koşullarını araştırdık ve bazı sonuçlar elde ettik. Yarı-değişmez submersiyonun liflerinin geometrisi hakkında bilgi sahibi olmak için, liflerin tamamen jeodezik olma durumunu inceledik. Lifleri tamamen umbilik kabul ederek, bazı sonuçlar elde ettik. Kanonik yapıların paralel olma tanımını verildi. Bu yapıların paralel olmaları durumunda, kanonik yapıların birbirleri arasındaki ilişkiler hakkında ve liflerin geometrisi hakkında bazı sonuçlar elde ettik. Ayrıca, yarı-değişmez submersiyonlar için de birinci varyasyonel formülünü tanımlayarak ve kullanarak liflerin hangi koşullar altında harmonik olduğuna dair yeni yaklaşım ve koşullar elde ettik.

Dördüncü bölümün devamında, noktasal yarı-eğik adı ile yeni tip submersiyon tanımladık. Bu submersiyon tipini kaynak manifoldunu yerel çarpım Riemann manifoldu alarak çalıştık. Tanımlanan bu submersiyon tipi için örneğe yer verdik. Benzer şekilde hemen hemen çarpım yapısını ve tanımdaki distribüsyonları göz önüne alarak bazı ayrışım yardımcı teoremleri elde ettik. Alt manifold teorisinde kullanılan Gauss ve Weingarten denklemlerinin noktasal yarı-eğik submersiyonlar için karşılıklarını elde ettik. Noktasal yarı-eğik submersiyon tanımında bahsedilen distribüsyonların hangi koşullar altında integrallenebileceğini araştırdık. Liflerin geometrileri ile ilgilenerek, bazı sonuçlar elde ettik. Kanonik yapıların paralelliği tanımlanarak çalışıldı. Ayrıca, liflerin tamamen umbilik olması koşulunda da bazı sonuçlar elde edilmiştir. Bu sonuçlara ilave olarak, noktasal yarı-eğik submersiyonlar için birinci varyasyonel formülü tanımlandı. Bu tanım yardımıyla, liflerin hangi koşullar altında harmonik olduğunu araştırmak adına bir yaklaşım sunmuş olduk.

Ve son olarak, ele alınan hemen hemen kompleks yapıya göre tüm submersiyon tiplerinin bir genelleştirmesi olan kapsamlı submersiyonu (Ronsse anlamında) tanımladık. Bu submersiyon tipi için kaynak manifoldu Kaehler manifold aldık. Öncelikle, bu tip submersiyonlar için örnekler verdik. Alt manifold teorisinde kullanılan Gauss ve Weingarten denklemlerinin kapsamlı submersiyonlar için karşılıklarını elde ettik. Bu denklemler kanıtlarda kullanıldı. Kapsamlı submersiyon tanımındaki distribüsyonlar için integrallenebilme koşullarını inceledik. Ayrıca, liflerin geometrisini anlayabilmek adına, liflerin tümel jeodezik olma koşulları da incelendi. Liflerin tümel umbilik olması durumunda minimal olması için bir koşul kanıtlandı. Bu tip submersiyonlar için, vektörlerin ayrışımında verilen kanonik yapılar paralel düşünüldü. Bu durumda, distribüsyonların karışık jeodezik olmaları ve kanonik yapıların paralelliği hakkında gerek ve yeter şartlar elde edildi.

Bu tezin sonrasında, gelecek çalışmalarda, tanımladığımız kapsamlı submersiyon için kaynak manifold, hedef manifold ve liflerin eğrilikleri incelenebilir ve aralarındaki ilişkiler araştırılabilir. Ayrıca, bir kapsamlı submersiyonun kaynak manifoldu için şu problem çalışılabilir: “Hangi koşullar altında kaynak manifoldu Einstein uzayı olur?”. Diğer taraftan, tüm tanımlanan submersiyonların Fizik’te karşılığının var olduğu bilinen bir gerçektir. Bilhassa, şu probleme cevap aranabilir: “Kapsamlı submersiyonun Fizik’teki karşılığı nedir?”. Disiplinlerarası çalışma adına etkili bir problem olacağı öngörülmektedir. Son olarak, submersiyon teorisinin istatistiksel makine öğrenmesi süreçlerinde karşılığı olduğu bilinmektedir. Tanımladığımız kapsamlı submersiyon ve istatistiksel makine öğrenmesi arasındaki ilişkiler çalışılabilir. Dahası, submersiyon teorisinin kullanıldığı daha başka alanlardaki kapsamlı submersiyonların karşılıkları araştırılıp çalışılabilir.

1. INTRODUCTION

The theory of submanifold is an important and interesting research area in differential geometry. As a smooth map between Riemannian manifolds, a submersion is one of the various ways to get a submanifold. With this method, a chance rises to compare the geometries of two manifolds.

In this sense, the notion of a Riemannian submersion was first introduced by O'Neill [1] and Gray [2], independently from each other. Watson considered Riemannian submersions between almost Hermitian manifolds with the name of almost Hermitian submersions [3]. In this case, the Riemannian submersion is also a complex mapping, and consequently, the vertical and horizontal distributions are invariant with respect to the almost complex structure of the total manifold of the submersion. Afterwards, almost Hermitian submersions have been studied for different subclasses of almost Hermitian manifolds, for example; see [4]. It is note that the Riemannian submersions were extended to several subclasses of almost contact manifolds which is called contact Riemannian submersions. Some of the studies related with almost Hermitian, contact Riemannian and Riemannian submersions are included in [5], for the further information we refer to [5].

Recently, Şahin introduced the notion of anti-invariant submersions from almost Hermitian manifolds onto Riemannian manifolds [6]. He studied such submersions from Kaehler manifolds onto Riemannian manifolds. In these circumstances, the fibers are anti-invariant with respect to the almost complex structure of the total manifold of the submersion. In [7, 8], it is mentioned that a Lagrangian submersion is a special case of an anti-invariant Riemannian submersion on which almost complex structure of the total manifold reverses the vertical and horizontal distributions. Latterly, it has been defined various new type of Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds such as slant submersion [9], semi-invariant submersion [7], pointwise slant submersion [10], hemi-slant submersion [11]. Note that, some of these submersions have been extended to the subclasses of almost contact manifolds, for instance see [12].

In this thesis, our goal is to define a generalization of all types of submersions that defined previously. To reach this aim, in the first step, we study on anti-invariant submersion and the special case of it, Lagrangian submersions, by taking the total manifold locally product Riemannian [13]. While we study this problem, we try to understand the theory of submersion with the help of [1, 6]. Also, we obtain some remarkable results.

In the second step, we study semi-invariant submersion, which is defined in [7], from a locally product Riemannian manifolds onto a Riemannian manifold [14]. In this case, the vertical distribution $ker\pi_*$ is a direct sum of two distributions, that is,

$$ker\pi_* = D \oplus D^{\perp}, \tag{1.1}$$

where D is invariant and $D^{\perp}$ is anti-invariant with respect to the almost product structure. That means, we have one more distribution to investigate.

In the third step, we consider the pointwise slant distribution and in the view of [10] define a new type of submersion so-called pointwise semi-slant submersion [15]. In this case, since the vertical distribution has a decomposition as

$$ker\pi_* = D \oplus D_\theta, \tag{1.2}$$

where D_θ is a pointwise slant distributon with pointwise slant angle θ. We improve our knowledge about pointwise angle and pointwise slant distribution.

As a final step, we construct a generalization of all kinds of submersions by taking the total manifold Kaehlerian. Three known generic submersion notions are given by Yano and Kon [16], Chen [17] and Ronsse [18]. By considering an idea in the theory of submanifold [18], we define a new type of submersion such that if the fibers of a submersion are generic submanifold (in the sense of Ronsse) of the vertical distribution, then the submersion is called a generic submersion (in the sense of Ronsse) [19]. By the way, in this work [19], Prof. Dr. Mukut Mani Tripathi contributed us and we studied in a cooperation with him. In the circumstances, the vertical distribution can be decomposed as

$$ker\pi_* = D^1 \oplus D^0 \oplus D^{\lambda_1} \oplus D^{\lambda_2} \oplus \ldots \oplus D^{\lambda_k}, \tag{1.3}$$

where D^1 is invariant, D^0 is anti-invariant, D^{λ_i} is pointwise slant distribution with slant function θ_i. In section 4, the generic submersion is studied deeply.

2. RIEMANNIAN MANIFOLDS

In this chapter, we give some fundamental definitions and theorems from [20] that we use throughout this thesis .

Let M be a differenatiable m-dimensional manifold. It is denoted that the algebra of differentiable functions on M by $\mathscr{F}(M)$ and the module of differentiable vector fields by $\Gamma(M)$, respectively. It can be seen that $\Gamma(M)$ is a vector space with respect to scalar multiplication and natural addition. Let ∇ be a map defined as

$$\nabla : \Gamma(M) \times \Gamma(M) \to \Gamma(M), \tag{2.1}$$

such that

$$\nabla_U f = Uf, \tag{2.2}$$

$$\nabla_{fU+gV} Z = f(\nabla_U Z) + g(\nabla_V Z), \tag{2.3}$$

and

$$\nabla_U (fZ + gV) = f\nabla_U Z + g\nabla_U V + (Uf)Z + (Ug)V, \tag{2.4}$$

for any vector fields U,V,Z and smooth functions f,g on M. ∇ is called a *linear connection*, ∇_U the *covariant derivative operator* and $\nabla_U V$ *covariant derivative of V with respect to U*. Let define a $(1,1)$-type tensor field ∇U by $(\nabla U)(V) = \nabla_U V$ for any V. Besides this, $\nabla_U g = Ug$ is the covariant derivative of g along U. For a $1-$form ω, the covariant derivative of it is defined by

$$(\nabla_U \omega)(V) = U(\omega(V)) - \omega(\nabla_U V). \tag{2.5}$$

The covariant derivative of a tensor T of type (r,s) along a vector field U is a tensor field $\nabla_U T$, of type (r,s), defined by, for any vector field U, r covariant vectors $\omega^1, \omega^2, ..., \omega^r$ and s contravariant vectors $V_1, V_2, ..., V_s$

$$\begin{aligned}(\nabla_U T)(\omega^1, \omega^2, ..., \omega^r, V_1, V_2, ..., V_s) = \quad & U(T(\omega^1, \omega^2, ..., \omega^r, V_1, V_2, ..., V_s)) \\ - \quad & \sum_{i=1}^{r} T(\omega^1, ..., \nabla_U \omega^i, ..., \omega^r, V_1, V_2, ..., V_s) \\ - \quad & \sum_{j=1}^{s} T(\omega^1, ..., \omega^r, V_1, ..., \nabla_U V_j, ..., V_s).\end{aligned} \tag{2.6}$$

One can say that the covariant derivative ∇T of tensor T is of type $(r,s+1)$. If a vector field U on M has constant covariant derivative along for any vector field V on M, then U is called parallel with respect to a linear connection ∇ that is $\nabla_V U = 0$. Similarly, a tensor field T is said to be parallel on M with respect to a linear connection ∇ if its covariant derivative is constant for along any vector field U on M that is $\nabla_U T = 0$. Let $\alpha : I \subset \mathbb{R}$ be a smooth curve with $(x^1, x^2, ..., x^n)$ coordinates. Then, the vector field X which is tangent to the curve α is given by

$$X = \frac{dx^j}{dt}\partial_j, \tag{2.7}$$

where $\partial_j = \frac{\partial}{\partial_j}$ and $(x^1, x^2, ..., x^n)$ is a local coordinate system.
Let $Z, V \in \Gamma(M)$. The Lie bracket $[Z,V]$ on M is defined by

$$[Z,V]\beta = Z(V\beta) - V(Z\beta). \tag{2.8}$$

For a function $\beta \in \mathscr{F}(M)$ and a vector field U on M, βU is a vector field on M which is given by $(\beta U)_p = \beta(p)U_p$, for some $p \in M$. It can be seen that Lie bracket $[,]$ is a skew-symmetric operator. Additionally, Lie bracket has the following properties:

$$[\beta Z, \gamma V] = \beta[Z,V] + \beta(Z\gamma)V - \gamma(V\beta)Z, \tag{2.9}$$

$$[[U,V],Z] + [[V,Z],U] + [[Z,U],V] = 0 \quad \text{(Jacobi's Identity)} \tag{2.10}$$

for $U,V,Z \in \Gamma(M)$ and $\beta, \gamma \in (F)(M)$.
A tensor field g of type $(0,2)$ is called a Riemannian metric if it satisfies the following conditions:

- $g_p : T_pM \times T_pM \to \mathbb{R}$ is positive definite bilinear form that is $g(U_p,U_p) \geq 0$, $(g(U_p,U_p) = 0 \Leftrightarrow U_p = 0)$.
- g is symmetric that is $g(U,V) = g(V,U)$ for any $U,V \in \Gamma(M)$,

In this case, (M,g) is called a *Riemannian manifold.*

Example 1. *Let consider the Euclidean space $\mathbb{R}^n$ with inner product*

$$g(U,V) = \sum_{i=1}^{n} u_i v_i, \tag{2.11}$$

where $U = (u_1, u_2, ..., u_n) \in \mathbb{R}^n$ and $V = (v_1, v_2, ..., v_n) \in \mathbb{R}^n$. Then, the inner product g is bilinear, symmetric and positive defined. Therefore, g is a Riemannian metric and the space $(\mathbb{R}^n, g)$ is a Riemannian manifold.

Let M be a real differentiable manifold with a linear connection ∇. A tensor field of M, denoted by R, of type $(1,3)$ is called *Riemannian curvature tensor* which is given by

$$R(U,V)Z = \nabla_U \nabla_V Z - \nabla_V \nabla_U - \nabla_{[U,V]} Z, \tag{2.12}$$

where $U,V,Z \in \Gamma(M)$.

A tensor T of type $(1,2)$ is called *torsion tensor* which is given by

$$T(U,V) = \nabla_U V - \nabla_V U - [U,V]. \tag{2.13}$$

∇ is called *torsion-free* or *symmetric connection* on M, if T vanishes. A linear connection ∇ on (M,g) is called a *compatible connection* if g is parallel with respect to ∇ i.e. for $U,V,Z \in \Gamma(M)$

$$(\nabla_U g)(Z,V) = U(g(Z,V)) - g(\nabla_U Z,V) - g(Z,\nabla_U V) = 0. \tag{2.14}$$

If we consider local coordinates, we obtain

$$g_{ij;k} = \partial_k g_{ij} - g_{ih}\Gamma^h_{jk} - g_{jh}\Gamma^h_{ik} = 0, \tag{2.15}$$

where

$$\Gamma^h_{ij} = \frac{1}{2} g^{hk}\{\partial_j g_{ki} + \partial_i g_{kj} - \partial_k g_{ij}\}, \quad \Gamma^h_{ij} = \Gamma^h_{ji}. \tag{2.16}$$

The coefficients Γ^h_{ji} are called *Christoffel symbols*. The following theorem is the essential conclusion of Riemannian geometry.

Theorem 1. *Let M be a Riemannian manifold. Then, there exists a unique linear connection ∇ on M such that the following conditions are hold:*

- ∇ *is symmetric*
- ∇ *is compatible with the Riemannian metric.*

The connection which is mentioned in the Theorem 1 above is called *Levi-Civita* (metric or Riemannian) connection on M. A metric connection ∇ has the following identity, which is called *Koszul formula*

$$\begin{aligned} 2g(\nabla_U V,Z) &= U(g(V,Z)) + V(g(U,Z)) - Z(g(U,V)) \\ &\quad + g([U,V],Z) + g([Z,U],V) - g([V,Z],U), \end{aligned} \tag{2.17}$$

where $U,V,Z \in \Gamma(M)$. Also, the following equations are called *Bianchi's identities*

$$R(U,V)Z + R(V,Z)U + R(Z,U)V = 0, \tag{2.18}$$

$$(\nabla_U R)(V,Z,W) + (\nabla_V R)(Z,U,W) + (\nabla_Z R)(U,V,W) = 0. \tag{2.19}$$

The Riemannian curvature tensor of type $(0,4)$ is given by

$$R(U,V,Z,W) = g(R(U,V)Z,W), \tag{2.20}$$

where $U,V,Z,W \in \Gamma(M)$. Furthermore, Riemannian curvature tensor has the following properties

$$R(U,V,Z,W) = -R(V,U,Z,W) \tag{2.21}$$

$$R(U,V,Z,W) = -R(U,V,W,Z), \tag{2.22}$$

$$R(U,V,Z,W) = R(V,U,W,Z). \tag{2.23}$$

A *tangent plane* to M is, for any $p \in M$, a 2-dimensional subspace of tangent space T_pM. For every tangent plane ρ in the tangent space, for any $p \in M$, T_pM, it is defined that

$$K(\rho) = K_p(U,V) = \frac{R(U,V,V,U)}{\|U\|^2\|V\|^2 - (g(U,V))^2}, \tag{2.24}$$

where U and V are any tangent vectors for ρ. The smooth function K appoints the each tangent plane ρ to a real number $K(\rho)$ is called a *sectional curvature* of M, which does not depend on choose of tangent vectors U and V. For all plane ρ in T_pM and for all points $p \in M$, if $K(\rho)$ is constant, then M is called a *space of constant curvature*. A Riemannian manifold of constant curvature is called *space form*. In this case, if M has constant curvature c, then curvature tensor field R is given by [21]

$$R(U,V)Z = c\{g(V,Z)U - g(U,Z)V\}. \tag{2.25}$$

Definition 1. *Let (M,g) be an $m-$dimensional Riemannian manifold and $f : (M,g) \mapsto C^\infty(M,\mathbb{R})$ be a function. The gradient of f on M, which is a vector field, defines as*

$$g(\nabla f, U) = df(U) = U(f), \tag{2.26}$$

where $U \in \Gamma(TM)$ and $\nabla f = grad f$.

2.1 Distributions

In this section, the concept of distribution and properties of distribution are given. Let M be an $m-$dimensional manifold. Define a map on T_pM as

$$\begin{aligned} D: M &\rightarrow \bigcup T_pM \\ p &\mapsto D_p \subset T_pM, \ \dim(D_p) = k. \end{aligned} \tag{2.27}$$

The map D is called a $k-$dimensional distribution. For any $U \in \Gamma(M)$, if $U_p \in D_p$, then U is said to belong to distribution D. If, for every single p, the subset D_p of T_pM has k linearly independent differentiable vectors, then D is said to be differentiable. In [22], the following examples are given.

Example 2. *[22] A vector field is a $1-$dimensional distribution on a manifold M.*

Example 3. *[22] Every vector subbundle of a vector bundle, which is defined on a manifold M, defines a distribution.*

Definition 2. *Let M be a C^∞-manifold and D be a $k-$dimensional distribution on M. If, for any $U,V \in \Gamma(D)$, $[V,U] \in \Gamma(D)$, then D is called involutive.*

Definition 3. *Let M be a C^∞-manifold, D be a $k-$dimensional distribution on M and $\bar{M} \subset M$ be a submanifold. If, for every point $p \in \bar{M}$, the tangent space of $\bar{M}$ and D_p are same, then $\bar{M}$ is called the integral manifold of the distribution D. Moreover, if $\bar{M}$ is the unique integral manifold of the distribution D, then $\bar{M}$ is called maximal integral manifold of the distribution D.*

We quote the following example from [22].

Example 4. *[22] Integral curve of a vector field is integral manifold of vector field which is $1-$dimensional distribution.*

Definition 4. *Let M be a C^∞-manifold and $\bar{M} \subset M$ be a submanifold. If, for any $p \in M$, the distribution D has a maximal integral manifold which contains p, then the distribution D is said to be integrable.*

Theorem 2. *(Frobenius Theorem) [23]*
Let D be distribution on M. D is integrable if and only if it is involutive. Furthermore, through every point $p \in M$ there passes a unique maximal integral manifold of D and every other integral manifold containing p is open submanifold of the maximal one.

One can see that, by the Frobenius theorem and the definition of Lie bracket, all the $1-$dimensional distributions are integrable. But, for the higher dimensions it is not valid.

Definition 5. *Let M be a manifold and ∇ be the connection on M. The distribution D is said to be parallel if, for any $U,V \in \Gamma(M)$, $\nabla_U V \in \Gamma(D)$.*

2.2 Locally Product Riemannian Manifolds

An m-dimensional C^∞-differentiable manifold M is called *almost product manifold* with *almost product structure* F which is a tensor field of type (1,1) satisfying

$$F^2 = identity, (F \neq \pm identity), \tag{2.28}$$

denoted by (M,F). If we put

$$P = \frac{1}{2}(I+F), \quad Q = \frac{1}{2}(I-F), \tag{2.29}$$

thus we obtain

$$P+Q = I, \quad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0 \tag{2.30}$$

and

$$F = P - Q. \tag{2.31}$$

Hence P and Q define globally complementary distributions. It is seen that F has the eigenvalues which are $+1$ or -1. An eigenvector, which corresponds to $+1$, lies in the P and an eigenvector, which corresponds to -1, lies in the Q. Hence, if F is of eigenvalue $+1$ with multiplicity a and eigenvalue -1 with multiplicity b, then P is of dimension a and Q is of dimension b.
Conversely, if M has two globally complemantary distributions P and Q with dimensions a and b, respectively, where $a+b=m$ and $a,b \geq 1$. Then, an almost product structure F can be defined on M as $F = P - Q$.

For any vector fields $E, G \in \Gamma(TM)$, if (M, F) has a relation with Riemannian metric g such that

$$g(FE, FG) = g(E, G), \tag{2.32}$$

then M is said to be an *almost product Riemannian manifold* [21].

Let ∇ be the Riemannian connection with respect to the metric g on M. Then M is called a *locally product Riemannian manifold* (briefly, *l.p.R.*) if for any $E \in \Gamma(TM)$ the following

$$\nabla_E F = 0 \tag{2.33}$$

holds [21].

3. COMPLEX MANIFOLDS

In this section, we give some fundamental definitions and basic concepts about complex manifolds [21].

Let M be a smooth manifold. A tensor field J is of type $(1,1)$ and which satisfies

$$J^2 = -identity \tag{3.1}$$

on M is called an *almost complex structure* and M is called an *almost complex manifold*. In this case, the manifold M is orientable and has even dimension. A manifold (M,J) is called an *almost Hermitian* manifold if (M,J) has a relation with Riemannian metric g such that for any $U,V \in \Gamma(M)$

$$g(V,U) = g(JV,JU). \tag{3.2}$$

The Nijenhuis tensor of J, denoted by N, is defined by

$$N(U,V) = [JU,JV] - J[JU,V] - J[U,JV] - [U,V], \tag{3.3}$$

a well-known theorem of Newlander-Nirenberg states that J is the almost complex structure associated to a complex manifold structure on M if and only if the Nijenhuis tensor of J vanishes, i.e., J is integrable.

The *Kaehler form* of an almost Hermitian manifold (M,J,g) is the smooth differential $2-$form defined by, for any $U,V \in \Gamma(TM)$

$$\Omega(U,V) = g(U,JV). \tag{3.4}$$

Let ∇ be the Levi-Civita connection on M with respect to g. It can be extended to the tensor algebra on M. Then, we have the following formulas [3].

$$(\nabla_U J)V = \nabla_U JV - J\nabla_U V, \tag{3.5}$$

$$(\nabla_U \Omega)(V,Z) = g(V,(\nabla_X J)Z), \tag{3.6}$$

$$d\Omega(U,V,Z) = (\nabla_U \Omega)(V,Z) + (\nabla_V \Omega)(Z,U) + (\nabla_Z \Omega)(U,V), \tag{3.7}$$

Let $\{e_1, e_2, ..., e_m, Je_1, Je_2, ..., Je_m\}$ be a J-frame of M. Then, co-differential of Ω is given by

$$(\delta\Omega)(U) = -\sum_{k=1}^{m} \left\{ (\nabla_{e_k}\Omega)(e_k, U) + (\nabla_{Je_k}\Omega)(Je_k, U) \right\}. \quad (3.8)$$

Now, we give the definitions of some classes of almost Hermitian manifolds [24] and [25]. Furthermore, A. Gray and L. M. Harvella [26] give a classification of almost Hermitian manifolds in 16 classes. Let denote the class of almost Hermitian manifolds by $\boldsymbol{AH}$. Then, for any (M, J, g) in $\boldsymbol{AH}$, some of the classes of $\boldsymbol{AH}$ are defined as in the following:

- If $\nabla_U J = 0$, then (M, J, g) is a *Kaehler*($\boldsymbol{K}$),
- if $d\Omega = 0$, then (M, J, g) is an *almost Kaehler*($\boldsymbol{AK}$),
- if $(\nabla_U J)U = 0$, then (M, J, g) is an *almost Tachibana*($\boldsymbol{AT}$),
- if $d\Omega^{(2,1)} = d\Omega^{(1,2)} = 0$, i.e., $(\nabla_U J)V + (\nabla_{JU} J)JV = 0$, then (M, J, g) is a *quasi Kaehler*($\boldsymbol{QK}$),
- if $\delta\Omega = 0$, then (M, J, g) is an *almost semi-Kaehler*($\boldsymbol{ASK}$),
- if $N = 0$, then (M, J, g) is a *Hermitian*($\boldsymbol{H}$),
- if $\delta\Omega = 0$ and $N = 0$, then (M, J, g) is a *semi-Kaehler*($\boldsymbol{SK}$).

Here, one can see that $\boldsymbol{K} = \boldsymbol{AK} \cap \boldsymbol{NK} = \boldsymbol{QK} \cap \boldsymbol{H}$. Between the classes of $\boldsymbol{AH}$ there is an inclusion relation as in the following:

$$\boldsymbol{K} \subset \begin{matrix} \boldsymbol{AK} \\ \boldsymbol{NK} \end{matrix} \subset \boldsymbol{QK} \subset \boldsymbol{ASK} \subset \boldsymbol{AH}. \quad (3.9)$$

Additionally, if it is assumed that $N = 0$, then

$$\boldsymbol{K} \subset \boldsymbol{SK} \subset \boldsymbol{H}. \quad (3.10)$$

4. RIEMANNIAN SUBMERSIONS

One of the most rising areas of the differential geometry is the theory of submanifold. Since, a submersion is a way to obtain a submanifold, many geometers have been concerning about the theory of submersion. The notion of Riemannian submersion was first defined by O'Neill [1] and Gray [2]. Later, Watson [3] studied Riemannian submersions between almost Hermitian manifolds.

Let (M,g) and (N,g_N) be Riemannian manifolds, where $dim(M) > dim(N)$.
A surjective mapping $\pi : (M,g) \to (N,g_N)$ is called a *Riemannian submersion* [1] if

(S1) π has maximal rank, and

(S2) π_*, restricted to $ker\pi_*^{\perp}$, is a linear isometry.

In this case, for each $q \in N$, $\pi^{-1}(q)$ is a k-dimensional submanifold of M and called a *fiber*, where $k = dim(M) - dim(N)$. A vector field on M is called *vertical* (resp. *horizontal*) if it is always tangent (resp. orthogonal) to fibers. A vector field X on M is called *basic* if X is horizontal and π-related to a vector field X_* on N, i.e., $\pi_* X_p = X_{*\pi(p)}$ for all $p \in M$. We will denote by $\mathscr{V}$ and $\mathscr{H}$ the projections on the vertical distribution $ker\pi_*$, and the horizontal distribution $ker\pi_*^{\perp}$, respectively. As usual, the manifold (M,g) is called *total manifold* and the manifold (N,g_N) is called *base manifold* of the submersion $\pi : (M,g) \to (N,g_N)$.

Lemma 1. *[1] Let (M,g) and (N,g_N) be Riemannian manifolds with a Riemannian submersion π between them. For any basic vector fields $\alpha, \beta \in \Gamma(TM)$, we obtain the following*

1. $g(\alpha,\beta) = g_N(\alpha_*,\beta_*) \circ \pi$,

2. $\pi_*(\mathscr{H}[\alpha,\beta]) = [\alpha_*,\beta_*]$,

3. $\pi_*(\mathscr{H}\nabla_\alpha \beta) = \nabla^*_{\alpha_*}(\beta_*)$,

where ∇^* is the Riemannian connection on N, $\pi_*(\alpha) = \alpha_*$ and $\pi_*(\beta) = \beta_*$.

To characterize the geometry of the fibers, O'Neill tensors T and A are defined as follows:

$$T_{\bar{U}}\bar{V} = \mathscr{V}\nabla_{\mathscr{V}\bar{U}}\mathscr{H}\bar{V} + \mathscr{H}\nabla_{\mathscr{V}\bar{U}}\mathscr{V}\bar{V}, \tag{4.1}$$

$$A_{\bar{U}}\bar{V} = \mathscr{V}\nabla_{\mathscr{H}\bar{U}}\mathscr{H}\bar{V} + \mathscr{H}\nabla_{\mathscr{H}\bar{U}}\mathscr{V}\bar{V} \tag{4.2}$$

for any vector fields $\bar{U}$ and $\bar{V}$ on M, where ∇ is the Levi-Civita connection of g. $T_{\bar{U}}$ and $A_{\bar{U}}$ are skew-symmetric operators on the tangent bundle of M reversing the vertical and the horizontal distributions [1]. We now summarize the properties of the tensor fields T and A. Let V,W be vertical and X,Y be horizontal vector fields on M, then we have

$$T_V W = T_W V, \tag{4.3}$$

$$A_X Y = -A_Y X = \frac{1}{2}\mathscr{V}[X,Y]. \tag{4.4}$$

Moreover, from (4.1) and (4.2), we obtain

$$\nabla_V W = T_V W + \hat{\nabla}_V W, \tag{4.5}$$

$$\nabla_V X = T_V X + \mathscr{H}\nabla_V X, \tag{4.6}$$

$$\nabla_X V = A_X V + \mathscr{V}\nabla_X V, \tag{4.7}$$

$$\nabla_X Y = \mathscr{H}\nabla_X Y + A_X Y, \tag{4.8}$$

where $\hat{\nabla}_V W = \mathscr{V}\nabla_V W$. If X is basic

$$\mathscr{H}\nabla_V X = A_X V.$$

Remark 1. *In this thesis, all the horizontal vector fields are considered as basic vector fields.*

We observe that T acts on the fibers as the second fundamental form while A acts on the horizontal distribution and measures of the obstruction to the integrability of this distribution.

Lemma 2. *[1] Let (M,g) and (N,g_N) be two Riemannian manifolds and $\pi:(M,g)\mapsto(N,g_N)$ be a Riemannian submersion. Then, for any U,V vertical and X,Y horizontal vector fields on M, the followings are obtained:*

$$(\nabla_U A)_V = -A_{T_U V}, \tag{4.9}$$

$$(\nabla_U T)_X = -T_{T_U X}, \tag{4.10}$$

$$(\nabla_X A)_U = -A_{A_X U}, \tag{4.11}$$

$$(\nabla_X T)_Y = -T_{A_X Y}. \tag{4.12}$$

Note that, if T (respectively A) vanishes, then it is said to be T (respectively A) is parallel.

For a submersion, the curvature relations are given as in the following:

Theorem 3. *[1] Let (M,g) and (N,g_N) be two Riemannian manifolds and $\pi : (M,g) \mapsto (N,g_N)$ be a Riemannian submersion. Then, for any U,V,W,Z vertical and X,Y,ξ,η horizontal vector fields*

$$\begin{aligned}
R(U,V,W,Z) &= \hat{R}(U,V,W,Z) - g(T_U W, T_V Z) + g(T_V W, T_U Z), && (4.13)\\
R(U,V,W,X) &= g((\nabla_U T)(V,W),X) - g((\nabla_V T)(U,W),X), && (4.14)\\
R(X,Y,\xi,\eta) &= R^*(X,Y,\xi,\eta) + 2g(A_\xi \eta, A_X Y) \\
&\quad + g(A_Y \eta, A_X \xi) - g(A_X \eta, A_Y \xi), && (4.15)\\
R(X,Y,\xi,U) &= -g((\nabla_\xi A)(X,Y),U) - g(T_U \xi, A_X Y) \\
&\quad - g(A_X \xi, T_U Y) + g(A_Y \xi, T_U X), && (4.16)\\
R(X,Y,U,V) &= -g((\nabla_U A)(X,Y),V) + g((\nabla_V A)(X,Y),U) \\
&\quad - g(A_X U, A_Y V) + g(A_X V, A_Y U) \\
&\quad + g(T_U X, T_V Y) - g(T_V X, T_U Y), && (4.17)\\
R(X,U,Y,V) &= -g((\nabla_X T)(U,V),Y) - g((\nabla_U A)(X,Y),V) \\
&\quad g(T_U X, T_V Y) - g(A_X U, A_Y V), && (4.18)
\end{aligned}$$

where R^*, $\hat{R}$ and R are Riemannian curvature tensor for base manifold, fibers and total manifold, respectively.

For a Riemannian submersion, the sectional curvature formulas are obtained by the above theorem, as in the following [1]: If it is assumed that K^*, $\hat{K}$ and K are sectional curvatures for base manifold, fibers and total manifold, respectively, then for any horizontal vector field X and vertical vector U field

$$\begin{aligned}
K(U,V) &= \hat{K}(U,V) + g(T_U V, T_U V) - g(T_U U, T_V V), && (4.19)\\
K(X,Y) &= K^*(X,Y) - 3g(A_X Y, A_X Y), && (4.20)\\
K(X,U) &= g((\nabla_X T)(U,U),X) + g(T_U X, T_U X) - g(A_X U, A_X U). && (4.21)
\end{aligned}$$

The fibers of a Riemannian submersion $\pi : (M,g) \to (N,g_N)$ is called *totally umbilical* if

$$T_V U = g(V,U)H, \quad (4.22)$$

for any $U,V \in \Gamma(ker\pi_*)$, where H is the mean curvature vector field of the fiber. Moreover if $H=0$, the fibers are called minimal.

The distribution D_1 is called *parallel along the distribution* D_2 if and only if for $V \in \Gamma(D_1), U \in \Gamma(D_2)$, $\nabla_U V \in D_1$.

Let π be a Riemannian submersion from a Riemannian manifold (M,g) onto a Riemannian manifold (N,g_N) . Then, we say that the fibers of π are *mixed geodesic*, if $T_X W = 0$, for all $X \in \Gamma(\mathscr{V})$, $W \in \Gamma(\mathscr{H})$, [7].

4.1 Anti-Invariant Submersions

The notion of the anti-invariant submersion from almost Hermitian manifolds onto Riemannian manifolds was defined first by Şahin [6]. We study anti-invariant and Lagrangian submersions from locally product Riemannian manifolds onto Riemannian manifolds. We first give a characterization theorem for Riemannian submersions. It is proved that the fibers of a Lagrangian submersion are always totally geodesic. We also consider the first variational formula of the anti-invariant Riemannian submersions and give a new condition for the harmonicity of such submersions.

In general, $g(F\bar{V},\bar{V}) \neq 0$ for any unit vector $\bar{V} \in \Gamma(T_pM)$ in a l.p.R. manifold M, contrary to almost Hermitian $(g(J\bar{V},\bar{V})=0)$ and almost contact $(g(\varphi\bar{V},\bar{V})=0)$ manifolds. However, we can establish that the almost product structure F in a l.p.R. manifold M such that $g(F\bar{V},\bar{V})=0$, for all $\bar{V} \in \Gamma(T_pM)$. In fact, if M is an even dimensional l.p.R. manifold with an orthonormal basis $\{e_1,...,e_n,e_{n+1},...,e_{2n}\}$, then we can define F by

$$F(e_i)=e_{n+i}, \quad F(e_{n+i})=e_i, \quad i \in \{1,2,\cdots,n\}. \tag{4.23}$$

Hence, we observe that the almost product structure F satisfies

$$g(Fe_j,e_j)=0, \quad j=\{1,2,\cdots,n,\cdots,2n\}. \tag{4.24}$$

Let $M_1(c_1)$ (resp. $M_2(c_2)$) be a real space form with sectional curvature c_1 (resp. c_2). Then, Riemannian curvature tensor R of locally product Riemannian manifold

$M = M_1(c_1) \times M_2(c_2)$ has the form

$$
\begin{aligned}
&R(\bar{U},\bar{V})\bar{W} \qquad (4.25)\\
&= \frac{1}{4}(c_1+c_2)\Big\{g(\bar{V},\bar{W})\bar{U}-g(\bar{U},\bar{W})\bar{V}+g(F\bar{V},\bar{W})F\bar{U}-g(F\bar{U},\bar{W})F\bar{V}\Big\}\\
&+\frac{1}{4}(c_1-c_2)\Big\{g(F\bar{V},\bar{W})\bar{U}-g(F\bar{U},\bar{W})\bar{V}+g(\bar{V},\bar{W})F\bar{U}-g(\bar{U},\bar{W})F\bar{V}\Big\},
\end{aligned}
$$

where $\bar{U},\bar{V},\bar{W} \in \Gamma(TM)$ [21]. In case of $c_1 = c_2 = c$, the Riemannian curvature tensor R of locally product Riemannian manifold $M(c) = M_1(c) \times M_2(c)$ becomes

$$R(\bar{U},\bar{V})\bar{W} = \frac{c}{2}\Big\{g(\bar{V},\bar{W})\bar{U}-g(\bar{U},\bar{W})\bar{V}+g(F\bar{V},\bar{W})F\bar{U}-g(F\bar{U},\bar{W})F\bar{V}\Big\}, \qquad (4.26)$$

where $\bar{U}, \bar{V}, \bar{W} \in \Gamma(TM(c))$.

Proposition 1. *Let $\pi : (M(c),g,F) \to (N,g_N)$ be a Riemannian submersion from a l.p.R manifold with $c \neq 0$ onto a Riemannian manifold. If the almost product structure F of $M(c)$ satisfies (4.24), then the fibers of π are invariant or anti-invariant with respect to F if and only if*

$$g((\nabla_U T)(V,W),X) = g((\nabla_V T)(U,W),X), \qquad (4.27)$$

where $U,V,W \in \Gamma(\ker\pi_*)$ and $X \in \Gamma(\ker\pi_*^\perp)$.

Proof. From (4.26), we have

$$R(U,V)W = \frac{c}{2}\Big\{g(V,W)U-g(U,W)V+g(FV,W)FU-g(FU,W)FV\Big\}, \qquad (4.28)$$

where $U,V,W \in \Gamma(\ker\pi_*)$. If the fibres of π are invariant or anti-invariant with respect to F, then it is not difficult to see that $R(U,V)W$ is vertical from (4.28). Hence, for any $X \in \Gamma(\ker\pi_*^\perp)$, we easily get

$$R(U,V,W,X) = g(R(U,V)W,X) = 0. \qquad (4.29)$$

Thus, (4.27) follows from (4.29) and the O'Neill curvature formula $\{1\}$ in [1]:

$$R(U,V,W,X) = g((\nabla_V T)(U,W),X) - g((\nabla_U T)(V,W),X). \qquad (4.30)$$

Conversely, if the equation (4.27) holds, then using the above O'Neill formula we see that $R(U,V)W$ is vertical. By putting $W = U$ in (4.28), we obtain

$$R(U,V)U = \frac{c}{2}\Big\{g(V,U)U-\|U\|^2V+g(FV,U)FU\Big\}. \qquad (4.31)$$

Since $R(U,V)U$ is vertical, $g(FV,U)FU$ is also vertical from (4.31). Thus, we conclude that either FU is vertical or $g(FV,U)=0$. It follows that either $F(\ker\pi_*)\subseteq\ker\pi_*$ or $F(\ker\pi_*)\subseteq(\ker\pi_*)^\perp$, i.e., either the fibers of π are invariant or anti-invariant with respect to F. $\square$

Let M be a locally product Riemanian manifold with Riemannian metric g and almost product structure F, and N be a Riemanian manifold with Riemannian metric g_N. Suppose that there exists a Riemannian submersion $\pi : M \to N$ such that the vertical distribution $\ker\pi_*$ is anti-invariant with respect to F, i.e., $F(\ker\pi_*)\subseteq\ker\pi_*^\perp$. Then, the Riemannian submersion π is called an *anti-invariant Riemannian submersion* [6].

In this case, we observe that $F(\ker\pi_*^\perp)\cap\ker\pi_*\neq\{0\}$. If we denote the complementary orthogonal distribution of $F(\ker\pi_*)$ in $\ker\pi_*^\perp$ by μ, then we write

$$\ker\pi_*^\perp = F\ker\pi_*\oplus\mu\,. \tag{4.32}$$

Let $FX\in\Gamma(F\ker\pi_*)$ and $Y\in\Gamma(\mu)$. Then, by (2.32), we see that

$$g(FX,FY)=g(X,Y)=0. \tag{4.33}$$

Therefore, μ is invariant distribution of $\ker\pi_*^\perp$ with respect to the almost product structure F. Thus, for any $X\in\Gamma(\ker\pi_*^\perp)$, we have

$$FX = BX + CX\,, \tag{4.34}$$

where $BX\in\Gamma(\ker\pi_*)$ and $CX\in\Gamma(\ker\pi_*^\perp)$.

Now, let π be an anti-invariant Riemannian submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). If $\dim(\ker\pi_*)=\dim(\ker\pi_*^\perp)$, then we call π a *Lagrangian submersion.* In that case, the almost product structure F of M reverses the vertical and horizontal distributions, i.e., $F(\ker\pi_*)=\ker\pi_*^\perp$, and $F(\ker\pi_*^\perp)=\ker\pi_*$. This case has been studied; see [6, 8, 27] for more details and examples.

We now examine how the almost product structure on M effects the tensor fields T and A of an anti-invariant submersion π from a l.p.R. (M,g,F) onto a Riemannian manifold (N,g_N).

Lemma 3. *Let π be an anti-invariant Riemannian submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then we have*

$$FT_VW = T_V FW\,, \tag{4.35}$$

$$F\hat{\nabla}_VW = \mathscr{H}\nabla_VFW\,, \tag{4.36}$$

$$B\mathscr{H}\nabla_VX = \hat{\nabla}_VBX + T_V CX\,, \tag{4.37}$$

$$FT_VX + C\mathscr{H}\nabla_VX = T_VBX + \mathscr{H}\nabla_VCX\,, \tag{4.38}$$

$$BA_XV = A_XFV, \tag{4.39}$$

$$CA_XV + F(\mathscr{V}\nabla_XV) = \mathscr{H}\nabla_XFV\,, \tag{4.40}$$

$$B\mathscr{H}\nabla_XY = \mathscr{V}\nabla_XBY + A_XCY\,, \tag{4.41}$$

$$FA_XY + C\mathscr{H}_XY = A_XBY + \mathscr{H}\nabla_XCY\,, \tag{4.42}$$

where $V,\,W \in \Gamma(\ker\pi_*)$, and $X,\,Y \in \Gamma(\ker\pi_*^{\perp})$.

Proof. Using (2.33) and (4.6) we have

$$F\nabla_VW = T_VFW + \mathscr{H}\nabla_VFW, \tag{4.43}$$

where $V,\,W \in \Gamma(\ker\pi_*)$. By using (4.5), we get

$$F\hat{\nabla}_VW + FT_VW = T_VFW + \mathscr{H}\nabla_VFW, \tag{4.44}$$

from (4.43). Taking the vertical and horizontal parts of this equation we obtain (4.35) and (4.36). The other assertions can be obtained in a similar way. □

Corollary 1. *Let π be a Lagrangian submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then*

$$T_VFE = FT_VE\,, \tag{4.45}$$

$$A_XFE = FA_XE\,, \tag{4.46}$$

where $V \in \Gamma(\ker\pi_*)$, $X \in \Gamma(\ker\pi_*^{\perp})$ and $E \in \Gamma(TM)$.

Proof. The first assertion follows from (4.35) and (4.38). The other follows from (4.39) and (4.42). □

Corollary 2. *Let π be a Lagrangian submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold $(N,g_{_N})$. Then,*

$$T_U FV = T_V FU\,, \tag{4.47}$$

$$A_X FY = -A_Y FX\,, \tag{4.48}$$

where U, $V \in \Gamma(\ker\pi_*)$ and X ,$Y \in \Gamma(\ker\pi_*^{\perp})$.

Proof. By using (4.3), the first assertion can be obtained from (4.45) and using (4.4), the other assertion follows from (4.46). □

Theorem 4. *Let π be a Lagrangian submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold $(N,g_{_N})$. Then the fibers of π are totally geodesic. In other words, $T_U = 0$, for any $U \in \ker\pi_*$.*

Proof. Let U, V and W be any vertical fields. Then, using (4.5), (4.6), (2.28)$\sim$(2.33) and (4.47), we have,

$$\begin{aligned} g(T_U FV,W) &= g(\nabla_U FV,W) = g(F\nabla_U V,W) = g(\nabla_U V,FW) \\ &= g(T_U V,FW) = -g(T_U FW,V) = -g(T_W FU,V) \\ &= g(T_W V,FU) = g(T_V W,FU) = -g(T_V FU,W) \\ &= -g(T_U FV,W)\,. \end{aligned} \tag{4.49}$$

Hence, it follows that $T_U FV = 0$. Since FV is an arbitrary horizontal vector field, we get $T_U(\ker\pi_*^{\perp}) = 0$. The property of skew-symmetry of T gives $T_U(\ker\pi_*) = 0$. Thus, we find $T_U = 0$. □

4.1.1 The first variational formula of anti-invariant submersions

In this section, we define the first variational formula for anti-invariant submersions from a l.p.R. manifold (M,g,F) onto a Riemannian manifold $(N,g_{_N})$ and by means of that form, we focus on a new approach to investigate whether an anti-invariant submersion is harmonic.

Let π be an anti-invariant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold $(N,g_{_N})$. Then, we can define the 1-form dual to the vector field $F\xi$, for $\xi \in \Gamma(\ker\pi_*^{\perp})$, such that

$$\sigma_\xi : \Gamma(\ker\pi_*) \mapsto \mathscr{F}(\pi_q^{-1}), q \in N$$

$$V \mapsto \sigma_\xi(V) = g(F\xi, V)$$

for all $V \in \Gamma(\ker \pi_*)$. In the view of [28] and [29], we define the followings:

The *Legendre variations* of any fiber of π, denoted by the set $\mathbb{L}$, where

$$\mathbb{L} = \{\xi \in \Gamma(\ker \pi_*^\perp) : d\sigma_\xi = 0, \textit{ i.e. }, \sigma_\xi \textit{ is closed}\},$$

the *Hamiltonian variations* of any fiber of π, denoted by the set $\mathbb{E}$,

$$\mathbb{E} = \{\xi \in \Gamma(\ker \pi_*^\perp) : \exists f \in \mathscr{F}(\pi_q^{-1}) \Rightarrow \sigma_\xi = df, \textit{ i.e. }, \sigma_\xi \textit{ is exact}\}$$

and the *harmonic variations* of any fiber of π are given by the set

$$\mathbb{H} = \{\xi \in \Gamma(\ker \pi_*^\perp) : \Delta\sigma_\xi = 0, \textit{ i.e. }, \sigma_\xi \textit{ is harmonic}\}.$$

By the definitions of differential and co-differential operators, we observe that

$$\mathbb{E} \subset \mathbb{L}\,, \ \mathbb{H} \subset \mathbb{L} \ \textit{and} \ \mathbb{E} \cap \mathbb{H} = \{0\}. \tag{4.50}$$

Now, we examine that under what conditions the 1-form σ_ξ, defined above, is a Legendre variation.

Lemma 4. *Let π be an anti-invariant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). The 1-form σ_ξ is a Legendre variation if and only if*

$$g(A_\xi U, FV) = -g(A_\xi V, FU), \tag{4.51}$$

where $U, V \in \Gamma(\ker \pi_*)$.

Proof. Let U, V be in $\Gamma(\ker \pi_*)$. Then, by the definition of differential, (4.6) and (2.32), we obtain

$$\begin{aligned}
(d\sigma_\xi)(U,V) =& Ug(F\xi, V) - Vg(F\xi, U) - g(F\xi, [U,V]) \\
=& Ug(\xi, FV) - Vg(\xi, FU) - g(\xi, F[U,V]) \\
=& g(\nabla_U \xi, FV) + g(\xi, \nabla_U FV) \\
& - g(\nabla_V \xi, FU) - g(\xi, \nabla_V FU) \\
& - g(\xi, F\nabla_U V) + g(\xi, F\nabla_V U) \\
=& g(\nabla_U \xi, FV) - g(\nabla_V \xi, FU) \\
=& g(\mathscr{H}\nabla_U \xi, FV) + g(\mathscr{H}\nabla_V \xi, FU).
\end{aligned} \tag{4.52}$$

Since, we assume ξ is basic, we get

$$(d\sigma_\xi)(U,V) = g(A_\xi U, FV) + g(A_\xi V, FU). \tag{4.53}$$

Thus, the assertion follows. □

Lemma 5. *For $\xi \in \Gamma(\mu)$, $\sigma_\xi \equiv 0$.*

Proof. Let $\xi \in \Gamma(\mu)$. Then $F\xi \in \Gamma(\mu)$. For any $V \in \Gamma(\ker\pi_*)$, we get

$$\sigma_\xi(V) = g(F\xi, V) = 0. \tag{4.54}$$

So, $\sigma_\xi \equiv 0$, for all $V \in \Gamma(\ker\pi_*)$. □

Remark 2. *Because of Lemma 5, throughout this subsection, we can assume that H belongs to $F(ker\pi_*)$. Here, H is the mean curvature vector field of the fibers.*

Proposition 2. *Let π be an anti-invariant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N) and f be a smooth function on a fiber. Then, $F(grad(f)) \in \mathbb{E}$.*

Proof. Let f be a smooth function on a fiber. For $\xi = F(grad(f))$, and any $V \in \Gamma(\ker\pi_*)$, we obtain

$$\sigma_\xi(V) = g(F\xi, V) = g(grad(f), V) = V[f] = df(V). \tag{4.55}$$

Thus, we get $\sigma_\xi = df$, i.e., $\xi \in \mathbb{E}$. □

Let π be an anti-invariant submersion with compact fibers from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N) and $\xi \in \Gamma(\ker\pi_*^\perp)$. The first variational formula of a fiber π_q^{-1}, for $q \in N$, is defined as follows [30]

$$\mathbf{V}'(\xi) = -k\int_{\pi_q^{-1}} g(\xi, H)*1, \tag{4.56}$$

where $k = \dim(\pi_q^{-1})$. We introduce the following terminology;

- If $\mathbf{V}'(\xi) = 0$, for all $\xi \in \mathbb{L}$, then π_q^{-1} is $\mathbb{L}-minimal$,
- If $\mathbf{V}'(\xi) = 0$, for all $\xi \in \mathbb{E}$, then π_q^{-1} is $\mathbb{E}-minimal$,
- If $\mathbf{V}'(\xi) = 0$, for all $\xi \in \mathbb{H}$, then π_q^{-1} is $\mathbb{H}-minimal$.

Remark 3. *One can easily see that if the fiber is minimal, then the fiber is* $\mathbb{L}, \mathbb{E}$ *and* $\mathbb{H}-$*minimal. On the other hand, because of the facts that* $\mathbb{E} \subset \mathbb{L}$ *and* $\mathbb{H} \subset \mathbb{L}$*, the fiber is* $\mathbb{E}-$*minimal and* $\mathbb{H}-$*minimal if it is* $\mathbb{L}-$*minimal.*

Theorem 5. *Let* π *be an anti-invariant submersion with compact fibers from a l.p.R. manifold* (M, g, F) *onto a Riemannian manifold* (N, g_N)*. Then,*

(a) *The fiber* π_q^{-1} *is* $\mathbb{L}-$*minimal if and only if* σ_H *is co-exact.*
(b) *The fiber* π_q^{-1} *is* $\mathbb{E}-$*minimal if and only if* σ_H *is co-closed.*
(c) *The fiber* π_q^{-1} *is* $\mathbb{H}-$*minimal if and only if* σ_H *is the sum of an exact and a co-exact 1-form.*

Proof. **(a)** $\Rightarrow$: Let the fiber π_q^{-1} is $\mathbb{L}-minimal$, then for any $\xi \in \mathbb{L}$, we have $g(H,\xi) = 0$ from (4.56). By the definition of the Hodge star operator [31], we have

$$\sigma_\xi \wedge \sigma_H(V_1, V_2, ..., V_k) = g(\xi, H) * 1(V_1, V_2, ..., V_k), \tag{4.57}$$

for $V_1, V_2, ..., V_k \in \Gamma(\ker \pi_*)$. From the definition of the global scalar product $(.|.)$ ([31]) on the module of all forms on the fiber, we get

$$(\sigma_\xi|\sigma_H) = \int_{\pi_q^{-1}} \sigma_\xi \wedge *\sigma_H = 0. \tag{4.58}$$

Denote by δ the codifferential operator [31] on the fiber π_q^{-1}. Since σ_ξ is closed, for any 2-form β on π_q^{-1}, we have

$$0 = (d\sigma_\xi|\beta) = (\sigma_\xi|\delta\beta). \tag{4.59}$$

Since π_q^{-1} is compact, by (4.58) and (4.59), we conclude that σ_H is co-exact.
$\Leftarrow$: Suppose that σ_H is co-exact, we have $\sigma_H = \delta\psi$ for some 2-form ψ. Then, for any $\xi \in \mathbb{L}$,

$$(\sigma_\xi|\sigma_H) = (\sigma_\xi|\delta\psi) = (d\sigma_\xi|\psi) = 0 \tag{4.60}$$

and then

$$\mathbf{V}'(\xi) = -k\int_{\pi_q^{-1}} g(H,\xi)*1 = -k\int_{\pi^{-1}(q)} (\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H) = 0, \tag{4.61}$$

i.e. π_q^{-1} is $\mathbb{L}-minimal$.
(b) $\Rightarrow$: Let the fiber π_q^{-1} be $\mathbb{E}-minimal$. Then, we have

$$0 = \mathbf{V}'(\xi) = -k\int_{\pi_q^{-1}} g(\xi,H)*1 = -k\int_{\pi_q^{-1}} (\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H), \tag{4.62}$$

that is, $(\sigma_\xi|\sigma_H) = 0$. Since $\xi \in \mathbb{E}$, $\sigma_\xi = df$ for some function f on the fiber π_q^{-1}. Thus,

$$(df|\sigma_H) = (f|\delta\sigma_H) = 0. \tag{4.63}$$

Hence it follows that $\delta\sigma_H = 0$, i.e. σ_H is co-closed.

$\Leftarrow$: Suppose that σ_H is co-closed. Let $\xi \in \mathbb{E}$, then there exists a function $f \in \mathscr{F}(\pi_q^{-1})$ such that $\sigma_\xi = df$. Hence, we have

$$(\sigma_\xi|\sigma_H) = (df|\sigma_H) = (f|\delta\sigma_H) = 0. \tag{4.64}$$

Therefore,

$$\mathbf{V}^{'}(\xi) = -k\int_{\pi_q^{-1}} g(H,\xi)*1 = -k\int_{\pi_q^{-1}}(\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H) = 0, \tag{4.65}$$

that is $\mathbf{V}^{'}(\xi) = 0$ for $\xi \in \mathbb{E}$, i.e. π_q^{-1} is $\mathbb{E}-minimal$.

(c) $\Rightarrow$: If the fiber π_q^{-1} is $\mathbb{H}-minimal$, then for $\xi \in \mathbb{H}$, we have

$$0 = \mathbf{V}^{'}(\xi) = -k\int_{\pi_q^{-1}} g(\xi,H)*1 = -k\int_{\pi_q^{-1}}(\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H). \tag{4.66}$$

It means that, σ_H is orthogonal to harmonic 1-forms on the fiber π_q^{-1}. Thus, by the Hodge decomposition theorem [31], we conclude that σ_H is the sum of an exact and a co-exact 1-form.

$\Leftarrow$: Let σ_H be the sum of an exact 1-form $\omega_1 = df$ and a co-exact 1-form $\omega_2 = \delta\psi$. For $\xi \in \mathbb{H}$, we have

$$\begin{aligned}(\sigma_\xi|\sigma_H) =&(\sigma_\xi|df + \delta\psi) = (\sigma_\xi|df) + (\sigma_\xi|\delta\psi)\\ =&(\delta\sigma_\xi|f) + (d\sigma_\xi|\psi) = 0,\end{aligned} \tag{4.67}$$

since $d\sigma_\xi = \delta\sigma_\xi = 0$. Thus,

$$\mathbf{V}^{'}(\xi) = -k\int_{\pi_q^{-1}} g(\xi,H)*1 = -k\int_{\pi_q^{-1}}(\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H), \tag{4.68}$$

that is, the fiber is $\mathbb{H}-minimal$. □

Now, if we give a restriction in Theorem 5, we get the following theorem.

Theorem 6. *Let π be an anti-invariant submersion with compact fibers from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). If $H \in \mathbb{L}$, then*

(a) *π_q^{-1} is $\mathbb{L}-minimal$ if and only if π_q^{-1} is minimal.*

(b) *π_q^{-1} is $\mathbb{E}-minimal$ if and only if σ_H is a harmonic variation.*

(c) *π_q^{-1} is $\mathbb{H}-minimal$ if and only if σ_H is an exact 1-form.*

Proof. **(a)** If the fiber π_q^{-1} is $\mathbb{L}-minimal$, then by Theorem 5-(a) we have, σ_H is co-exact. Hence σ_H is co-closed. Taking into account the fact that $d\sigma_H = 0$, we deduce that σ_H is harmonic. But this is a contradiction because of the Hodge decomposition theorem [31]. So, σ_H must be zero. Hence we conclude that $H = 0$. The converse is clear.

(b) $\Rightarrow$: If the fiber π_q^{-1} is $\mathbb{E}-minimal$, then we have $\delta\sigma_H = 0$ from Theorem 5-(b). Since $d\sigma_H = 0$, σ_H is also harmonic, i.e. $\Delta\sigma_H = 0$.

$\Leftarrow$: If σ_H is harmonic, then σ_H is co-closed. By Theorem 5-(b), the fiber π_q^{-1} is $\mathbb{E}-minimal$.

(c) $\Rightarrow$: Assume that π_q^{-1} is $\mathbb{H}-minimal$. Then, from Theorem 5-(c), σ_H is the sum of an exact 1-form and a co-exact 1-form. On the other hand, the condition $H \in \mathbb{L}$ implies that σ_H is orthogonal to every co-exact 1-form on π_q^{-1}. Thus,, σ_H must be exact.

$\Leftarrow$: Let σ_H be an exact 1-form. For $\xi \in \mathbb{H}$, we obtain

$$\begin{aligned} \mathbf{V}'(\xi) &= -k\int_{\pi_q^{-1}} g(\xi, H)*1 = -k\int_{\pi_q^{-1}} (\sigma_\xi \wedge *\sigma_H) \\ &= -k(\sigma_\xi|\sigma_H) = (\sigma_\xi|df) = (\delta\sigma_\xi|f) = 0, \end{aligned} \tag{4.69}$$

that is, π_q^{-1} is $\mathbb{H}-minimal$. □

Remark 4. *The method that considering the basis to investigate the harmonicity of a submersion, while the total manifold is taken as a l.p.R. manifold, is not always easy. Since a l.p.R. manifold is not always even dimensional, choosing a basis and using it is not easy. On the other hand, it is well known that, the fibers of a submersion is minimal if and only if the submersion is harmonic. Now, we give the following corollary which is a new approach to investigate the harmonicity of a submersion. By Theorem 6-(a), we get the next result.*

Corollary 3. *Let π be an anti-invariant submersion with compact fibers from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). If $H \in \mathbb{L}$, then, π is harmonic if and only if π_q^{-1} is $\mathbb{L}-minimal$.*

Lemma 6. *Let π be an anti-invariant submersion with compact fibers from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then,*

$$\delta\sigma_H = 0 \Leftrightarrow \Sigma_i g(A_H E_i, F E_i) = 0, \tag{4.70}$$

where $\{E_1, E_2, ..., E_m\}$ is a local basis of $\ker\pi_*$.

Proof. For any $\{E_1, E_2, ..., E_m\}$, we have

$$\delta\sigma_H = 0 \Leftrightarrow \Sigma_i g(\nabla_{E_i} FH, E_i) = 0. \tag{4.71}$$

Using (4.8), we get

$$\Rightarrow \ \delta\sigma_H = 0 \Leftrightarrow \ \Sigma_i g(\nabla_{E_i} H, FE_i) = \Sigma_i g(A_H E_i, FE_i) = 0. \tag{4.72}$$

Thus, the assertion follows from the skew-symmetry and symmetry properties of the O'Neill tensor A. □

4.2 Semi-Invariant Submersions

The notion of the semi-invariant submersion from almost Hermitian manifold onto Riemannian manifolds was first defined by Şahin [7]. We study semi-invariant submersions from locally product Riemannian manifolds onto Riemannian manifolds. We also give a characterization theorem for the proper semi-invariant submersions with totally umbilical fibers and find some results for such submersions with parallel canonical structures. Moreover, we define first variational formula on the fibers of a semi-invariant submersion and by the virtue of that, we prove a new theorem which has a condition for the harmonicity of a semi-invariant submersion.
The definition of a semi-invariant submersion from a locally product Riemannian manifold onto a Riemannian manifold as in the following:

Definition 6. *Let (M, g, F) be a l.p.R. manifold and (N, g_N) be a Riemannian manifold. A Riemannian submersion $\pi : (M, g, F) \to (N, g_N)$ is called* semi-invariant submersion, *if there is a distribution $D \subset ker\pi_*$ such that*

$$ker\pi_* = D \oplus D^\perp, \quad FD = D, \quad FD^\perp \subset ker\pi_*^\perp, \tag{4.73}$$

where $D^\perp$ is the orthogonal complement of D in $ker\pi_*$. In this case, the horizontal distribution $ker\pi_*^\perp$ can be decomposed as

$$ker\pi_*^\perp = FD^\perp \oplus \mu, \tag{4.74}$$

where μ is the orthogonal complementary distribution of $FD^\perp$ in $ker\pi_*^\perp$, and it is invariant with respect to F. A semi-invariant submersion is called proper *if both $D \neq \{0\}$ and $D^\perp \neq \{0\}$.*

We give the following example.

Example. Consider the Euclidean 6-space $\mathbb{R}^6$ with standart metric g. Define the almost product structure F on $(\mathbb{R}^6, g)$ by

$$F(\frac{\partial}{\partial x_1}) = \frac{\partial}{\partial x_2}, \quad F(\frac{\partial}{\partial x_2}) = \frac{\partial}{\partial x_1}, \quad F(\frac{\partial}{\partial x_3}) = \frac{\partial}{\partial x_4}, \tag{4.75}$$

$$F(\frac{\partial}{\partial x_4}) = \frac{\partial}{\partial x_3}, \quad F(\frac{\partial}{\partial x_5}) = \frac{\partial}{\partial x_5}, \quad F(\frac{\partial}{\partial x_6}) = -\frac{\partial}{\partial x_6}, \tag{4.76}$$

where $(x_1, x_2, \cdots, x_6)$ are natural coordinates of $\mathbb{R}^6$.

Now, we define a map $\pi : \mathbb{R}^6 \to \mathbb{R}^3$ by

$$\pi(x_1, \cdots, x_6) = (\frac{x_1 - x_2}{\sqrt{2}}, \ \frac{x_3 - x_4}{\sqrt{2}}, \ \frac{x_5 - x_6}{\sqrt{2}}). \tag{4.77}$$

Then the map π is a proper semi-invariant submersion such that

$$ker\pi_* = D \oplus D^{\perp} \tag{4.78}$$

where

$$D = \text{span}\{\partial_1 + \partial_2, \ \partial_3 + \partial_4\}, \tag{4.79}$$

and

$$D^{\perp} = \text{span}\{\partial_5 + \partial_6\}. \tag{4.80}$$

Moreover,

$$ker\pi_*^{\perp} = FD^{\perp} \oplus \mu, \tag{4.81}$$

where

$$\mu = \text{span}\{\partial_1 - \partial_2, \partial_3 - \partial_4\}. \tag{4.82}$$

Let π be a semi-invariant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). For any $V \in \Gamma(ker\pi_*)$, we put

$$FV = \phi V + \omega V, \tag{4.83}$$

where $\phi V \in \Gamma(ker\pi_*)$ and $\omega V \in \Gamma(ker\pi_*^{\perp})$. Also, for $\xi \in \Gamma(ker\pi_*^{\perp})$ we write

$$F\xi = B\xi + C\xi, \tag{4.84}$$

where $B\xi \in \Gamma(ker\pi_*)$ and $C\xi \in \Gamma(ker\pi_*^{\perp})$. Then, using (4.73) and (4.83), we get the following:

Lemma 7. *Let π be a semi-invariant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, we have*

$$(a)\quad \phi D = D, \qquad (b)\quad \phi D^{\perp} = \{0\}, \quad (c)\quad \omega D = \{0\}, \qquad (d)\quad \omega D^{\perp} = FD^{\perp}.$$

Proof. For any $X \in \Gamma(D)$, by (4.83), we have $FX = \phi X + \omega X$. On the other hand, with the help of (4.73), $FX \in \Gamma(D)$, i.e., $\omega X = 0$. Thus, we obtain $\phi D = D$.
Moreover, for any $U \in \Gamma(D^{\perp})$, by (4.83), we obtain $FU = \phi U + \omega U$. Beside this, by using (4.73), $FU \in \Gamma(ker\pi_*^{\perp})$, i.e., $\phi U = 0$. Therefore, we get $\phi D^{\perp} = \{0\}$. To prove (c) and (d), the same method above can be used. □

Also, using (4.74) and (4.84), we get the following result.

Lemma 8. *Let π be a semi-invariant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, we have*

$$(a)\quad B(FD^{\perp}) = D^{\perp}, \qquad (b)\quad B\mu = \{0\}, \quad (c)\quad C(FD^{\perp}) = \{0\}, \qquad (d)\quad C\mu = \mu.$$

We now examine how the almost product structure on M effects the O'Neill's tensors T and A of a semi-invariant submersion $\pi : (M,g,F) \to (N,g_N)$.

Lemma 9. *Let π be a semi-invariant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, we have*

$$\hat{\nabla}_V \phi W + T_V \omega W = \phi \hat{\nabla}_V W + BT_V W, \tag{4.85}$$

$$T_V \phi W + \mathscr{H}\nabla_V \omega W = \omega \hat{\nabla}_V W + CT_V W, \tag{4.86}$$

$$\mathscr{V}\nabla_\xi \eta + A_\xi C\eta = \phi A_\xi \eta + B\mathscr{H}\nabla_\xi \eta, \tag{4.87}$$

$$A_\xi B\eta + \mathscr{H}\nabla_\xi C\eta = \omega A_\xi \eta + C\mathscr{H}\nabla_\xi \eta, \tag{4.88}$$

$$\hat{\nabla}_V B\xi + T_V C\xi = \phi T_V \xi + B\mathscr{H}\nabla_V \xi, \tag{4.89}$$

$$T_V B\xi + \mathscr{H}\nabla_V C\xi = \omega T_V \xi + C\mathscr{H}\nabla_V \xi, \tag{4.90}$$

$$\mathscr{V}A_\xi \phi V + A_\xi \omega V = BA_\xi V + \phi \mathscr{V}\nabla_\xi V, \tag{4.91}$$

$$A_\xi \phi V + \mathscr{H}\nabla_\xi \omega V = CA_\xi V + \omega \mathscr{V}\nabla_\xi V, \tag{4.92}$$

where $V,W \in \Gamma(ker\pi_*)$ and $\xi,\eta \in \Gamma(ker\pi_*^{\perp})$.

Proof. Using (4.5)∼(4.8), (2.33), (4.83) and (4.84), we can easily obtain all assertions. □

Lemma 10. *Let π be a semi-invariant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, we have*

$$g(T_X FY, FV) = -g(T_Y FX, FV) \tag{4.93}$$

where $V \in \Gamma(D)$, and $X, Y \in \Gamma(D^\perp)$.

Proof. Let $X, Y \in \Gamma(D^\perp)$, and $V \in \Gamma(D)$. Then using (4.3), (4.5) and (2.28) we have

$$\begin{aligned} g(T_X FY, FV) &= -g(T_X FV, FY) \\ &= -g(T_{FV} X, FY) = -g(\nabla_{FV} X, FY) \\ &= -g(\nabla_{FV} FX, Y) = -g(T_{FV} FX, Y) \\ &= g(T_{FV} Y, FX) = g(T_Y FV, FX) \\ &= -g(T_Y FX, FV). \end{aligned} \tag{4.94}$$

This completes the proof. □

4.2.1 Integrability of distributions

Now, we investigate the necessary and sufficient conditions for the integrability of all distributions including vertical and horizontal distributions of the semi-invariant submersion π from l.p.R. manifold(M,g,F) onto a Riemannian manifold (N,g_N).

Theorem 7. *Let π be a semi-invariant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then the anti-invariant distribution $D^\perp$ is integrable if and only if*

$$g(T_X FY, FV) = 0, \tag{4.95}$$

where $X, Y \in \Gamma(D^\perp)$ and $V \in \Gamma(D)$.

Proof. Let $X, Y \in \Gamma(D^\perp)$ and $V \in \Gamma(D)$. By (4.6) and (2.33), we have

$$\begin{aligned} g([X,Y],V) &= g(F[X,Y],FV) \\ &= g(F\nabla_X Y - F\nabla_Y X, FV) \\ &= g(\nabla_X FY - \nabla_Y FX, FV) \\ &= g(T_X FY - T_Y FX, FV). \end{aligned} \tag{4.96}$$

From (4.93), we get

$$g([X,Y],V) = 2g(T_X FY, FV), \tag{4.97}$$

which completes the proof. □

Theorem 8. *Let π be a semi-invariant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, the invariant distribution D is integrable if and only if*

$$T_V FW = T_W FV \tag{4.98}$$

for all $V,W \in \Gamma(D)$.

Proof. For $V,W \in \Gamma(D)$, using (4.86), we have

$$T_V \phi W = \omega \hat{\nabla}_V W + CT_V W \tag{4.99}$$

By using (4.3), (4.5) and (4.99), we get

$$T_V FW - T_W FV = \omega[V,W] \tag{4.100}$$

Thus, our assertion comes from Lemma 7-(c) and (4.100). □

4.2.2 Totally geodesicness of the fibers

Proposition 3. *Let π be a semi-invariant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then*

$$g(T_V FW, \xi) = g(T_V W, F\xi), \tag{4.101}$$

for $V,W \in \Gamma(D)$, and $\xi \in \Gamma(\mu)$.

Proof. For $V,W \in \Gamma(D)$ and $\xi \in \Gamma(\mu)$, using (4.86), we have

$$\begin{aligned} g(T_V FW, \xi) &= g(\omega \hat{\nabla}_V W + CT_V W,\ \xi) \\ &= g(\omega \hat{\nabla}_V W, \xi) + g(CT_V W, \xi). \end{aligned}$$

Using (4.83) and (4.84), we obtain

$$\begin{aligned} g(T_V FW, \xi) &= g(F \hat{\nabla}_V W, \xi) + g(FT_V W, \xi) \\ &= g(\hat{\nabla}_V W, F\xi) + g(T_V W, F\xi). \end{aligned}$$

Since $g(\hat{\nabla}_V W, F\xi) = 0$, we get

$$g(T_V FW, \xi) = g(T_V W, F\xi).$$

□

Theorem 9. *Let π be a semi-invariant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, the invariant distribution D defines a totally geodesic foliation on $ker\pi_*$ if and only if*

$$g(T_V W, FX) = 0 \tag{4.102}$$

for $V, W \in \Gamma(D)$ and $X \in \Gamma(D^\perp)$.

Proof. The invariant distribution D defines a totally geodesic foliation on $ker\pi_*$ if and only if $g(\hat{\nabla}_V W, X) = 0$, for $V, W \in \Gamma(D)$ and $X \in \Gamma(D^\perp)$. Here, using (2.28) and (4.5), we have

$$g(\hat{\nabla}_V W, X) = g(\nabla_V W, X) = g(\nabla_V FW, FX) = g(T_V FW, FX). \tag{4.103}$$

Hence, (4.102) follows. □

Theorem 10. *Let π be a semi-invariant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, the anti-invariant distribution $D^\perp$ defines a totally geodesic foliation on $ker\pi_*$ if and only if*

$$g(T_X FY, FV) = 0, \tag{4.104}$$

for $X, Y \in \Gamma(D^\perp)$ and $V \in \Gamma(D)$.

Proof. Let $X, Y \in \Gamma(D^\perp)$ and $V \in \Gamma(D)$. Using (4.5), (4.6) and (2.28), we have

$$g(\hat{\nabla}_X Y, V) = g(\nabla_X Y, V) = g(\nabla_X FY, FV) = g(T_X FY, FV). \tag{4.105}$$

Since the anti-invariant distribution $D^\perp$ defines a totally geodesic foliation on $ker\pi_*$ if and only if $g(\hat{\nabla}_X Y, V) = 0$, the assertion follows. □

By Theorem 9 and Theorem 10, we have the following result.

Corollary 4. *Let π be a semi-invariant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, the vertical distribution $ker\pi_*$ is a locally product $M_D \times M_{D^\perp}$ if and only if*

$$g(T_U FW, FX) = 0 \tag{4.106}$$

for $W \in \Gamma(D)$, $X \in \Gamma(D^\perp)$ and $U \in \Gamma(ker\pi_*)$, where M_D and $M_{D^\perp}$ are integral manifolds of the distributions D and $D^\perp$, respectively.

It is well known that the vertical distribution $ker\pi_*$ of a Riemannian submersion is always integrable. We now give a necessary and sufficient condition for the integrability of the horizontal distribution $(ker\pi_*)^\perp$ of a semi-invariant submersion π from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N).

Theorem 11. *Let π be a semi-invariant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, the horizontal distribution $ker\pi_*^\perp$ is integrable and totally geodesic if and only if*

$$\phi(\mathscr{V}\nabla_\xi B\eta + A_\xi C\eta) + B(A_\xi B\eta + \mathscr{H}\nabla_\xi C\eta) = 0, \tag{4.107}$$

where $\xi, \eta \in \Gamma(ker\pi_*^\perp)$.

Proof. By (4.4), we know that the horizontal distribution $ker\pi_*^\perp$ is integrable and totally geodesic if and only if $A \equiv 0$, i.e. $A_\xi \eta = 0, \forall \xi, \eta \in \Gamma(ker\pi_*^\perp)$. On the other hand, from (4.8), this is equivalent to $\mathscr{V}\nabla_\xi \eta = 0$. Here, using (2.32), (2.33), (4.83) and (4.84), we have

$$\begin{aligned} \nabla_\xi \eta = F\nabla_\xi F\eta = F(\nabla_\xi B\eta + \nabla_\xi C\eta) \\ = BA_\xi B\eta + B\mathscr{H}\nabla_\xi C\eta + \phi\mathscr{V}\nabla_\xi B\eta + \phi A_\xi C\eta \\ + CA_\xi B\eta + C\mathscr{H}\nabla_\xi C\eta + \omega\mathscr{V}\nabla_\xi B\eta + \omega A_\xi C\eta. \end{aligned} \tag{4.108}$$

Taking the the vertical part of this equation, we get

$$\mathscr{V}\nabla_\xi \eta = \phi(\mathscr{V}\nabla_\xi B\eta + A_\xi C\eta) + B(A_\xi B\eta + \mathscr{H}\nabla_\xi C\eta). \tag{4.109}$$

Hence, our assertion follows. □

With a similar method, we have that:

Theorem 12. *Let π be a semi-invariant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, the vertical distribution $ker\pi_*$ defines a totally geodesic foliation if and only if*

$$\omega(\hat{\nabla}_U \phi V + T_U \omega V) + C(T_U \phi V + \mathscr{H}\nabla_U \omega V) = 0 \tag{4.110}$$

for all $U,V \in \Gamma(ker\pi_*)$.

Corollary 5. *Let π be a semi-invariant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, M is a locally product $M_{ker\pi_*} \times M_{ker\pi_*^{\perp}}$ if and only if (4.107) and (4.110) hold, where $M_{ker\pi_*}$ and $M_{ker\pi_*^{\perp}}$ are integral manifolds of the distributions $ker\pi_*$ and $ker\pi_*^{\perp}$, respectively.*

It is well known that the vertical distribution $ker\pi_*$ defines a totally geodesic foliation if and only if $T \equiv 0$ and the horizontal distribution $ker\pi_*^{\perp}$ defines a totally geodesic foliation if and only if $A \equiv 0$. On the other hand, we know that a Riemannian submersion $\pi : (M,g) \to (N,g_N)$ is totally geodesic if and only if both O'Neill's tensors T and A vanish [32]. Thus, by Theorem 11 and Theorem 12, we have the following result.

Theorem 13. *Let π be a semi-invariant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, π is a totally geodesic map if and only if (4.107) and (4.110) hold.*

4.2.3 Semi-invariant submersions with totally umbilical fibers

Theorem 14. *Let π be a proper semi-invariant submersion with totally umbilical fibers from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). If $dim(D^{\perp}) > 1$, then the fibers of π are totally geodesic or the mean curvature vector field H belongs to μ.*

Proof. The case that the fibers of π are totally geodesic is obvious. Let us consider the other case. Since $dim(D^{\perp}) > 1$, then we can choose $X,Y \in \Gamma(D^{\perp})$ such that the set $\{X, Y\}$ is orthonormal. By using (4.5), (2.28), (4.83) and (4.84), we have

$$T_X FY + \mathscr{H}\nabla_X FY = \nabla_X FY = F\nabla_X Y = \phi\hat{\nabla}_X Y + \omega\hat{\nabla}_X Y + BT_X Y + CT_X Y \tag{4.111}$$

Hence, we obtain

$$g(T_X FY, X) = g(\phi \hat{\nabla}_X Y + BT_X Y, X). \tag{4.112}$$

Here, using (2.28) and (4.83), we get

$$g(T_X FY, X) = g(T_X Y, FX). \tag{4.113}$$

Thus, using (4.22) and (4.113), we find

$$g(H, FY) = g(T_X X, FY) = -g(T_X FY, X) = -g(T_X FY, X)$$

$$= -g(T_X Y, FX) = -g(X,Y)g(H, FX) = 0, \tag{4.114}$$

since $g(X,Y) = 0$. So, we deduce that $H \perp FD^{\perp}$. Therefore, it follows $H \in \mu$ from (4.74). □

Corollary 6. *Let π be a proper semi-invariant submersion with totally umbilical fibers from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). If $ker\pi_*^{\perp} = FD^{\perp}$, i.e. $\mu = \{0\}$, then the fibers of π are totally geodesic.*

4.2.4 Semi-invariant submersions with parallel canonical structures

In this section, we study semi-invariant submersions from l.p.R. manifolds onto Riemannian manifolds with parallel canonical structures.

Let π be semi-invariant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, we define

$$(\nabla_U \phi)V = \hat{\nabla}_U \phi V - \phi \hat{\nabla}_U V, \tag{4.115}$$

$$(\nabla_U \omega)V = \mathscr{H} \nabla_U \omega V - \omega \hat{\nabla}_U V, \tag{4.116}$$

$$(\nabla_U B)\xi = \hat{\nabla}_U B\xi - B\mathscr{H} \nabla_U \xi, \tag{4.117}$$

$$(\nabla_U C)\xi = \mathscr{H} \nabla_U C\xi - C\mathscr{H} \nabla_U \xi, \tag{4.118}$$

where $U, V \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma(ker\pi_*^{\perp})$.

We say that ϕ (resp. ω, B or C) is *parallel* if $\nabla\phi = 0$ (resp. $\nabla\omega = 0$, $\nabla B = 0$ or $\nabla C = 0$).

Lemma 11. *Let π be a semi-invariant submersion with parallel canonical structures from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then for any $U,V \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma(ker\pi_*^{\perp})$, we have*

$$(\nabla_U \phi)V = BT_U V - T_U \omega V, \tag{4.119}$$

$$(\nabla_U \omega)V = CT_U V - T_U \phi V, \tag{4.120}$$

$$(\nabla_U B)\xi = \phi T_U \xi - T_U C\xi, \tag{4.121}$$

$$(\nabla_U C)\xi = \omega T_U \xi - T_U B\xi. \tag{4.122}$$

Proof. (4.119) follows from (4.115) and (4.85), (4.120) follows from (4.116) and (4.86), (4.121) follows from (4.117) and (4.89) and (4.122) follows from (4.118) and (4.90). □

Theorem 15. *Let π be a semi-invariant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). If ϕ is parallel, i.e. $\nabla\phi = 0$ then, the following two facts hold:*

$$\textit{Each leaf of the anti-invariant distribution } D^{\perp} \textit{ is totally geodesic.} \tag{4.123}$$

$$\textit{The fibres of } \pi \textit{ are mixed geodesic.} \tag{4.124}$$

Proof. Let ϕ be parallel. Then, for any $X,Y \in \Gamma(D^{\perp})$, from (4.115) we have

$$\phi \hat{\nabla}_X Y = 0, \tag{4.125}$$

since $\phi Y = 0$. By Lemma 7-(b), it follows that $\hat{\nabla}_X Y \in D^{\perp}$, so we obtain (4.123) .
On the other hand, for any $Z \in \Gamma(D)$ and $X \in \Gamma(D^{\perp})$, from (4.119) we have

$$BT_Z X = T_Z \omega X. \tag{4.126}$$

Since $\omega X = FX$, from (4.126), we get

$$B^2 T_Z X = BT_Z \omega X = T_Z \omega^2 X = T_Z X. \tag{4.127}$$

But, using (4.3) and Lemma 7-(c), we have

$$B^2 T_Z X = B^2 T_X Z = BT_X \omega Z = 0. \tag{4.128}$$

From (4.127) and (4.128), we find $T_Z X = 0$ which proves (4.124). □

Theorem 16. *Let π be a semi-invariant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). If ω is parallel, i.e. $\nabla\omega = 0$, then the following three facts hold:*

$$\text{Each leaf of the invariant distribution } D \text{ is totally geodesic.} \tag{4.129}$$

$$\text{The fibers of } \pi \text{ are mixed geodesic.} \tag{4.130}$$

$$T_{ker\pi_*}D^{\perp} \subset FD^{\perp}. \tag{4.131}$$

Proof. Let ω be parallel. Then, for any $U,Z \in \Gamma(D)$, from (4.116), we have $\omega\hat{\nabla}_U Z = 0$, since $\omega Z = 0$. By Lemma 7-(c), it follows that $\hat{\nabla}_U Z \in D$, so we get (4.129). On the other hand, for any $Z \in \Gamma(D)$ and $X \in \Gamma(D^{\perp})$, we have

$$CT_X Z = T_X \phi Z \tag{4.132}$$

from (4.120). Since $\phi Z = FZ$, we get

$$C^2 T_X Z = CT_X \phi Z = CT_X \phi^2 Z = T_X Z. \tag{4.133}$$

from (4.126). Hence, using (4.3) and Lemma 7-(b), we obtain

$$C^2 T_X Z = C^2 T_Z X = CT_Z \phi X. \tag{4.134}$$

Thus, the assertion (4.130) follows from (4.133) and (4.134).
Now, take $\xi \in \Gamma(\mu)$. Then, for any $V \in \Gamma(ker\pi_*)$, using (4.120), we get

$$g(T_V X,\xi) = g(FT_V X,F\xi) = g(CT_V X,F\xi) = g(T_V \phi X,F\xi) = 0, \tag{4.135}$$

since μ is invariant with respect to F, that is, we find

$$g(T_V X,\xi) = 0. \tag{4.136}$$

Thus, by (4.136) and (4.74), the assertion (4.131) follows. □

Proposition 4. *Let π be a semi-invariant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, ω is parallel if and only if B is parallel.*

Proof. Let ω be parallel. For any $U,V \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma(ker\pi_*^{\perp})$, using (2.32) and (4.120), we have

$$g(\phi T_U \xi,V) = g(FT_U \xi,V) = g(T_U \xi,\phi V) = -g(T_U \phi V,\xi) = -g(CT_U V,\xi)$$

$$= -g(FT_UV,\xi) = -g(T_UV,F\xi) = -g(T_UV,C\xi) = g(T_UC\xi,V), \tag{4.137}$$

that is; $g(\phi T_U\xi,V) = g(T_UC\xi,V)$. So, by (4.121), we find B is parallel.
The converse can be calculated in a similar way. □

4.2.5 First variational formula of a semi-invariant submersion

In this subsection, we investigate the first variational formula of a semi-invariant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). We use the definitions which are given in Subsection 4.1.1.

We start our study by investigating the conditions that under which a 1-form σ_ξ is a Legendre variation.

Lemma 12. *Let π be a semi-invariant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). The 1-form σ_ξ is a Legendre variation if and only if*

$$g(T_U\xi,\phi V) - g(T_V\xi,\phi U) = g(A_\xi U,\omega V) - g(A_\xi V,\omega U) \tag{4.138}$$

for all $U,V \in \Gamma(ker\pi_*)$.

Proof. Let $U,V \in \Gamma(ker\pi_*)$. Then, by the definition of differential, (4.6) and (2.32), we obtain

$$\begin{aligned}
(d\sigma_\xi)(U,V) =& Ug(F\xi,V) - Vg(F\xi,U) - g(F\xi,[U,V]) \\
=& Ug(\xi,FV) - Vg(\xi,FU) - g(\xi,F[U,V]) \\
=& g(\nabla_U\xi,FV) + g(\xi,\nabla_UFV) \\
& - g(\nabla_V\xi,FU) - g(\xi,\nabla_VFU) \\
& - g(\xi,F\nabla_UV) + g(\xi,F\nabla_VU) \\
=& g(\nabla_U\xi,\phi V+\omega V) - g(\nabla_V\xi,\phi U+\omega U) \\
=& g(\nabla_U\xi,\phi V) + g(\nabla_U\xi,\omega V) \\
& - g(\nabla_V\xi,\phi U) + g(\nabla_V\xi,\omega U) \\
=& g(T_U\xi,\phi V) + g(\mathscr{H}\nabla_U\xi,\omega V) \\
& - g(T_V\xi,\phi U) + g(\mathscr{H}\nabla_V\xi,\omega U).
\end{aligned} \tag{4.139}$$

Since we assume ξ is basic, we obtain

$$(d\sigma_\xi)(U,V) = g(T_U\xi,\phi V) + g(A_\xi U,\omega V) - g(T_V\xi,\phi U) + g(A_\xi V,\omega U). \tag{4.140}$$

Thus, the assertion follows. □

Lemma 13. *For $\xi \in \Gamma(\mu)$, $\sigma_\xi \equiv 0$.*

Proof. Let $\xi \in \Gamma(\mu)$. Then, $F\xi \in \Gamma(\mu)$. For any $V \in \Gamma(ker\pi_*)$, we get

$$\sigma_\xi(V) = g(F\xi,V) = 0. \tag{4.141}$$

So, $\sigma_\xi \equiv 0$, for all $V \in \Gamma(ker\pi_*)$. □

Remark 5. *Because of Lemma 13, throughout this subsection, we can assume that H belongs to $\Gamma(\omega D^\perp)$.*

Proposition 5. *Let π be a semi-invariant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N) and f be a smooth function on a fiber. Then, $F(grad(f)|_{\omega D^\perp}) \in \mathbb{E}$.*

Proof. Let f be a smooth function on a fiber. For $\xi = F(grad(f)|_{\omega D^\perp})$, and any $V \in \Gamma(ker\pi_*)$, we obtain

$$\sigma_\xi(V) = g(F\xi,V) = g(grad(f),V) = V[f] = df(V). \tag{4.142}$$

Thus, we get $\sigma_\xi = df$, i.e., $\xi \in \mathbb{E}$. □

Theorem 17. *Let π be a semi-invariant submersion with compact fibers from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then,*

(a) *The fiber π_q^{-1} is $\mathbb{L}-minimal$ if and only if σ_H is co-exact.*
(b) *The fiber π_q^{-1} is $\mathbb{E}-minimal$ if and only if σ_H is co-closed.*
(c) *The fiber π_q^{-1} is $\mathbb{H}-minimal$ if and only if σ_H is the sum of an exact and a co-exact 1-form.*

Proof. **(a)** $\Rightarrow$: Let the fiber π_q^{-1} be $\mathbb{L}-minimal$, then for any $\xi \in \mathbb{L}$, we have $g(H,\xi) = 0$ from (4.56). By the definition of the Hodge star operator [31], we have

$$\sigma_\xi \wedge \sigma_H(V_1,V_2,...,V_k) = g(\xi,H)*1(V_1,V_2,...,V_k),$$

for $V_1, V_2, ..., V_k \in \Gamma(ker\pi_*)$. From the definition of the global scalar product $(.|.)$ [31] on the module of all forms on the fiber, we get

$$(\sigma_\xi|\sigma_H) = \int_{\pi_q^{-1}} \sigma_\xi \wedge *\sigma_H = 0. \tag{4.143}$$

Denote by δ the codifferential operator [31] on the fiber π_q^{-1}. Since σ_ξ is closed, for any 2-form β on π_q^{-1}, we have

$$0 = (d\sigma_\xi|\beta) = (\sigma_\xi|\delta\beta). \tag{4.144}$$

Since π_q^{-1} is compact, by (4.143) and (4.144), we conclude that σ_H is co-exact.
$\Leftarrow$: Suppose that σ_H is co-exact, we have $\sigma_H = \delta\psi$ for some 2-form ψ. Then, for any $\xi \in \mathbb{L}$,

$$(\sigma_\xi|\sigma_H) = (\sigma_\xi|\delta\psi) = (d\sigma_\xi|\psi) = 0 \tag{4.145}$$

and then

$$\mathbf{V}'(\xi) = -k\int_{\pi_q^{-1}} g(H,\xi)*1 = -k\int_{\pi^{-1}(q)} (\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H) = 0,$$

i.e. π_q^{-1} is $\mathbb{L}-minimal$.
(b) $\Rightarrow$: Let the fiber π_q^{-1} be $\mathbb{E}-minimal$. Then, we have

$$0 = \mathbf{V}'(\xi) = -k\int_{\pi_q^{-1}} g(\xi,H)*1 = -k\int_{\pi_q^{-1}} (\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H),$$

that is, $(\sigma_\xi|\sigma_H) = 0$. Since for $\xi \in \mathbb{E}$, $\sigma_\xi = df$ for some function f on the fiber π_q^{-1}. Thus,

$$(df|\sigma_H) = (f|\delta\sigma_H) = 0.$$

Hence it follows that $\delta\sigma_H = 0$, i.e. σ_H is co-closed.
$\Leftarrow$: Suppose that σ_H is co-closed. Let $\xi \in \mathbb{E}$, then there exists a function $f \in \mathscr{F}(\pi_q^{-1})$ such that $\sigma_\xi = df$. Hence, we have

$$(\sigma_\xi|\sigma_H) = (df|\sigma_H) = (f|\delta\sigma_H) = 0. \tag{4.146}$$

Therefore,

$$\mathbf{V}'(\xi) = -k\int_{\pi_q^{-1}} g(H,\xi)*1 = -k\int_{\pi_q^{-1}} (\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H) = 0, \tag{4.147}$$

that is $\mathbf{V}'(\xi) = 0$ for $\xi \in \mathbb{E}$, i.e. π_q^{-1} is $\mathbb{E}-minimal$.
(c) $\Rightarrow$: If the fiber π_q^{-1} is $\mathbb{H}-minimal$, then for $\xi \in \mathbb{H}$, we have

$$0 = \mathbf{V}'(\xi) = -k\int_{\pi_q^{-1}} g(\xi,H)*1 = -k\int_{\pi_q^{-1}} (\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H). \tag{4.148}$$

It means that, σ_H is orthogonal to harmonic 1-forms on the fiber π_q^{-1}. Thus, by the Hodge decomposition theorem, we conclude that σ_H is the sum of an exact and a co-exact 1-form.

$\Leftarrow$: Let σ_H be the sum of an exact 1-form ω_1 such that $\omega_1 = df$ and a co-exact 1-form ω_2 such that $\omega_2 = \delta\psi$. For $\xi \in \mathbb{H}$, we have

$$\begin{aligned}(\sigma_\xi|\sigma_H) =&(\sigma_\xi|df+\delta\psi) = (\sigma_\xi|df)+(\sigma_\xi|\delta\psi)\\ =&(\delta\sigma_\xi|f)+(d\sigma_\xi|\psi) = 0,\end{aligned} \tag{4.149}$$

since $d\sigma_\xi = \delta\sigma_\xi = 0$. Thus,

$$\mathbf{V}'(\xi) = -k\int_{\pi_q^{-1}} g(\xi,H)*1 = -k\int_{\pi_q^{-1}}(\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H), \tag{4.150}$$

that is, the fiber is $\mathbb{H}-minimal$. □

If we give a restriction of Theorem 17, we have the following results.

Theorem 18. *Let π be a semi-invariant submersion with compact fibers from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). If $H \in \mathbb{L}$, then*

(a) *π_q^{-1} is $\mathbb{L}-minimal$ if and only if π_q^{-1} is minimal.*

(b) *π_q^{-1} is $\mathbb{E}-minimal$ if and only if σ_H is a harmonic variation.*

(c) *π_q^{-1} is $\mathbb{H}-minimal$ if and only if σ_H is an exact 1-form.*

Proof. **(a)** If the fiber π_q^{-1} is $\mathbb{L}-minimal$, then by Theorem 17-(a) we have, σ_H is co-exact. Hence σ_H is co-closed. Taking into account the fact that $d\sigma_H = 0$, we deduce that σ_H is harmonic. But this is a contradiction because of Hodge de Rham decomposition theorem. So, σ_H must be zero. Hence we conclude that $H = 0$. The converse is clear.

(b) $\Rightarrow$: If the fiber π_q^{-1} is $\mathbb{E}-minimal$, then we have $\delta\sigma_H = 0$ from Theorem 17-(b). Since $d\sigma_H = 0$, σ_H is also harmonic, i.e. $\Delta\sigma_H = 0$.

$\Leftarrow$: If σ_H is harmonic, then σ_H is co-closed. By Theorem 17-(b), the fiber π_q^{-1} is $\mathbb{E}-minimal$.

(c) $\Rightarrow$: Assume that π_q^{-1} is $\mathbb{H}-minimal$. Then, from Theorem 17-(c), σ_H is the sum of an exact 1-form and a co-exact 1-form. On the other hand, the condition $H \in \mathbb{L}$ implies that σ_H is orthogonal to every co-exact 1-form on π_q^{-1}. Thus, σ_H must be exact.

$\Leftarrow$: Let σ_H be an exact 1-form. For $\xi \in \mathbb{H}$, we obtain

$$\begin{aligned}\mathbf{V}^{'}(\xi) &= -k\int_{\pi_q^{-1}} g(\xi,H)*1 = -k\int_{\pi_q^{-1}}(\sigma_\xi \wedge *\sigma_H) \\ &= -k(\sigma_\xi|\sigma_H) = (\sigma_\xi|df) = (\delta\sigma_\xi|f) = 0, \end{aligned} \tag{4.151}$$

that is, π_q^{-1} is $\mathbb{H}-$*minimal*. □

Remark 6. *The method that considering the basis to investigate the harmonicity of a submersion, while the total manifold is taken as a l.p.R. manifold, is not always easy. Since a l.p.R. manifold is not always even dimensional, choosing a basis and using it is not easy. On the other hand, it is well known that, the fibers of a submersion is minimal if and only if the submersion is harmonic. Now, we give the following corollary which is a new approach to investigate the harmonicity of a submersion. By Theorem 18-(a), we have the following result.*

Corollary 7. *Let π be a semi-invariant submersion with compact fibers from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). If $H \in \mathbb{L}$, then, π is harmonic if and only if π_q^{-1} is $\mathbb{L}-$minimal.*

Lemma 14. *Let π be a semi-invariant submersion with compact fibers from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then,*

$$\delta\sigma_H = 0 \Leftrightarrow \Sigma_i g(T_{\phi E_i}E_i,H) = -\Sigma_i g(A_{\omega E_i}E_i,H), \tag{4.152}$$

where $\{E_1,E_2,...,E_m\}$ is a local basis of $ker\pi_*$.

Proof. By the definition of the co-differential of a 1-form, we have

$$\delta\sigma_H = 0 \Leftrightarrow \Sigma_i g(\nabla_{E_i}FH,E_i) = 0.$$

Here, we assume that H is basic. Then using (4.83), (4.6) and Remark 1, we get

$$\begin{aligned}\Rightarrow \delta\sigma_H = 0 \Leftrightarrow\ & \Sigma_i g(\nabla_{E_i}H,FE_i) = 0 \Leftrightarrow \Sigma_i g(\nabla_{E_i}H,\phi E_i + \omega E_i) \\ =\ & \Sigma_i g(\nabla_{E_i}H,\phi E_i) + \Sigma_i g(\nabla_{E_i}H,\omega E_i) \\ =\ & \Sigma_i g(T_{E_i}H,\phi E_i) + \Sigma_i g(A_H E_i,\omega E_i) = 0. \end{aligned} \tag{4.153}$$

Thus, the assertion follows from the skew-symmetry and symmetry properties of the O'Neill tensor A and T. □

4.3 Pointwise Semi-Slant Submersions

The notion of the pointwise slant submersion from almost Hermitian manifolds onto Riemannian manifolds was first defined by Lee and Şahin [10]. In this section, we construct on the idea of pointwise slant submersion and define a new type of submersion which is called pointwise semi-slant submersion.

Definition 7. *[10] Let π be a Riemannian submersion from an almost Hermitian manifold (M,g,J) onto a Riemannian manifold (N,g_N). If, at each given point $p \in M$, the Wirtinger angle $\theta(V)$ between JV and the space $(ker\pi_*)_p$ is independent of the choice of the non-zero vector $V \in (ker\pi_*)$, then we say that π is a* pointwise slant submersion*. In this case, the angle θ can be regarded as a function on M, which is called the* slant function *of the pointwise slant submersion.*

Now, we define the pointwise semi-slant submersion.

Definition 8. *Let (M,g,F) be a l.p.R. manifold and (N,g_N) be a Riemannian manifold. A Riemannian submersion $\pi : (M,g,F) \to (N,g_N)$ is called a* pointwise semi-slant Riemannian submersion*, if there is a distribution $D \subset ker\pi_*$ such that*

$$ker\pi_* = D \oplus D_\theta\,, \quad FD = D\,, \tag{4.154}$$

where D_θ is orthogonal complement of D in $ker\pi_*$ and the angle $\theta = \theta(X)$ between FX and the space $(D_\theta)_p$ is independent of the choice of non-zero vector $X \in \Gamma((D_\theta)_p)$ for $p \in M$, i.e. θ is a function on M, which is called slant function of the pointwise semi-slant submersion. We say that π is proper *if the slant function is $\theta \neq 0$ and $\theta \neq \pi/2$.*

Remark 7. *From now on, in this section, instead of using the term "pointwise semi-slant Riemannian submersion", we will briefly use the term "pointwise semi-slant submersion".*

In this case, for any $V \in \Gamma(ker\pi_*)$, we have

$$V = PV + QV, \tag{4.155}$$

where $PV \in \Gamma(D)$ and $QV \in \Gamma(D_\theta)$.

For $V \in \Gamma(ker\pi_*)$, we have

$$FV = \phi V + \omega V, \tag{4.156}$$

where $\phi V \in \Gamma(ker\pi_*)$ and $\omega V \in \Gamma(ker\pi_*^\perp)$.

For $\xi \in \Gamma(ker\pi_*^\perp)$, we have

$$F\xi = B\xi + C\xi, \tag{4.157}$$

where $B\xi \in \Gamma(ker\pi_*)$ and $C\xi \in \Gamma(ker\pi_*^\perp)$.

For any $E \in \Gamma(TM)$, we obtain

$$E = \mathscr{V}E + \mathscr{H}E, \tag{4.158}$$

where $\mathscr{V}E \in \Gamma(ker\pi_*)$ and $\mathscr{H}E \in \Gamma(ker\pi_*^\perp)$.

Therefore, the horizontal distribution $(ker\pi_*)^\perp$ is decomposed as

$$ker\pi_*^\perp = \omega D_\theta \oplus \mu\,, \tag{4.159}$$

where μ is the orthogonal complementary distribution of ωD_θ in $(ker\pi_*^\perp)$, and it is invariant with respect to F.

Example. Consider the Euclidean 6-space $\mathbb{R}^6$ with usual metric g. Define the almost product structure F on $(\mathbb{R}^6, g)$ by

$$F\partial_1 = \partial_2, \quad F\partial_2 = \partial_1, \quad F\partial_3 = \partial_4, \quad F\partial_4 = \partial_3, \quad F\partial_5 = \partial_5, \quad F\partial_6 = -\partial_6,$$

where $\partial_i = \frac{\partial}{\partial x_i}$, $i = 1, \cdots, 6$ and $(x_1, x_2, \cdots, x_6)$ are natural coordinates of $\mathbb{R}^6$. Now, we define a map $\pi : \mathbb{R}^6 \to \mathbb{R}^3$ by

$$\pi(x_1, \cdots, x_6) = \big(f_1, f_2, f_3\big),$$

where

$$\begin{aligned} f_1 &= (x_1 + (\sqrt{2}-1)x_2 - x_3 + x_4 + x_6), \\ f_2 &= (\frac{(x_1)^2}{2} + (\sqrt{2}-1)x_2 - \frac{(x_3)^2}{2} + x_4 - x_6), \\ f_3 &= (x_1 + (\sqrt{2}-1)x_2 - x_3 - x_4 + x_6), \end{aligned} \tag{4.160}$$

and $x_1 \neq x_3$. Then, the Jacobian matrix of π is:

$$\begin{pmatrix} 1 & \sqrt{2}-1 & -1 & 1 & 0 & 1 \\ x_1 & \sqrt{2}-1 & -x_3 & 1 & 0 & -1 \\ 1 & \sqrt{2}-1 & -1 & -1 & 0 & 1 \end{pmatrix}. \tag{4.161}$$

Since the rank of this matrix is equaled to 3, the map π is a submersion. After some calculations, we see that

$$ker\pi_* = D \oplus D_\theta, \tag{4.162}$$

where

$$D = span\{\partial_5\}, \tag{4.163}$$

and

$$D_\theta = span\{\frac{1}{\sqrt{2}}\partial_1 + \frac{1}{\sqrt{2}}\partial_2 + \partial_3, x_3\partial_1 + x_1\partial_3\}. \tag{4.164}$$

Moreover, the slant function of D_θ is $\theta = \arccos(\frac{1}{2}\frac{x_3}{\sqrt{(x_1)^2+(x_3)^2}})$. By direct calculation, we see that π satisfies the condition **(S2)** of the definition of Riemannian submersion. Hence the map π is a proper pointwise semi-slant submersion with the slant function θ.

Using (2.28), (4.156) and (4.157), we get the following useful facts.

Lemma 15. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, we have*

$$\textbf{(a) } \phi^2 + B\omega = I, \qquad \textbf{(b) } \omega\phi + C\omega = 0,$$

$$\textbf{(c) } \phi B + BC = 0, \qquad \textbf{(d) } \omega B + C^2 = I,$$

where I is the identity operator on TM.

Proof. For any $V \in \Gamma(ker\pi_*)$, by (2.28), we have

$$F^2V = V. \tag{4.165}$$

Using (4.156) and (4.157), we obtain

$$F^2V = V = F(FV) = F(\phi V + \omega V) = \phi^2 V + \omega\phi V + B\omega V + C\omega V. \tag{4.166}$$

If (4.166) is considered as decomposed into the vertical and horizontal parts, we obtain the following: $\phi^2 + B\omega = I$ and $\omega\phi + C\omega = 0$.

(c) and (d) can be proved with the same method above. □

By using (4.154)∼(4.159), we get the following two results.

Lemma 16. *Let π be a pointwise semi-slant Riemannian submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, we have*

$$\textbf{(a)} \quad \phi D = D \qquad \textbf{(b)} \quad \phi D_\theta \subset D_\theta \quad \textbf{(c)} \quad \omega D = \{0\}.$$

Lemma 17. *Let π be a pointwise semi-slant Riemannian submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, we have*

(a) $B(FD_\theta)=D_\theta$ ***(b)*** $B\mu=\{0\}$ ***(c)*** $C(FD_\theta)=\omega D_\theta$ ***(d)*** $C\mu=\mu$.

Now we investigate the effect of the almost product structure F on the O'Neill's tensors T and A of a pointwise semi-slant Riemannian submersion
$\pi:(M,g,F)\to(N,g_N)$.

Lemma 18. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, we have*

$$\hat{\nabla}_V\phi W+T_V\omega W=\phi\hat{\nabla}_V W+BT_V W, \tag{4.167}$$

$$T_V\phi W+\mathscr{H}\nabla_V\omega W=\omega\hat{\nabla}_V W+CT_V W, \tag{4.168}$$

$$\mathscr{V}\nabla_\xi B\eta+A_\xi C\eta=\phi A_\xi\eta+B\mathscr{H}\nabla_\xi\eta, \tag{4.169}$$

$$A_\xi B\eta+\mathscr{H}\nabla_\xi C\eta=\omega A_\xi\eta+C\mathscr{H}\nabla_\xi\eta, \tag{4.170}$$

$$\hat{\nabla}_V B\xi+T_V C\xi=\phi T_V\xi+B\mathscr{H}\nabla_V\xi, \tag{4.171}$$

$$T_V B\xi+\mathscr{H}\nabla_V C\xi=\omega T_V\xi+C\mathscr{H}\nabla_V\xi, \tag{4.172}$$

$$\mathscr{V}\nabla_\xi\phi V+A_\xi\omega V=BA_\xi V+\phi\mathscr{V}\nabla_\xi V, \tag{4.173}$$

$$A_\xi\phi V+\mathscr{H}\nabla_\xi\omega V=CA_\xi V+\omega\mathscr{V}\nabla_\xi V, \tag{4.174}$$

where $V,W\in\Gamma(ker\pi_*)$, and $\xi,\eta\in\Gamma(ker\pi_*^\perp)$.

Proof. For any $V\in\Gamma(ker\pi_*)$ and $\xi\in\Gamma(ker\pi_*^\perp)$, using (2.33), we have

$$F\nabla_\xi V=\nabla_\xi FV.$$

Hence, using (4.7), (4.8), (4.156) and (4.157), we obtain

$$BA_\xi V+CA_\xi V+\phi\mathscr{V}\nabla_\xi V+\omega\mathscr{V}\nabla_\xi V=A_\xi\phi V+\mathscr{V}\nabla_\xi\phi V+A_\xi\omega V+\mathscr{H}\nabla_\xi\omega V. \tag{4.175}$$

Taking the vertical and horizontal parts of this equation, we get (4.173) and (4.174). The other assertions can be obtained by using (4.5)∼(4.8), (4.156) and (4.157). □

Proposition 6. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, we obtain*

$$\phi^2 X = \cos^2\theta X, \tag{4.176}$$

for $X \in \Gamma(D_\theta)$, where θ denotes the slant function.

Proof. For any non-zero $X \in \Gamma(D_\theta)$ we can write following equations:

$$\cos\theta = \frac{g(FX,\phi X)}{|FX||\phi X|} = \frac{g(X,\phi^2 X)}{|X||\phi X|} \text{ and } \cos\theta = \frac{|\phi X|}{|FX|}. \tag{4.177}$$

Then, we obtain

$$\cos^2\theta = \frac{g(X,\phi^2 X)}{|X||\phi X|}\frac{|\phi X|}{|FX|}. \tag{4.178}$$

Therefore, we get the equality

$$g(\cos^2\theta X, X) = g(X,\phi^2 X), \tag{4.179}$$

which gives the assertion. □

Remark 8. *We easily observe that the converse of the Proposition 6 also holds.*

Now we give a theorem for pointwise semi-slant submersions, which has similar idea with the Theorem 4.2. in [33].

Theorem 19. *Let π be a Riemannian submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, π is a proper pointwise semi-slant submersion if and only if there exists a constant $\lambda \in [0,1]$ such that*
(a) $D^{'} = \{x \in D^{'} \mid \phi^2 X = \lambda X\}$,
(b) *For any $X \in \Gamma(TM)$, orthogonal to $D^{'}$, $\omega X = 0$.*
Moreover, in this case $\lambda = \cos^2\theta$, where θ denotes the slant function.

Proof. Let $\pi : (M,g,F) \to (N,g_N)$ be a pointwise semi-slant submersion. Then, $\lambda = \cos^2\theta$ and $D^{'} = D_\theta$. By the definition of the pointwise semi-slant submersion, $\omega X = 0$, where X belongs to orthogonal complement of $D^{'}$.
Conversely, **(a)** and **(b)** imply that $TM = D \oplus D^{'}$. Since $\phi D^{'} \subseteq D^{'}$, from **(b)**, D is an invariant distribution. Thus, π is a pointwise semi-slant submersion. □

4.3.1 Integrability of distributions

In this subsection, we investigate the integrability conditions for invariant and slant distributions.

Theorem 20. *Let π be a pointwise semi-slant Riemannian submersion from an almost product Riemannian manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, the invariant distribution D is integrable if and only if*

$$\phi(\hat{\nabla}_V W - \hat{\nabla}_W V) \in D \tag{4.180}$$

for $V,W \in \Gamma(D)$.

Proof. For $V,W \in \Gamma(D)$ and $X \in \Gamma(D_\theta)$, we know $[V,W] \in D$ if and only if $F[V,W] \in D$. Then, by (4.156) we obtain,

$$\begin{aligned} g(F[V,W],X) =& g(F(\nabla_V W - \nabla_W V),X) \\ =& g(F(T_V W + \hat{\nabla}_V W - T_W V - \hat{\nabla}_W V),X) \\ =& g(\phi(\hat{\nabla}_V W - \hat{\nabla}_W V),X). \end{aligned} \tag{4.181}$$

Thus, $[V,W] \in D$ if and only if $\phi(\hat{\nabla}_V W - \hat{\nabla}_W V) \in D$. □

In a similar way, we get the following theorem.

Theorem 21. *Let π be a pointwise semi-slant Riemannian submersion from an almost product Riemannian manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, the slant distribution D_θ is integrable if and only if*

$$\phi(\hat{\nabla}_X Y - \hat{\nabla}_Y X) \in D_\theta$$

for $X,Y \in \Gamma(D_\theta)$.

If we consider the total manifold l.p.R. instead of almost product Riemannian, we obtain the following results.

Lemma 19. *Let π be a proper pointwise semi-slant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, we have the following:*

$$g(\nabla_V W,X) = \csc^2\theta\{g(T_V W,\omega\phi X) + g(T_V \phi W,\omega X)\}, \tag{4.182}$$

$$g(\nabla_X Y,V) = \csc^2\theta\{g(T_X \omega\phi Y,V) + g(T_X \omega Y,\phi V)\}, \tag{4.183}$$

where θ is the slant function, $V,W \in \Gamma(D)$ and $X,Y \in \Gamma(D_\theta)$.

Proof. Let $V,W \in \Gamma(D)$ and $X,Y \in \Gamma(D_\theta)$. Then, by using (2.32) and (4.156), we obtain

$$\begin{aligned} g(\nabla_V W,X) =& g(\nabla_V FW,FX) \\ =& g(\nabla_V FW,\phi X)+g(\nabla_V FW,\omega X) \\ =& g(\nabla_V W,\phi^2 X)+g(\nabla_V W,\omega\phi X)+g(\nabla_V \phi W,\omega X). \end{aligned} \tag{4.184}$$

If we regard (4.176), (4.5) and (4.6) for the last expression, we get the following equality

$$\Rightarrow (1-\cos^2\theta)g(\nabla_V W,X) = g(T_V W,\omega\phi X)+g(T_V\phi X,\omega X). \tag{4.185}$$

Thus, we obtain the first assertion.
For the second equation we apply the same idea. Let $X,Y \in \Gamma(D_\theta)$ and $V \in \Gamma(D)$. Then by using (2.32) and (4.156), we get

$$\begin{aligned} g(\nabla_X Y,V) =& g(\nabla_X FY,FV) \\ =& g(\nabla_X \phi Y,FV)+g(\nabla_X \omega Y,FV) \\ =& g(\nabla_X \phi^2 Y,V)+g(\nabla_X \omega\phi Y,V)+g(\nabla_X \omega Y,FV). \end{aligned} \tag{4.186}$$

If we consider (4.176),(4.5) and (4.6) with the last equation, we get the following

$$\begin{aligned} g(\nabla_X Y,V) =& g(\nabla_X(\cos^2\theta)Y,V)+g(\nabla_X\omega\phi Y,V)+g(\nabla_X\omega Y,FV) \\ =& g(-(\sin 2\theta)(X\theta)Y,V)+g(\cos^2\theta\nabla_X Y,V)+g(T_X\omega\phi Y,V) \\ &+g(T_X\omega Y,\phi V). \end{aligned} \tag{4.187}$$

Therefore, since $g(-(\sin 2\theta)(X\theta)Y,V)=0$, we get the assertion. □

Theorem 22. *Let π be a proper pointwise semi-slant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, the invariant distribution D is integrable if and only if*

$$g(T_V\phi W - T_W\phi V,\omega X)=0 \tag{4.188}$$

for $V,W \in \Gamma(D)$ and $X \in \Gamma(D_\theta)$.

Proof. Let $V,W \in \Gamma(D)$ and $X \in \Gamma(D_\theta)$. Then, by Lemma 19 and (4.3), we have

$$\begin{aligned} g([V,W],X) =& g(\nabla_V W, X) - g(\nabla_W V, X) \\ =& \csc^2\theta\{g(T_V W, \omega\phi X) + g(T_V \phi W, \omega X) \\ & - g(T_W V, \omega\phi X) + g(T_W \phi V, \omega X)\} \\ =& \csc^2\theta\{g(T_V \phi W, \omega X) - g(T_W \phi V, \omega X)\}. \end{aligned} \tag{4.189}$$

Therefore, D is integrable if and only if $g(T_V\phi W - T_W\phi V, \omega X) = 0$. □

In the same way, we examine the slant distribution.

Theorem 23. *Let π be a proper pointwise semi-slant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, the slant distribution D_θ is integrable if and only if*

$$g(T_X\omega\phi Y - T_Y\omega\phi X, V) = g(T_Y\omega X - T_X\omega Y, \phi V) \tag{4.190}$$

for $X,Y \in \Gamma(D_\theta)$ and $V \in \Gamma(D_\theta)$.

Proof. Let $X,Y \in \Gamma(D_\theta)$ and $V \in \Gamma(D)$. By using Lemma 19, we obtain

$$\begin{aligned} g([X,Y],V) =& \csc^2\theta\{g(T_X\omega\phi Y, V) + g(T_X\omega Y, \phi V) \\ & - g(T_Y\omega\phi X, V) + g(T_Y\omega X, \phi V)\}. \end{aligned} \tag{4.191}$$

Thus, slant distribution D_θ is integrable if and only if
$g(T_X\omega\phi Y - T_Y\omega\phi X, V) = g(T_Y\omega X - T_X\omega Y, \phi V)$. □

4.3.2 Totally geodesicness of the fibers

Now, we focus on the geometry of the fibers and the distributions that is mentioned in the definition of a pointwise semi-slant submersion.

Proposition 7. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, $ker\pi_*$ defines a totally geodesic foliation if and only if*

$$C(T_V\phi W + \mathscr{H}\nabla_V\omega W) + \omega(\hat{\nabla}_V\phi W + T_V\omega W) = 0 \tag{4.192}$$

for $V,W \in \Gamma(ker\pi_*)$.

Proof. For $V,W \in \Gamma(ker\pi_*)$, by using (4.5), (4.6) and (4.156), we get

$$\begin{aligned}\nabla_V W =& F\nabla_V FW = F(\nabla_V \phi W + \nabla_V \omega W)\\ =& F(T_V \phi W + \hat{\nabla}_V \phi W + T_V \omega W + \mathscr{H}\nabla_V \omega W)\\ =& BT_V \phi W + CT_V \phi W + \phi \hat{\nabla}_V \phi W + \omega \hat{\nabla}_V \phi W\\ &+ \phi T_V \omega W + \omega T_V \omega W + B\mathscr{H}\nabla_V \omega W + C\mathscr{H}\nabla_V \omega W. \end{aligned} \tag{4.193}$$

Therefore, $ker\pi_*$ defines a totally geodesic foliation if and only if
$C(T_V \phi W + \mathscr{H}\nabla_V \omega W) + \omega(\hat{\nabla}_V \phi W + T_V \omega W) = 0.$ □

Proposition 8. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, $ker\pi_*^{\perp}$ defines a totally geodesic foliation if and only if*

$$B(A_\xi B\eta + \mathscr{H}\nabla_\xi C\eta) + \phi(\mathscr{V}\nabla_\xi B\eta + A_\xi C\eta) = 0 \tag{4.194}$$

for $\xi,\eta \in \Gamma(ker\pi_*^{\perp})$.

Proof. This proof can be done using the techniques of the proof of Proposition 7 . □

In the view of Proposition 7 and Proposition 8, we obtain the following result.

Corollary 8. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, M is a locally product $M_{ker\pi_*} \times M_{ker\pi_*^{\perp}}$ if and only if (4.192) and (4.194) hold, where $M_{ker\pi_*}$ and $M_{ker\pi_*^{\perp}}$ are integral manifolds of the distributions $ker\pi_*$ and $ker\pi_*^{\perp}$, respectively.*

Proposition 9. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, the invariant distribution D defines a totally geodesic foliation on $ker\pi_*$ if and only if for $U,V \in \Gamma(D)$,*

$$Q(BT_U \phi V + \phi \hat{\nabla}_U \phi V) = 0 \text{ and } (CT_U \phi V + \omega \hat{\nabla}_U \phi V) = 0. \tag{4.195}$$

Proof. For $U,V \in \Gamma(D)$, from (4.5), (4.6), (4.156) and (4.157) we obtain

$$\begin{aligned}\nabla_U V =& F\nabla_U FV = F(\nabla_U \phi V + \nabla_U \omega W)\\ =& F(\nabla_U \phi V) = F(T_U \phi V + \hat{\nabla}_U \omega V)\\ =& BT_U \phi V + CT_U \phi V + \phi \hat{\nabla}_U \omega V + \omega \hat{\nabla}_U \omega V. \end{aligned} \tag{4.196}$$

Therefore, we obtain the assertion. □

Proposition 10. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, the slant distribution D_θ defines a totally geodesic foliation on kerπ_* if and only if for*
$X,Y \in \Gamma(D_\theta)$,

$$P(B(T_X\phi Y + \mathscr{H}\nabla_X \omega Y) + \phi(T_X \omega Y + \hat{\nabla}_X \phi Y)) = 0 \tag{4.197}$$

and

$$\omega(\hat{\nabla}_X \phi Y + T_X \omega Y) + C(T_X \phi Y + \mathscr{H}\nabla_X \omega Y) = 0. \tag{4.198}$$

Proof. The argument is the same as the proof of Proposition 9. □

By Proposition 9 and Proposition 10 we have the following result.

Corollary 9. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, the vertical distribution kerπ_* is a locally product $M_D \times M_{D_\theta}$ if and only if (4.195),* (4.197) *and (4.198) hold, where M_D and M_{D_θ} are intergral manifolds of D and D_θ, respectively.*

Theorem 24. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, π is a totally geodesic map if and only if*

$$\omega(\hat{\nabla}_V \phi W + T_U \omega W) + C(T_V \phi W + \mathscr{H}\nabla_V \omega W) = 0 \tag{4.199}$$

and

$$\omega(\hat{\nabla}_V B\xi + T_V C\xi) + C(T_V B\xi + \mathscr{H}\nabla_V C\xi) = 0 \tag{4.200}$$

for $V,W \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma(ker\pi_*^\perp)$.

Proof. Since π is a Riemannian submersion, we have

$$(\nabla\pi_*)(\xi,\eta) = 0, \; for \; \xi,\eta \in \Gamma(ker\pi_*^\perp).$$

For $V,W \in \Gamma(ker\pi_*)$, we obtain

$$
\begin{aligned}
(\nabla\pi_*)(V,W) &= \nabla^{\pi}_{V}(\pi_*W) - \pi_*\nabla_V W \\
&= -\pi_*(F\nabla_V FW) = -\pi_*(F(\nabla_V \phi W + \nabla_V \omega W) \\
&= -\pi_*(F(T_V\phi W + \hat{\nabla}_V \phi W + T_V \omega W + \mathscr{H}\nabla_V \omega W) \\
&= -\pi_*(BT_V\phi W + CT_V\phi W + \phi\hat{\nabla}_V \phi W + \omega\hat{\nabla}_V \phi W \\
&\quad + \phi T_V \omega W + \omega T_V \omega W + B\mathscr{H}\nabla_V \omega W + C\mathscr{H}\nabla_V \omega W) \\
&= -\pi_*(CT_V\phi W + \omega\hat{\nabla}_V \phi W + \omega T_V \omega W + C\mathscr{H}\nabla_V \omega W). \qquad (4.201)
\end{aligned}
$$

Thus,

$(\nabla\pi_*)(V,W) = 0 \Leftrightarrow \omega(\hat{\nabla}_V \phi W + T_V \omega W) + C(T_V \phi W + \mathscr{H}\nabla_V \omega W) = 0.$

By a similar way above, for $V \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma(ker\pi_*^{\perp})$, we get

$$(\nabla\pi_*)(V,\xi) = 0 \Leftrightarrow \omega(\hat{\nabla}_V B\xi + T_V C\xi) + C(T_V B\xi + \mathscr{H}\nabla_V C\xi) = 0.$$

□

4.3.3 Parallel canonical structures and totally umbilical case of the fibers

Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N) . We can define

$$(\nabla_U \phi)V = \hat{\nabla}_U \phi V - \phi\hat{\nabla}_U V, \qquad (4.202)$$

$$(\nabla_U \omega)V = \mathscr{H}\nabla_U \omega V - \omega\hat{\nabla}_U V, \qquad (4.203)$$

$$(\nabla_U B)\xi = \hat{\nabla}_U B\xi - B\mathscr{H}\nabla_U \xi, \qquad (4.204)$$

$$(\nabla_U C)\xi = \mathscr{H}\nabla_U C\xi - C\mathscr{H}\nabla_U \xi, \qquad (4.205)$$

where $U,V \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma(ker\pi_*^{\perp})$.

We say that ϕ (resp. ω, B or C) is *parallel* if $\nabla\phi = 0$ (resp. $\nabla\omega = 0$, $\nabla B = 0$ or $\nabla C = 0$).

Lemma 20. *Let π be a pointwise semi-slant submersion with parallel canonical structures from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N) . Then for any $U,V \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma(ker\pi_*^{\perp})$, we have*

$$(\nabla_U \phi)V = BT_U V - T_U \omega V, \qquad (4.206)$$

$$(\nabla_U \omega)V = CT_U V - T_U \phi V, \tag{4.207}$$

$$(\nabla_U B)\xi = \phi T_U \xi - T_U C\xi, \tag{4.208}$$

$$(\nabla_U C)\xi = \omega T_U \xi - T_U B\xi. \tag{4.209}$$

Proof. All of the equations follow from Lemma 18 and (4.202)∼(4.205). □

Theorem 25. *Let π be a proper pointwise semi-slant submersion with totally umbilical fibers from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). If $dim(D_\theta) \geq 2$ and ϕ is parallel, then the fibers of π are totally geodesic or the mean curvature vector field H belongs to μ.*

Proof. The case of totally geodesic fibers is obvious. Let us assume the other case. Since $dim(D_\theta) \geq 2$, then we can choose $X,Y \in \Gamma(D_\theta)$ such that the set $\{X, Y\}$ is orthonormal. By using (2.32), (2.33), (4.156), (4.157), (4.5) and (4.6), we have

$$\nabla_X FY = F\nabla_X Y \tag{4.210}$$

$$\nabla_X \phi Y + \nabla_X \omega Y = F(T_X Y + \hat{\nabla}_X Y) \tag{4.211}$$

$$T_X \phi Y + \hat{\nabla}_X \phi Y + T_X \omega Y + \mathscr{H}\nabla_X \omega Y = BT_X Y + CT_X Y + \phi\hat{\nabla}_X Y + \omega\hat{\nabla}_X Y. \tag{4.212}$$

Therefore, we obtain

$$g(\hat{\nabla}_X \phi Y + T_X \omega Y, X) = g(BT_X Y + \phi\hat{\nabla}_X Y, X) \tag{4.213}$$

$$g(\phi\hat{\nabla}_X Y - \hat{\nabla}_X \phi Y, X) = g(T_X \omega Y - BT_X Y, X) \tag{4.214}$$

$$g((\nabla_X \phi)Y, X) = g(FT_X Y - T_X FY, X). \tag{4.215}$$

Since ϕ is parallel, we get

$$g(FT_X Y, X) = g(T_X FY, X). \tag{4.216}$$

Thus, using (4.22) and (4.216), we have

$$\begin{aligned} g(H,FY) =& g(T_X X, FY) = -g(T_X FY, X) = -g(FT_X Y, X) \\ =& -g(T_X Y, FX) = -g(X,Y)g(H,FX) = 0, \end{aligned} \tag{4.217}$$

since $g(X,Y) = 0$. So, we deduce that $H \perp \omega D_\theta$. Therefore, it follows $H \in \mu$ from (4.159). □

Corollary 10. *Let π be a proper pointwise semi-slant submersion with totally umbilical fibers from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). If $(ker\pi_*)^{\perp} = \omega D_\theta$, i.e. $\mu = \{0\}$ and ϕ is parallel, then the fibers of π are totally geodesic.*

Theorem 26. *Let π be a proper pointwise semi-slant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). If ω is parallel, i.e. $\nabla\omega = 0$, then the fibers of π are mixed geodesic.*

Proof. Let ω be parallel, then for any $U,V \in \Gamma(ker\pi_*)$ from (4.207), we have

$$CT_U V = T_U \phi V. \tag{4.218}$$

Using (4.218), we obtain

$$C^2 T_U V = T_U \phi^2 V. \tag{4.219}$$

If we put $U = W \in \Gamma(D)$ and $V = X \in \Gamma(D_\theta)$ in (4.219) and using (4.176), we get

$$C^2 T_W X = \cos^2\theta T_W X. \tag{4.220}$$

On the other hand, using the symmetry property of T on $\Gamma(ker\pi_*)$ and (4.218), we have

$$C^2 T_W X = C^2 T_X W = T_X \phi^2 W = T_X W, \tag{4.221}$$

that is

$$C^2 T_W X = T_X W. \tag{4.222}$$

Since submersion π is proper, from (4.220) and (4.222), it follows that

$$T_X W = 0. \tag{4.223}$$

□

4.3.4 The first variational formula of a pointwise semi-slant submersion

In this section, we give a different approach to check whether a pointwise semi-slant submersion is harmonic. We use the definitions from Subsection 4.1.1 for pointwise semi-slant submersions.

Now, we investigate the conditions under which the 1-form σ_ξ is a Legendre variation.

Lemma 21. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). The 1-form σ_ξ is a Legendre variation if and only if*

$$g(T_U\xi,\phi V)-g(T_V\xi,\phi U)=g(A_\xi U,\omega V)-g(A_\xi V,\omega U), \tag{4.224}$$

for all $U,V\in\Gamma(ker\pi_*)$.

Proof. Let U,V be in $ker\pi_*$. Then, by the definition of differential, (4.6) and (2.32), we obtain

$$\begin{aligned}(d\sigma_\xi)(U,V)=&Ug(F\xi,V)-Vg(F\xi,U)-g(F\xi,[U,V])\\ =&Ug(\xi,FV)-Vg(\xi,FU)-g(\xi,F[U,V])\\ =&g(\nabla_U\xi,FV)+g(\xi,\nabla_UFV)\\ &-g(\nabla_V\xi,FU)-g(\xi,\nabla_VFU)\\ &-g(\xi,F\nabla_UV)+g(\xi,F\nabla_VU)\\ =&g(\nabla_U\xi,\phi V+\omega V)-g(\nabla_V\xi,\phi U+\omega U)\\ =&g(\nabla_U\xi,\phi V)+g(\nabla_U\xi,\omega V)\\ &-g(\nabla_V\xi,\phi U)+g(\nabla_V\xi,\omega U)\\ =&g(T_U\xi,\phi V)+g(\mathscr{H}\nabla_U\xi,\omega V)\\ &-g(T_V\xi,\phi U)+g(\mathscr{H}\nabla_V\xi,\omega U).\end{aligned} \tag{4.225}$$

Since we assume ξ is basic, we have

$$\begin{aligned}(d\sigma_\xi)(U,V)=&g(T_U\xi,\phi V)+g(A_\xi U,\omega V)\\ &-g(T_V\xi,\phi U)+g(A_\xi V,\omega U).\end{aligned} \tag{4.226}$$

Thus, the assertion follows. □

Lemma 22. *For $\xi \in \Gamma(\mu)$, $\sigma_\xi \equiv 0$.*

Proof. Let $\xi \in \Gamma(\mu)$. Then, $F\xi \in \Gamma(\mu)$. For any $V \in \Gamma(ker\pi_*)$, we get

$$\sigma_\xi(V) = g(F\xi, V) = 0. \tag{4.227}$$

So, $\sigma_\xi \equiv 0$, for all $V \in \Gamma(ker\pi_*)$. □

Remark 9. *Because of Lemma 22, throughout this subsection, we can assume that H belongs to $\Gamma(\omega D_\theta)$.*

Proposition 11. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N) and f be a smooth function on a fiber. Then, $F(grad(f)|_{\omega D_\theta}) \in \mathbb{E}$.*

Proof. Let f be a smooth function on a fiber. For $\xi = F(grad(f)|_{\omega D_\theta})$, and any $V \in \Gamma(ker\pi_*)$, we obtain

$$\sigma_\xi(V) = g(F\xi, V) = g(grad(f), V) = V[f] = df(V). \tag{4.228}$$

Thus, we get $\sigma_\xi = df$, i.e. $\xi \in \mathbb{E}$. □

Theorem 27. *Let π be a pointwise semi-slant submersion with compact fibers from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then,*
(a) *The fiber π_q^{-1} is $\mathbb{L}-minimal$ if and only if σ_H is co-exact.*
(b) *The fiber π_q^{-1} is $\mathbb{E}-minimal$ if and only if σ_H is co-closed.*
(c) *The fiber π_q^{-1} is $\mathbb{H}-minimal$ if and only if σ_H is the sum of an exact and a co-exact 1-form.*

Proof. **(a)** $\Rightarrow$: Let the fiber π_q^{-1} is $\mathbb{L}-minimal$, then for any $\xi \in \mathbb{L}$, we have $g(H, \xi) = 0$ from (4.56). By the definition of the Hodge star operator [31], we have

$$\sigma_\xi \wedge \sigma_H(V_1, V_2, ..., V_k) = g(\xi, H) {*1}(V_1, V_2, ..., V_k), \tag{4.229}$$

for $V_1, V_2, ..., V_k \in \Gamma(ker\pi_*)$. From the definition of the global scalar product $(.|.)$ (see [31]) on the module of all forms on the fiber, we get

$$(\sigma_\xi|\sigma_H) = \int_{\pi_q^{-1}} \sigma_\xi \wedge *\sigma_H = 0. \tag{4.230}$$

Denote by δ the codifferential operator [31] on the fiber π_q^{-1}. Since σ_ξ is closed, for any 2-form β on π_q^{-1}, we have

$$0 = (d\sigma_\xi|\beta) = (\sigma_\xi|\delta\beta). \tag{4.231}$$

Since π_q^{-1} is compact, by (4.230) and (4.231) we conclude that σ_H is co-exact. $\Leftarrow$: Suppose that σ_H is co-exact, we have $\sigma_H = \delta\psi$ for some 2-form ψ. Then, for any $\xi \in \mathbb{L}$,

$$(\sigma_\xi|\sigma_H) = (\sigma_\xi|\delta\psi) = (d\sigma_\xi|\psi) = 0 \tag{4.232}$$

and then

$$\mathbf{V}^{'}(\xi) = -k\int_{\pi_q^{-1}} g(H,\xi)*1 = -k\int_{\pi^{-1}(q)} (\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H) = 0, \tag{4.233}$$

i.e. π_q^{-1} is $\mathbb{L}-minimal$.

(b) $\Rightarrow$: Let the fiber π_q^{-1} be $\mathbb{E}-minimal$. Then, we have

$$0 = \mathbf{V}^{'}(\xi) = -k\int_{\pi_q^{-1}} g(\xi,H)*1 = -k\int_{\pi_q^{-1}} (\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H),$$

that is, $(\sigma_\xi|\sigma_H) = 0$. Since $\xi \in \mathbb{E}$, $\sigma_\xi = df$ for some function f on the fiber π_q^{-1}. Thus,

$$(df|\sigma_H) = (f|\delta\sigma_H) = 0. \tag{4.234}$$

Hence it follows that $\delta\sigma_H = 0$, i.e. σ_H is co-closed.

$\Leftarrow$: Suppose that σ_H is co-closed. Let $\xi \in \mathbb{E}$, then there exists a function $f \in \mathscr{F}(\pi_q^{-1})$ such that $\sigma_\xi = df$. Hence, we have

$$(\sigma_\xi|\sigma_H) = (df|\sigma_H) = (f|\delta\sigma_H) = 0. \tag{4.235}$$

Therefore,

$$\mathbf{V}^{'}(\xi) = -k\int_{\pi_q^{-1}} g(H,\xi)*1 = -k\int_{\pi_q^{-1}} (\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H) = 0, \tag{4.236}$$

that is $\mathbf{V}^{'}(\xi) = 0$ for $\xi \in \mathbb{E}$, i.e. π_q^{-1} is $\mathbb{E}-minimal$.

(c) $\Rightarrow$: If the fiber π_q^{-1} is $\mathbb{H}-minimal$, then for $\xi \in \mathbb{H}$, we have

$$0 = \mathbf{V}^{'}(\xi) = -k\int_{\pi_q^{-1}} g(\xi,H)*1 = -k\int_{\pi_q^{-1}} (\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H). \tag{4.237}$$

It means that, σ_H is orthogonal to harmonic 1-forms on the fiber π_q^{-1}. Thus, by the Hodge decomposition theorem [31], we conclude that σ_H is the sum of an exact and a co-exact 1-form.

$\Leftarrow$: Let σ_H be the sum of an exact 1-form ω_1 such that $\omega_1 = df$ and a co-exact 1-form ω_2 such that $\omega_2 = \delta\psi$. For $\xi \in \mathbb{H}$, we have

$$\begin{aligned}(\sigma_\xi|\sigma_H) =&(\sigma_\xi|df+\delta\psi) = (\sigma_\xi|df)+(\sigma_\xi|\delta\psi)\\ =&(\delta\sigma_\xi|f)+(d\sigma_\xi|\psi) = 0,\end{aligned} \tag{4.238}$$

since $d\sigma_\xi = \delta\sigma_\xi = 0$. Thus,

$$\mathbf{V}'(\xi) = -k\int_{\pi_q^{-1}} g(\xi,H)*1 = -k\int_{\pi_q^{-1}}(\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H), \tag{4.239}$$

that is, the fiber is $\mathbb{H}-minimal$. □

Theorem 28. *Let π be a pointwise semi-slant submersion with compact fibers from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). If $H \in \mathbb{L}$, then*
(a) *π_q^{-1} is $\mathbb{L}-minimal$ if and only if π_q^{-1} is minimal.*
(b) *π_q^{-1} is $\mathbb{E}-minimal$ if and only if σ_H is a harmonic variation.*
(c) *π_q^{-1} is $\mathbb{H}-minimal$ if and only if σ_H is an exact 1-form.*

Proof. **(a)** If the fiber π_q^{-1} is $\mathbb{L}-minimal$, then by Theorem 27-(a) we have, σ_H is co-exact. Hence σ_H is co-closed. Taking into account the fact that $d\sigma_H = 0$, we deduce that σ_H is harmonic. But this is a contradiction because of Hodge decomposition theorem [31]. So, σ_H must be zero. Hence we conclude that $H = 0$. The converse is clear.

(b) $\Rightarrow$: If the fiber π_q^{-1} is $\mathbb{E}-minimal$, then we have $\delta\sigma_H = 0$ from Theorem 27-(b). Since $d\sigma_H = 0$, σ_H is also harmonic, i.e. $\Delta\sigma_H = 0$.

$\Leftarrow$: If σ_H is harmonic, then σ_H is co-closed. By Theorem 27-(b), the fiber π_q^{-1} is $\mathbb{E}-minimal$.

(c) $\Rightarrow$: Assume that π_q^{-1} is $\mathbb{H}-minimal$. Then, from Theorem 27-(c), σ_H is the sum of an exact 1-form and a co-exact 1-form. On the other hand, the condition $H \in \mathbb{L}$ implies that σ_H is orthogonal to every co-exact 1-form on π_q^{-1}. Thus, σ_H must be exact.

$\Leftarrow$: Let σ_H be an exact 1-form. For $\xi \in \mathbb{H}$, we obtain

$$\begin{aligned}\mathbf{V}'(\xi) =& -k\int_{\pi_q^{-1}} g(\xi,H)*1 = -k\int_{\pi_q^{-1}}(\sigma_\xi \wedge *\sigma_H)\\ =& -k(\sigma_\xi|\sigma_H) = (\sigma_\xi|df) = (\delta\sigma_\xi|f) = 0,\end{aligned} \tag{4.240}$$

that is, π_q^{-1} is $\mathbb{H}-minimal$. □

Remark 10. *The method that considering the basis to investigate the harmonicity of a submersion, while the total manifold is taken as a l.p.R. manifold, is not always easy. Since a l.p.R. manifold is not always even dimensional, choosing a basis and using it is not easy. On the other hand, it is well known that, the fibers of a submersion is minimal if and only if the submersion is harmonic. Now, we give a new approach for harmonicity of a pointwise semi-slant submersion. By Theorem 28-(a), we obtain the following result.*

Corollary 11. *Let π be a pointwise semi-slant submersion with compact fibers from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). If $H \in \mathbb{L}$, then π is harmonic if and only if π_q^{-1} is $\mathbb{L}-minimal$.*

Lemma 23. *Let π be a pointwise semi-slant submersion with compact fibers from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then,*

$$\delta\sigma_H = 0 \Leftrightarrow \Sigma_i g(T_{\phi E_i}E_i, H) = \Sigma_i g(A_{\omega E_i}E_i, H), \tag{4.241}$$

where $\{E_1, E_2, ..., E_m\}$ is a local basis of D_θ.

Proof.

$$\delta\sigma_H = 0 \Leftrightarrow \Sigma_i g(\nabla_{E_i}FH, E_i) = 0. \tag{4.242}$$

Using (2.33),

$$\begin{aligned} \Rightarrow \delta\sigma_H = 0 \Leftrightarrow & \Sigma_i g(\nabla_{E_i}H, FE_i) \Leftrightarrow \Sigma_i g(\nabla_{E_i}H, \phi E_i + \omega E_i) \\ = & \Sigma_i g(\nabla_{E_i}H, \phi E_i) + \Sigma_i g(\nabla_{E_i}H, \omega E_i) \\ = & \Sigma_i g(T_{E_i}H, \phi E_i) + \Sigma_i g(A_H E_i, \omega E_i). \end{aligned} \tag{4.243}$$

Thus, the assertion follows from the skew-symmetry and symmetry properties of the O'Neill tensors A and T. □

4.4 Generic Submersions

This section is the main part of our thesis. Until this section, we improved our knowledge about invariant, anti-invariant and pointwise slant distributions. And now, we construct a generalization for the Riemannian submersions and study on it. We define generic submersion from Kaehler manifolds onto Riemannian manifolds.

Let $\bar{M}$ be an almost Hermitian manifold with Riemannian metric g and almost complex structure J, and M be a Riemannian manifold isometrically immersed in $\bar{M}$. For any $\bar{V} \in \Gamma(TM)$, we write

$$J\bar{V} = P\bar{V} + F\bar{V}, \tag{4.244}$$

where $P\bar{V} \in \Gamma(TM)$ and $F\bar{V} \in \Gamma(TM^{\perp})$. By (3.1) and (4.244), we have $g(P^2\bar{U},\bar{V}) = g(\bar{U},P^2\bar{V})$ for all $\bar{U},\bar{V} \in \Gamma(TM)$. It means that P^2 is symmetric operator on the tangent space $T_pM, p \in M$. Therefore its eigenvalues are real and diagonalizable. Moreover, its eigenvalues are in the closed interval $[-1,0]$. For each point $p \in M$, we may set

$$D_p^{\lambda} = ker\{P^2 + \lambda^2(p)I\}_p \tag{4.245}$$

where I is the identity transformation and $\lambda(p)$ belongs to the closed real interval $[0,1]$ such that $-\lambda^2(p)$ is an eigenvalue of P_p^2. Since P^2 is symmetric and diagonalizable, there is some integer k such that $-\lambda_1^2(p), -\lambda_2^2(p), ..., -\lambda_k^2(p)$ are distinct eigenvalues of P^2 and T_pM can be decomposed as the direct sum of the mutually orthogonal $P-$invariant eigenspaces, i.e.

$$T_pM = D_p^{\lambda_1} \oplus D_p^{\lambda_2} \oplus ... \oplus D_p^{\lambda_k}. \tag{4.246}$$

Note that $D_p^1 = kerF_p$ and $D_p^0 = kerP_p$. Here D_p^1 is the maximal $J-$invariant subspace of T_pM and D_p^0 is the maximal anti$-J-$invariant subspace of T_pM.
Ronsse defined the generic and skew CR-submanifolds of an almost Hermitian manifold as follows [18].

Definition 9. *[18] A submanifold M of an almost Hermitian manifold $\bar{M}$ is called a* generic submanifold *of M if there are k functions $\lambda_1, \lambda_2, ..., \lambda_k$ defined on M with values in the open interval $(0,1)$ such that the following two conditions hold:*

- *$-\lambda_1^2, ..., -\lambda_k^2$ are distinct eigenvalues of P^2 at $p \in M$ with*

$$T_pM = D_p^1 \oplus D_p^0 \oplus D_p^{\lambda_1} \oplus D_p^{\lambda_2} \oplus ... \oplus D_p^{\lambda_k}, \tag{4.247}$$

  where $D_p^1 = kerF_p$, $D_p^0 = kerP_p$ and $D_p^{\lambda_i} = ker(P^2 + \lambda_i^2(p)I)_p$, $i \in \overline{1,k}$,

- *the dimensions of $D_p^1, D_p^0, D_p^{\lambda_1}, ..., D_p^{\lambda_k}$ are independent of $p \in M$.*

Moreover, if each λ_i is constant, then M is called a skew CR-submanifold.

It is seen that, the distributions D_p^0, D_p^1 and $D_p^{\lambda_i}$ in Definition 9 state the same idea with *totally real*, *holomorphic* [34] and *pointwise slant distribution* [35], respectively.
Note that, such submanifolds were also studied by Tripathi [36] for generalized complex space forms.
We construct a new type of submersion, which is generalization of all kinds of submersions, by considering the idea of Ronsse [18].

Definition 10. *Let (M,J,g) be an almost Hermitian manifold, (N,g_N) be a Riemannian manifold and $\pi : (M,J,g) \mapsto (N,g_N)$ be a Riemannian submersion. Then, we say that π is a generic submersion if the fibers of the submersion π are generic submanifold (in the sense of Ronsse [18]) of M.*

Remark 11. *To be more clear, in this thesis generic submersion means generic submersion in the sense of Ronsse.*

In this case, there are k functions $\lambda_1, \lambda_2, ..., \lambda_k$ defined on the fibers with values in the open interval $(0,1)$ such that $ker\pi_*$ is decomposed as

$$ker\pi_* = D^1 \oplus D^0 \oplus D^{\lambda_1} \oplus D^{\lambda_2} \oplus ... \oplus D^{\lambda_k}, \tag{4.248}$$

where D^1 is invariant, D^0 is anti-invariant, D^{λ_i} is pointwise slant distribution with slant function θ_i and $-\lambda_i^2$ is a distinct eigenvalue of P^2 for each $i = \overline{1,k}$.
If each λ_i is a constant for $i = \overline{1,k}$, then π is called a *skew CR-submersion*.

Remark 12. *Each distribution D^{λ_i} has the slant function θ_i for $i = \overline{1,k}$. Since for any unit vector $Z_i \in \Gamma(D^{\lambda_i})$, $g(PZ_i, JZ_i) = g(PZ_i, PZ_i) = \theta_i^2$ it is known that $-\lambda_i^2 = -\cos^2\theta_i$. From now on, to avoid confusion, we denote the distributions D^{λ_i} by D^{θ_i} for $i = \overline{1,k}$.*

In the view of Remark 12, the decomposition of $\ker\pi_*$ can be written as follows:

$$ker\pi_* = D^1 \oplus D^0 \oplus D^{\theta_1} \oplus D^{\theta_2} \oplus ... \oplus D^{\theta_k} \tag{4.249}$$

where D^1 is invariant, D^0 is anti-invariant, D^{θ_i} is pointwise slant distribution with slant function θ_i for $i = \overline{1,k}$.
Let π be a generic submersion from an almost Hermitian manifold (M,g,J) onto a Riemannian manifold (N,g_N). Then, for $V \in \Gamma(ker\pi_*)$, we set

$$JV = PV + FV, \tag{4.250}$$

where $PV \in \Gamma(ker\pi_*)$ and $FV \in \Gamma(ker\pi_*^{\perp})$. Also, for $\xi \in \Gamma(ker\pi_*^{\perp})$, we put

$$J\xi = t\xi + f\xi, \tag{4.251}$$

where $t\xi \in \Gamma(ker\pi_*)$ and $f\xi \in \Gamma(ker\pi_*^{\perp})$. Therefore, the horizontal distribution $ker\pi_*^{\perp}$ can be decomposed as

$$ker\pi_*^{\perp} = \underline{D}^1 \oplus \underline{D}^0 \oplus \underline{D}^{\theta_1} \oplus \underline{D}^{\theta_2} \oplus ... \oplus \underline{D}^{\theta_k}, \tag{4.252}$$

where $\underline{D}^1 = Ker(t)$, $\underline{D}^0 = Ker(f)$, $FD^{\theta} = \underline{D}^{\theta}$ and $t\underline{D}^{\theta} = D^{\theta}$, $\theta \in \{\theta_1, \theta_2, ..., \theta_k\}$.

Remark 13. *By defining generic submersion, we give a generalization for submersions from an almost Hermitian manifold onto a Riemannian submersion. Here are the some generalizations:*
A generic submersion from an almost Hermitian manifold becomes

- *an anti-invariant submersion [6] if $k = 0$ and $D^1 = \{0\}$,*
- *a semi-invariant submersion [7] if $k = 0$,*
- *a hemi-slant submersion [11] if $D^1 = \{0\}$ and $k = 1$ (θ_1 is constant),*
- *a proper slant submersion [9] if $D^1 = \{0\}$, $D^0 = \{0\}$ and $k = 1$ (θ_1 is constant),*
- *a proper semi-slant submersion [37] if $D^0 = \{0\}$ and $k = 1$ (θ_1 is constant),*
- *a proper pointwise slant submersion [10] if $D^1 = \{0\}$, $D^0 = \{0\}$ and $k = 1$,*

Example 5. *Let $\mathbb{R}^8$ be 8-dimensional Euclidean space. We choose Kaehler structure on $\mathbb{R}^8$. Namely, $(\mathbb{R}^8, g, J)$ is a Kaehler manifold with Eucliedan metric g on $\mathbb{R}^8$ and canonical complex structure J. Consider the map*
$\pi : \mathbb{R}^8 \longmapsto \mathbb{R}^3$ defined by

$$\pi(x_1, ..., x_8) = \Big(\frac{x_1 - x_4}{\sqrt{2}}, x_2, x_5\Big) \tag{4.253}$$

Then, the Jacobian matrix of π is:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{4.254}$$

Since the rank of this matrix is equal to 3, the map π is a submersion. By direct calculation, we observe that

$$ker\pi_* = D^1 \oplus D^0 \oplus D^\theta \tag{4.255}$$

where

$$D^1 = span\left\{ \partial_7, \partial_8 \right\} \tag{4.256}$$

$$D^0 = span\left\{ \partial_6 \right\} \tag{4.257}$$

$$\tag{4.258}$$

and

$$D^\theta = span\left\{ \frac{1}{\sqrt{2}}(\partial_1 + \partial_4), \partial_3 \right\}. \tag{4.259}$$

*Moreover, the slant function of D^θ is $\theta = \frac{\pi}{4}$. After some calculations, we see that π satisfies the condition **(S2)**, which is in the definition of Riemannian submersion. Therefore, the map π is a skew CR-submersion with the constant slant function θ.*

Example 6. *Let $\mathbb{R}^{4k+6}$ be $(4k+6)$-dimensional Euclidean space. We choose the usual Kaehler structure on $\mathbb{R}^{4k+6}$. Namely, $(\mathbb{R}^{4k+6}, g, J)$ is a Kaehler manifold with Euclidean metric g on $\mathbb{R}^{4k+6}$ and canonical complex structure J. Consider the map $\pi : \mathbb{R}^{4k+6} \longmapsto \mathbb{R}^{2k+3}$ defined by*

$$\begin{aligned}
&\pi(x_1, x_2, ..., x_{4k+6}) = (f_1, f_2, ..., f_{2k+3}) \\
&f_1 = \frac{x_1 + x_2}{\sqrt{2}}, \\
&f_2 = \frac{x_3 + x_5}{\sqrt{2}}, \\
&f_3 = \frac{x_4 + x_6}{\sqrt{2}}, \\
&f_4 = \cos(x_7) - \sin(x_{10}), \\
&f_5 = x_8, \\
&\vdots \\
&f_{2i+2} = \cos(x_{4i+3}) - \sin(x_{4i+6}), \\
&f_{2i+3} = x_{4i+4}, \\
&\vdots \\
&f_{2k+2} = \cos(x_{4k+3}) - \sin(x_{4k+6}), \\
&f_{2k+3} = x_{4k+4},
\end{aligned} \tag{4.260}$$

where $x_{4i+4} \neq x_{4j+4}$ *for* $i \neq j$. *Since the Jacobian matrix of* π *is of rank* $2k+3$, *the map* π *is a submersion. After some calculations, for* $i = \overline{1,k}$ *we see that*

$$ker\pi_* = D^1 \oplus D^0 \oplus D^{\theta_1} \oplus D^{\theta_2} \oplus \ldots \oplus D^{\theta_k}, \tag{4.261}$$

where,

$$D^1 = span\left\{X = \frac{1}{\sqrt{2}}(-\partial_3 + \partial_5), Y = \frac{1}{\sqrt{2}}(-\partial_4 + \partial_6)\right\}, \tag{4.262}$$

$$D^0 = span\left\{V = \frac{1}{\sqrt{2}}(-\partial_1 + \partial_2)\right\} \tag{4.263}$$

and

$$D^{\theta_i} = span\left\{Z_i = -\sin(x_{4i+3})\,\partial_{4i+3} + \cos(x_{4i+6})\,\partial_{4i+6}, W_i = \partial_{4i+5}\right\}. \tag{4.264}$$

Moreover, the slant function of the pointwise slant distribution D^{θ_i} *is* $\theta_i = x_{4i+4}$ *for* $i = \overline{1,k}$. *By direct calculation, we observe that* π *satisties condition* ***(S2)***. *Hence the map* π *is a generic submersion.*

Example 7. *Let* $\mathbb{R}^{10}$ *be 10-dimensional Euclidean space. Define the map* $\pi : \mathbb{R}^{10} \longmapsto \mathbb{R}^5$ *as follows*

$$\pi(x_1, x_2, \ldots, x_{10}) = \left(\cos x_1 - \sin x_4, x_2, \frac{x_5 + x_6}{\sqrt{2}}, \frac{x_7 + x_9}{\sqrt{2}}, \frac{x_8 + x_{10}}{\sqrt{2}}\right). \tag{4.265}$$

Then the map π *is a generic submersion with slant function* $\theta_1 = x_4$ *such that*

$$ker\pi_* = D^1 \oplus D^0 \oplus D^{\theta_1}, \tag{4.266}$$

where

$$D^1 = span\left\{X = \frac{1}{\sqrt{2}}(-\partial_7 + \partial_9), Y = \frac{1}{\sqrt{2}}(-\partial_8 + \partial_{10})\right\}, \tag{4.267}$$

$$D^0 = span\left\{V = \frac{1}{\sqrt{2}}(-\partial_5 + \partial_6)\right\} \tag{4.268}$$

and

$$D^{\theta_1} = span\left\{Z = -\sin(x_1)\,\partial_1 + \cos(x_4)\,\partial_4, W = \partial_3\right\}. \tag{4.269}$$

We give some useful identities, which are obtained by means of the definition of generic submersion and the complex structure of Kaehler manifold.

Lemma 24. *Let π be a generic submersion from an almost Hermitian manifold (M,g,J) onto a Riemannian manifold (N,g_N). Then we have,*

$$\textbf{(a)}\ P^2+tF=-I, \qquad \textbf{(b)}\ FP+fF=0,$$

$$\textbf{(c)}\ Pt+tf=0, \qquad \textbf{(d)}\ Ft+f^2=-I,$$

where I is the identity operator on TM.

Proof. All the identities could be obtain by simple calculations with the help of (3.1), (3.2), (4.250) and (4.251). □

If we consider some vector fields specific in Lemma 24, which are involved in the definition of generic submanifold, we get the following Corollary.

Corollary 12. *Let π be a generic submersion from an almost Hermitian manifold (M,g,J) onto a Riemannian manifold (N,g_N). Then we have*

$$\textbf{(a)}\ P^2X=-X, \qquad \textbf{(b)}\ tFZ_i=-\sin^2\theta_i Z_i,$$

$$\textbf{(c)}\ FPX=0, \qquad \textbf{(d)}\ fFU=0,$$

$$\textbf{(e)}\ tFU=-U, \qquad \textbf{(f)}\ P^2Z_i+tFZ_i=-Z_i,$$

$$\textbf{(g)}\ FPZ_i+fFZ_i=0, \qquad \textbf{(h)}\ P^2Z_i=-\cos^2\theta_i Z_i,$$

where $X\in\Gamma(D^1)$, $U\in\Gamma(D^0)$, $Z_i,W_i\in\Gamma(D^{\theta_i})$ for $i=\overline{1,k}$.

By considering the decomposition of $ker\pi_*$ and $ker\pi_*^{\perp}$ with the equations (4.250) and (4.251), we obtain the following Lemma.

Lemma 25. *Let π be a generic submersion from an almost Hermitian manifold (M,g,J) onto a Riemannian manifold (N,g_N). Then, for $i=\overline{1,k}$ we have*

$$PD^1=D^1, \quad PD^0=\{0\}, \quad PD^{\theta_i}\subseteq D^{\theta_i}, \tag{4.270}$$

$$t\underline{D}^0\subseteq D^0, \quad t\underline{D}^1=\{0\}, \quad t\underline{D}^{\theta_i}\subseteq D^{\theta_i}, \tag{4.271}$$

$$f\underline{D}^{\theta_i}\subseteq \underline{D}^{\theta_i}, \quad fD^0=\{0\}, \quad f\underline{D}^1=\underline{D}^1, \tag{4.272}$$

$$FD^1=\{0\}. \tag{4.273}$$

From now on, we will focus on the generic submersions whose total manifolds are Kaehler manifolds. We start by examining the effect of the complex structure J on the O'Neill tensor fields T and A and get a lot of results.

Lemma 26. *Let π be a generic submersion from a Kaehler manifold (M,g,J) onto a Riemannian manifold (N,g_N). Then, we obtain*

$$tT_{E_1}E_2+P\hat{\nabla}_{E_1}E_2 = T_{E_1}FE_2+\hat{\nabla}_{E_1}PE_2, \tag{4.274}$$

$$fT_{E_1}E_2+F\hat{\nabla}_{E_1}E_2 = T_{E_1}PE_2+A_{FE_2}E_1, \tag{4.275}$$

$$\hat{\nabla}_E t\xi+T_E f\xi = PT_E\xi+tA_\xi E, \tag{4.276}$$

$$T_E t\xi+A_{f\xi}E = FT_E\xi+fA_\xi E, \tag{4.277}$$

where $E,E_1,E_2\in\Gamma(ker\pi_*)$ and $\xi\in\Gamma(ker\pi_*^\perp)$.

Proof. By the definition of a Kaehler manifold, for any $E_1,E_2\in\Gamma(ker\pi_*)$, we have

$$J\nabla_{E_1}E_2=\nabla_{E_1}JE_2. \tag{4.278}$$

With the help of (4.5) and (4.250), we get

$$\Rightarrow J(T_{E_1}E_2+\hat{\nabla}_{E_1}E_2)=\nabla_{E_1}PE_2+\nabla_{E_1}FE_2. \tag{4.279}$$

Now, by (4.5), (4.6), (4.250) and (4.251), we obtain

$$\begin{aligned}\Rightarrow tT_{E_1}E_2+fT_{E_1}E_2+P\hat{\nabla}_{E_1}E_2+F\hat{\nabla}_{E_1}E_2 &= T_{E_1}PE_2+\hat{\nabla}_{E_1}PE_2\\ &+ T_{E_1}FE_2+\mathscr{H}\nabla_{E_1}FE_2.\end{aligned} \tag{4.280}$$

In the view of Remark 1, by separating the last equation into the horizontal and vertical parts, we obtain the assertions (4.274) and (4.275). To get (4.276) and (4.277), the same idea should be applied. □

Now, we give remarkable lemmas which are equivalent to Gauss and Weingarten equations for generic submersions.

Lemma 27. *Let π be a generic submersion from a Kaehler manifold (M,g,J) onto a Riemannian manifold (N,g_N). Then, we have the following equations*

$$g(\nabla_X Y,Z_i)=\csc^2\theta_i g(T_Y FPZ_i-T_{PY}FZ_i,X), \tag{4.281}$$

$$g(\nabla_X Y,U)=-g(T_{PY}FU,X), \tag{4.282}$$

$$g(\nabla_X U,Y)=g(T_{PY}FU,X), \tag{4.283}$$

where $X,Y\in\Gamma(D^1)$, $U\in\Gamma(D^0)$ and $Z_i\in\Gamma(D^{\theta_i})$ for $i=\overline{1,k}$.

Proof. Let $X,Y \in \Gamma(D^1)$ and $Z_i \in \Gamma(D^{\theta_i})$ for $i=\overline{1,k}$. From (3.1), (3.2), (4.250) and Lemma 25, we have

$$\begin{aligned} g(\nabla_X Y, Z_i) &= g(\nabla_X JY, JZ_i) \\ &= g(\nabla_X JY, PZ_i) + g(\nabla_X JY, FZ_i) \\ &= -g(\nabla_X Y, P^2 Z_i) - g(\nabla_X Y, FPZ_i) + g(\nabla_X PY, FZ_i). \end{aligned} \tag{4.284}$$

Then, by Corollary 12 and (4.5), we get (4.281)

$$\begin{aligned} g(\nabla_X Y, Z_i) &= -g(\nabla_X Y, -\cos^2\theta_i Z_i) - g(T_X Y, FPZ_i) + g(T_X PY, FZ_i) \\ &\Rightarrow \sin^2\theta_i g(\nabla_X Y, Z_i) = g(T_X PY, FZ_i) - g(T_X Y, FPZ_i) \\ &\Rightarrow g(\nabla_X Y, Z_i) = \csc^2\theta_i g(T_Y FPZ_i - T_{PY} FZ_i, X). \end{aligned} \tag{4.285}$$

To prove (4.282), let $X,Y \in \Gamma(D^1)$ and $U \in \Gamma(D^0)$. By (3.1), (3.2), (4.250), Proposition 25 and the properties of T, we prove the following

$$\begin{aligned} g(\nabla_X Y, U) &= g(\nabla_X JY, JU) = g(\nabla_X PY, FU) \\ &= g(T_X PY, FU) = -g(T_{PY} FU, X). \end{aligned} \tag{4.286}$$

Finally, for the assertion (4.283), we could apply the similar idea which is used above.

□

Lemma 28. *Let π be a generic submersion from a Kaehler manifold (M,g,J) onto a Riemannian manifold (N,g_N). Then, we have*

$$g(\nabla_U V, X) = g(T_{PX} FV, U), \tag{4.287}$$

$$g(\nabla_U V, Z_i) = \csc^2\theta_i \left[g(A_{FV} U, FZ_i) - g(T_U V, FPZ_i) \right], \tag{4.288}$$

$$g(\nabla_U X, Z_i) = \csc^2\theta_i \, g(T_X FPZ_i - T_{PX} FZ_i, U), \tag{4.289}$$

where $X \in \Gamma(D^1)$, $U,V \in \Gamma(D^0)$ and $Z_i \in \Gamma(D^{\theta_i})$ for $i=\overline{1,k}$.

Proof. Let $X \in \Gamma(D^1)$ and $U,V \in \Gamma(D^0)$. Then, by Lemma 25, $JV = FV$ and $JX = PX$. So, by (3.1) and (3.2) we obtain

$$g(\nabla_U V, X) = g(\nabla_U FV, PX). \tag{4.290}$$

If we consider (4.5) and the properties of the tensor T, we get

$$g(\nabla_U V, X) = g(T_{PX} FV, U). \tag{4.291}$$

To obtain (4.288), assume that $U,V \in \Gamma(D^0)$ and $Z_i \in \Gamma(D^{\theta_i})$ for $i = \overline{1,k}$. If we use (3.1), (3.2), (4.250), (4.5), (4.6), Lemma 25 and the properties of the tensor T with the fact that FV can be considered basic, then we get

$$\begin{aligned} g(\nabla_U V, Z_i) &= g(\nabla_U JV, JZ_i) \\ &= g(\nabla_U JV, PZ_i) + g(\nabla_U JV, FZ_i) \\ &= -g(\nabla_U V, P^2 Z_i) - g(\nabla_U V, FPZ_i) + g(\nabla_U FV, FZ_i) \\ &= g(\nabla_U V, \cos^2\theta_i Z_i) - g(T_U V, FPZ_i) + g(\mathscr{H}\nabla_U FV, FZ_i) \quad (4.292) \end{aligned}$$

$$\begin{aligned} &\Rightarrow \quad (1-\cos^2\theta_i) g(\nabla_U V, Z_i) = g(A_{FV}U, FZ_i) - g(T_U V, FPZ_i) \\ &\Rightarrow \quad \sin^2\theta_i g(\nabla_U V, Z_i) = -g(A_{FV} FZ_i, U) + g(T_V FPZ_i, U). \quad (4.293) \end{aligned}$$

Thus, we obtain (4.288). To get (4.289), we use the same idea with the proof of (4.288). □

Lemma 29. *Let π be a generic submersion from a Kaehler manifold (M,g,J) onto a Riemannian manifold (N,g_N). Then, we obtain*

$$g(\nabla_{Z_i} W_i, X) = -\csc^2\theta_i\, g(T_{PX} FW_i - T_X fPW_i, Z_i), \quad (4.294)$$

$$g(\nabla_{Z_i} W_i, U) = -\csc^2\theta_i g(T_U FPW_i + A_{FW_i} FU, Z_i), \quad (4.295)$$

$$g(\nabla_{Z_i} W_i, Z_j) = \csc^2\theta_i g(T_{PZ_j} FW_i - T_{Z_j} FPW_i - A_{FW_i} FZ_j, Z_i), \quad (4.296)$$

where $X \in \Gamma(D^1)$, $U \in \Gamma(D^0)$, $Z_i, W_i \in \Gamma(D^{\theta_i})$ and $Z_j \in \Gamma(D^{\theta_j})$ $(i \neq j)$ $i,j = \overline{1,k}$.

Proof. Let $X \in \Gamma(D^1)$ and $Z_i, W_i \in \Gamma(D^{\theta_i})$ for $i = \overline{1,k}$ so $g(W_i, X) = 0$. From (3.1), (3.2), (4.6), (4.250), Corollary 12 and (4.270), we have

$$\begin{aligned} g(\nabla_{Z_i} W_i, X) &= g(\nabla_{Z_i} JW_i, JX) \\ &= g(\nabla_{Z_i} PW_i, JX) + g(\nabla_{Z_i} FW_i, JX) \\ &= -g(\nabla_{Z_i} JPW_i, X) + g(T_{Z_i} FW_i, PX) \\ &= -g(\nabla_{Z_i} P^2 W_i, X) - g(\nabla_{Z_i} FPW_i, X) + g(T_{Z_i} FW_i, PX) \\ &= g(\nabla_{Z_i}(\cos^2\theta_i W_i), X) - g(T_{Z_i} FPW_i, X) + g(T_{Z_i} FW_i, PX) \\ &= -g(\sin 2\theta_i (Z_i\theta_i) W_i, X) + \cos^2\theta_i g(\nabla_{Z_i} W_i, X) \\ &- g(T_{Z_i} FPW_i, X) + g(T_{Z_i} FW_i, PX) \\ &= \cos^2\theta_i g(\nabla_{Z_i} W_i, X) + g(T_{Z_i} FW_i, PX) - g(T_{Z_i} PX, FW_i). \quad (4.297) \end{aligned}$$

$$\Rightarrow (1-\cos^2\theta_i)g(\nabla_{Z_i}W_i, X) = g(T_{Z_i}FW_i, PX) - g(T_{Z_i}PX, FW_i). \tag{4.298}$$

So, we obtain (4.294). To obtain (4.295) and (4.296), the same method can be used.

□

4.4.1 Integrability of distributions

In this subsection, some conditions are given for the integrability of distributions which are mentioned in the definition of generic submersion. First, we give some helpful lemmas.

Lemma 30. *Let π be a generic submersion from a Kaehler manifold (M, g, J) onto a Riemannian manifold (N, g_N). Then, we obtain*

$$g(A_{FV}FZ_i, U) = g(A_{FU}FZ_i, V) \tag{4.299}$$

for $U, V \in \Gamma(D^0)$ and $Z_i \in \Gamma(D^{\theta_i})$ for $i = \overline{1,k}$.

Proof. For $U, V \in \Gamma(D^0)$ and $Z_i \in \Gamma(D^{\theta_i})$, if we consider (3.1), (3.2), (4.27), (4.8), Lemma 25 and skew-symmetry property of tensor A, we get

$$\begin{aligned} g(A_{FU}FZ_i, V) &= -g(A_{FZ_i}FU, V) = -g(\nabla_{FZ_i}FU, V) = -g(\nabla_{FZ_i}JU, V) \\ &= g(\nabla_{FZ_i}U, JV) = g(\nabla_{JZ_i}U, FV) = g(A_{FZ_i}U, FV) \\ &= -g(A_{FZ_i}FV, U) = g(A_{FU}FZ_i, V). \end{aligned} \tag{4.300}$$

□

Lemma 31. *Let π be a generic submersion from a Kaehler manifold (M, g, J) onto a Riemannian manifold (N, g_N). Then we obtain*

$$g(T_{PX}FV, U) = g(T_{PX}FU, V) \tag{4.301}$$

for $U, V \in \Gamma(D^0)$ and $X \in \Gamma(D^1)$.

Proof. For $U, V \in \Gamma(D^0)$ and $X \in \Gamma(D^1)$, if we consider (3.1), (3.2), (4.5), (4.6), Lemma 25 and the properties of T, we have

$$\begin{aligned} g(T_{PX}FV, U) &= -g(T_{PX}U, FV) = -g(\nabla_{PX}U, JV) = g(\nabla_{PX}JU, V) \\ &= g(\nabla_{PX}FU, V) = g(T_{PX}FU, V). \end{aligned} \tag{4.302}$$

□

We examine the integrability of the distributions D^0, D^1 and D^{θ_i} for $i=\overline{1,k}$. Since the second fundamental form of the fibers of a generic submersions is T and the fibers of submersions are CR-submanifolds, following conclusions could be obtained from Lemma 1.1 of [18].

Theorem 29. *Let π be a generic submersion from a Kaehler manifold (M,g,J) onto a Riemannian manifold (N,g_N). Then, D^0 is always integrable.*

Proof. Let $X\in\Gamma(D^1)$, $U,V\in\Gamma(D^0)$ and $Z_i\in\Gamma(D^{\theta_i})$ for $i=\overline{1,k}$. With the help of (4.287), (4.288) Lemma 30 and Lemma 31 we get the assertion as in the following:

$$\begin{aligned} g([U,V],X) &= g(\nabla_U V,X)-g(\nabla_V U,X)\\ &= g(T_{PX}FV,U)-g(T_{PX}FU,V)=0\\ g([U,V],Z_i) &= g(\nabla_U V,Z_i)-g(\nabla_V U,Z_i)\\ &= \csc^2\theta_i\{g(T_V FPZ_i-A_{FV}FZ_i,U)-g(T_U FPZ_i-A_{FU}FZ_i,V)\}\\ &= \csc^2\theta_i\{g(A_{FV}FZ_i,U)-g(A_{FU}FZ_i,V)\}=0. \end{aligned} \tag{4.303}$$

□

Theorem 30. *Let π be a generic submersion from a Kaehler manifold (M,g,J) onto a Riemannian manifold (N,g_N). Then, D^1 is integrable if and only if the following two conditions hold:*

$$g(T_{PY}X-T_{PX}Y,FU)=0, \tag{4.304}$$

$$g(T_{PY}X-T_{PX}Y,FZ_i)=0 \tag{4.305}$$

for $X,Y\in\Gamma(D^1)$, $U\in\Gamma(D^0)$ and $Z_i\in\Gamma(D^{\theta_i})$ for $i=\overline{1,k}$.

Proof. Let $X,Y\in\Gamma(D^1)$, $U\in\Gamma(D^0)$ and $Z_i\in\Gamma(D^{\theta_i})$ for $i=\overline{1,k}$. Then, by (4.282) and the properties of tensor field T, we get

$$\begin{aligned} g([X,Y],U) &= g(\nabla_X Y,U)-g(\nabla_Y X,U)\\ &= -g(T_{PY}FU,X)+g(T_{PX}FU,Y)\\ &= g(T_{PY}X-T_{PX}Y,FU). \end{aligned} \tag{4.306}$$

So, we obtain the first condition. For the second condition, we apply the same idea; by (4.281) and the properties of tensor field T, we get

$$\begin{aligned} g([X,Y],Z_i) &= g(\nabla_X Y, Z_i) - g(\nabla_Y X, Z_i) \\ &= \csc^2\theta_i\{g(T_Y FPZ_i - T_{PY}FZ_i, X) - g(T_X FPZ_i - T_{PX}FZ_i, Y)\} \\ &= \csc^2\theta_i\{g(T_{PX}FZ_i, Y) - g(T_{PY}FZ_i, X)\} \\ &= \csc^2\theta_i\{g(T_{PY}X - T_{PX}Y, FZ_i)\}. \end{aligned} \tag{4.307}$$

Therefore we have the assertion. □

Theorem 31. *Let π be a generic submersion from a Kaehler manifold (M,g,J) onto a Riemannian manifold (N,g_N). Then, D^{θ_i} for each $i=\overline{1,k}$ is integrable if and only if the following three conditions hold:*

$$g(T_{PX}FW_i - T_X FPW_i, Z_i) = g(T_{PX}FZ_i - T_X FPZ_i, W_i) \tag{4.308}$$

$$g(T_U FPW_i + A_{FW_i}FU, Z_i) = g(T_U FPZ_i + A_{FZ_i}FU, W_i) \tag{4.309}$$

$$\begin{aligned} &g(T_{PZ_j}FW_i - T_{Z_j}FPW_i - A_{FW_i}FZ_j, Z_i) \\ &= g(T_{PZ_j}FZ_i - T_{Z_j}FPZ_i - A_{FZ_i}FZ_j, W_i) \end{aligned} \tag{4.310}$$

for $X \in \Gamma(D^1)$, $U \in \Gamma(D^0)$, $Z_i, W_i \in \Gamma(D^{\theta_i})$ and $Z_j \in \Gamma(D^{\theta_j}),(i \neq j)$ $i,j=\overline{1,k}$.

Proof. Let $X \in \Gamma(D^1)$, $U \in \Gamma(D^0)$, $Z_i, W_i \in \Gamma(D^{\theta_i})$ and $Z_j \in \Gamma(D^{\theta_j}),(i \neq j)$ $i,j=\overline{1,k}$. By (4.294), (4.295), (4.296) and Remark 1, we get following three equalities

$$\begin{aligned} g([Z_i,W_i],X) &= g(\nabla_{Z_i}W_i, X) - g(\nabla_{W_i}Z_i, X) \\ &= \csc^2\theta_i\big\{g(T_{PX}FW_i - T_X FPW_i, Z_i) \\ &- g(T_{PX}FZ_i - T_X FPZ_i, W_i)\big\}. \end{aligned} \tag{4.311}$$

$$\begin{aligned} g([Z_i,W_i],U) &= g(\nabla_{Z_i}W_i, U) - g(\nabla_{W_i}Z_i, U) \\ &= -\csc^2\theta_i\big\{g(T_U FPW_i + A_{FW_i}FU, Z_i) \\ &- T_U FPZ_i + A_{FZ_i}FU, W_i)\big\} \end{aligned} \tag{4.312}$$

$$\begin{aligned} g([Z_i,W_i],Z_j) &= g(\nabla_{Z_i}W_i, Z_j) - g(\nabla_{W_i}Z_i, Z_j) \\ &= \csc^2\theta_i\big\{g(T_{PZ_j}FW_i - T_{Z_j}FPW_i - A_{FW_i}FZ_j, Z_i) \\ &- g(T_{PZ_j}FZ_i - T_{Z_j}FPZ_i - A_{FZ_i}FZ_j, W_i)\big\}. \end{aligned} \tag{4.313}$$

So, with the help of last three equalities, we conclude that D^{θ_i} is integrable if and only if (4.308)∼(4.310) hold. □

4.4.2 Totally geodesicness of the fibers

We investigate the geometry of the fibers for a generic submersion. Some conditions are given for totally geodesicness.

Theorem 32. *Let π be a generic submersion from a Kaehler manifold (M,g,J) onto a Riemannian manifold (N,g_N). Then, the vertical distribution $ker\pi_*$ defines a totally geodesic foliation if and only if*

$$f(T_{E_1}PE_2 + A_{FE_2}E_1) + F(\hat{\nabla}_{E_1}PE_2 + T_{E_1}FE_2) = 0 \tag{4.314}$$

for $E_1, E_2 \in \Gamma(ker\pi_*)$.

Proof. Let E_1 and E_2 be in $ker\pi_*$. By (3.1), (3.2), (4.5), (4.6), (4.250) and (4.251), we have

$$\begin{aligned}
\nabla_{E_1}E_2 &= -J\nabla_{E_1}JE_2 = -J(\nabla_{E_1}PE_2 + \nabla_{E_1}FE_2) \\
&= -J(T_{E_1}PE_2 + \hat{\nabla}_{E_1}PE_2 + T_{E_1}FE_2 + \mathscr{H}\nabla_{E_1}FE_2) \\
&= -fT_{E_1}PE_2 - tT_{E_1}PE_2 - P\hat{\nabla}_{E_1}PE_2 - F\hat{\nabla}_{E_1}PE_2 \\
&\quad - PT_{E_1}FE_2 - FT_{E_1}FE_2 - t\mathscr{H}\nabla_{E_1}FE_2 - f\mathscr{H}\nabla_{E_1}FE_2.
\end{aligned} \tag{4.315}$$

$ker\pi_*$ defines a totally geodesic foliation if and only if the horizontal part of the last equation vanishes so we obtain the assertion. □

Remark 14. *By (4.4) and (4.8), $A_\xi = 0$ for any horizontal vector field ξ. That means, the integrability and totally geodesicness of the horizontal distribution $ker\pi_*^\perp$ are equal to each other.*

Theorem 33. *Let π be a generic submersion from a Kaehler manifold (M,g,J) onto a Riemannian manifold (N,g_N). Then, $ker\pi_*^\perp$ is totally geodesic (integrable) if and only if*

$$t(A_\xi t\eta + \mathscr{H}\nabla_\xi f\eta) + P(\mathscr{V}\nabla_\xi t\eta + A_\xi f\eta) = 0 \tag{4.316}$$

for $\xi, \eta \in \Gamma(ker\pi_*^\perp)$.

Corollary 13. *Let π be a generic submersion from a Kaehler manifold (M,g,J) onto a Riemannian manifold (N,g_N). Then, M is a locally product $M_{ker\pi_*} \times M_{ker\pi_*^{\perp}}$ if and only if (4.314) and (4.316) hold, where $M_{ker\pi_*}$ and $M_{ker\pi_*^{\perp}}$ are integral manifolds of the distributions $ker\pi_*$ and $ker\pi_*^{\perp}$, respectively.*

It is known that the horizontal distribution $(ker\pi_*)^{\perp}$ defines a totally geodesic foliation if and only if $A \equiv 0$. Also $ker\pi_*$ defines a totally geodesic foliation if and only if $T \equiv 0$. On the other hand, we know that a Riemannian submersion $\pi : (M,g) \mapsto (N,g_N)$ is totally geodesic if and only if both O'Neill tensors T and A vanish, [32]. Thus, by Theorem 32 and Theorem 33, we have the following result.

Corollary 14. *Let π be a generic submersion from a Kaehler manifold (M,g,J) onto a Riemannian manifold (N,g_N). Then, π is a totally geodesic map if and only if (4.314) and (4.316) hold.*

Theorem 34. *Let π be a generic submersion from a Kaehler manifold (M,g,J) onto a Riemannian manifold (N,g_N). Then, the invariant distribution D^1 defines a totally geodesic foliation on* $\ker\pi_*$ *if and only if the following two facts hold:*

$$g(T_{PY}FU,X) = 0, \tag{4.317}$$

$$g(T_Y FPZ_i - T_{PY}FZ_i, X) = 0, \tag{4.318}$$

where $X,Y \in \Gamma(D^1)$, $U \in \Gamma(D^0)$ and $Z_i \in \Gamma(D^{\theta_i})$ for $i = \overline{1,k}$.

Proof. Let $X,Y \in \Gamma(D^1)$, $U \in \Gamma(D^0)$ and $Z_i \in \Gamma(D^{\theta_i})$. Then, by (4.281) and (4.282)

$$g(\hat{\nabla}_X Y,U) = g(\nabla_X Y,U) = -g(T_{PY}FU,X), \tag{4.319}$$

$$\begin{aligned} g(\hat{\nabla}_X Y,Z_i) &= g(\nabla_X Y,Z_i) \\ &= \csc^2\theta_i\{g(T_Y FPZ_i - T_{PY}FZ_i,X)\}. \end{aligned} \tag{4.320}$$

So, we obtain the assertion. □

Theorem 35. *Let π be a generic submersion from a Kaehler manifold (M,g,J) onto a Riemannian manifold (N,g_N). Then, the anti-invariant distribution D^0 defines a totally geodesic foliation on* $\ker\pi_*$ *if and only if the following two conditions hold:*

$$g(T_{PX}FV,U) = 0, \tag{4.321}$$

$$g(T_V FPZ_i - A_{FV}FZ_i, U) = 0, \tag{4.322}$$

where $U,V \in \Gamma(D^0)$, $X \in \Gamma(D^1)$ and $Z_i \in \Gamma(D^{\theta_i})$ for $i = \overline{1,k}$.

Proof. Let $U,V \in \Gamma(D^0)$, $X \in \Gamma(D^1)$ and $Z_i \in \Gamma(D^{\theta_i})$ for $i = \overline{1,k}$. By (4.287) and (4.288), we obtain

$$g(\hat{\nabla}_U V, X) = g(\nabla_U V, X) = g(T_{PX}FV, U), \tag{4.323}$$

$$\begin{aligned} g(\hat{\nabla}_U V, Z_i) &= g(\nabla_U V, Z_i) \\ &= \csc^2\theta_i\{g(T_V FPZ_i - A_{FV}FZ_i, U)\}. \end{aligned} \tag{4.324}$$

Therefore, we get the assertion. □

Theorem 36. *Let π be a generic submersion from a Kaehler manifold (M,g,J) onto a Riemannian manifold (N,g_N). Then, the pointwise slant distribution D^{θ_i} defines a totally geodesic foliation on* $\ker\pi_*$ *if and only if the following conditions hold:*

$$g(T_{PX}FW_i - T_X FPW_i, Z_i) = 0, \tag{4.325}$$

$$g(T_U FPW_i + A_{FW_i}FU, Z_i) = 0, \tag{4.326}$$

$$g(T_{PZ_j}FW_i - T_{Z_j}FPW_i - A_{FW_i}FZ_j, Z_i) = 0, \tag{4.327}$$

where $X \in \Gamma(D^1)$, $U \in \Gamma(D^0)$, $Z_i, W_i \in \Gamma(D^{\theta_i})$ and $Z_j \in \Gamma(D^{\theta_j})$, $(i \neq j)$ $i,j = \overline{1,k}$.

Proof. Let $X \in \Gamma(D^1)$, $U \in \Gamma(D^0)$, $Z_i, W_i \in \Gamma(D^{\theta_i})$ and $Z_j \in \Gamma(D^{\theta_j})$,$(i \neq j)$ $i,j = \overline{1,k}$. If we use the equations (4.294) $\sim$ (4.296), we get

$$\begin{aligned} g(\hat{\nabla}_{Z_i} W_i, X) &= g(\nabla_{Z_i} W_i, X) \\ &= \csc^2\theta_i g(T_{PX}FW_i - T_X FPW_i, Z_i), \end{aligned} \tag{4.328}$$

$$\begin{aligned} g(\hat{\nabla}_{Z_i} W_i, U) &= g(\nabla_{Z_i} W_i, U) \\ &= \csc^2\theta_i g(T_U FPW_i + A_{FW_i}FU, Z_i), \end{aligned} \tag{4.329}$$

and

$$\begin{aligned} g(\hat{\nabla}_{Z_i} W_i, Z_j) &= g(\nabla_{Z_i} W_i, Z_j) \\ &= \csc^2\theta_i g(T_{PZ_j}FW_i - T_{Z_j}FPW_i - A_{FW_i}FZ_j, Z_i). \end{aligned} \tag{4.330}$$

So, we obtain the assertion. □

Corollary 15. *Let π be a generic submersion from a Kaehler manifold (M,g,J) onto a Riemannian manifold (N,g_N). Then, the vertical distribution $\ker\pi_*$ is a locally product $M_{D^1} \times M_{D^0} \times M_{D^{\theta_1}} \times \ldots \times M_{D^{\theta_k}}$ if and only if (4.317), (4.318), (4.321), (4.322), (4.325), (4.326) and (4.327) hold, where M_{D^1}, M_{D^0} and $M_{D^{\theta_i}}$ are integral manifolds of the distributions D^1,D^0 and D^{θ_i} for $i = \overline{1,k}$, respectively.*

4.4.3 Totally umbilical case of fibers

Theorem 37. *Let π be a generic submersion from a Kaehler manifold (M,g,J) onto a Riemannian manifold (N,g_N). If the fibers of π are totally umbilical, then $D^0=\{0\}$ or $\dim D^0=1$ or $H\perp \underline{D}^0$. Moreover, if $\underline{D}^{\theta_i}$ is parallel along to $ker\pi_*$ for $i=\overline{1,k}$, then $H\in \underline{D}^1$.*

Proof. If $D^0=\{0\}$ or $\dim D^0=1$, then the result is obvious. So, consider the case $\dim D^0>1$. Let U and V be vector fields in D^0 such that $g(U,V)=0$ and
$\|U\|=\|V\|=1$. Then, by (3.1), (3.2), (4.5), (4.22) and Corollary 12-(e), we get

$$
\begin{aligned}
g(H,JU) =& g(H,FU)=g(T_VV,FU)=g(\nabla_VV,FU)\\
=& g(\nabla_VJV,JFU)=g(\nabla_VFV,tFU)=g(T_VU,FV)\\
=& g(V,U)g(H,FV)=0. \qquad (4.331)
\end{aligned}
$$

i.e. $H\perp\underline{D}^0$. Now, suppose that $\underline{D}^{\theta_i}$ is parallel along to $ker\pi_*$ i.e. $\nabla_{ker\pi_*}\underline{D}^{\theta_i}\in\underline{D}^{\theta_i}$. Let Z_i be any vector field in D^{θ_i} for $i=\overline{1,k}$, E be any vector field in $ker\pi_*$ such that $\|E\|=1$. Then, with the help of (4.6) and (4.22), we obtain

$$
\begin{aligned}
g(H,FZ_i) &= g(T_EE,FZ_i)=-g(T_EFZ_i,E)\\
&= -g(\nabla_EFZ_i,E)=0. \qquad (4.332)
\end{aligned}
$$

i.e. $H\perp\underline{D}^{\theta_i}$. Therefore, $H\in\underline{D}^1$. ☐

Thus, we reach the following result.

Corollary 16. *Let π be a generic submersion from a Kaehler manifold (M,g,J) with totally umbilical fibers onto a Riemannian manifold (N,g_N). If $\underline{D}^1=\{0\}$ and $\underline{D}^{\theta_i}$ is parallel along to $ker\pi_*$, then the fibers are minimal.*

4.4.4 Parallel canonical structures

In this section, we study on parallel canonical structures and give some remarkable results for a generic submersion.

Let π be a generic submersion from a Kaehler manifold (M,g,J) onto a Riemannian manifold (N, g_N). Then we define

$$(\nabla_{E_1} P)E_2 = \hat{\nabla}_{E_1} PE_2 - P\hat{\nabla}_{E_1} E_2, \tag{4.333}$$

$$(\nabla_{E_1} F)E_2 = \mathscr{H}\nabla_{E_1} FE_2 - F\hat{\nabla}_{E_1} E_2, \tag{4.334}$$

$$(\nabla_{E_1} t)\xi = \hat{\nabla}_{E_1} t\xi - t\mathscr{H}\nabla_{E_1}\xi, \tag{4.335}$$

$$(\nabla_{E_1} f)\xi = \mathscr{H}\nabla_{E_1} f\xi - f\mathscr{H}\nabla_{E_1}\xi, \tag{4.336}$$

where $E_1, E_2 \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma(ker\pi_*^{\perp})$. Then, we say that if $\nabla P \equiv 0$, then P is *parallel*,

if $\nabla F \equiv 0$, then F is *parallel*,

if $\nabla t \equiv 0$, then t is *parallel*,

if $\nabla f \equiv 0$, then f is *parallel*.

Lemma 32. *Let π be a generic submersion from a Kaehler manifold (M, g, J) onto a Riemannian manifold (N, g_N). Then for any $E_1, E_2 \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma(ker\pi_*^{\perp})$, we obtain*

$$(\nabla_{E_1} P)E_2 = tT_{E_1}E_2 - T_{E_1}FE_2, \tag{4.337}$$

$$(\nabla_{E_1} F)E_2 = fT_{E_1}E_2 - T_{E_1}PE_2, \tag{4.338}$$

$$(\nabla_{E_1} t)\xi = PT_{E_1}\xi - T_{E_1}f\xi, \tag{4.339}$$

$$(\nabla_{E_1} f)\xi = FT_{E_1}\xi - T_{E_1}t\xi. \tag{4.340}$$

Proof. Obviously, it can be seen that using the equations (4.274), (4.275), (4.276) and (4.277) can be proven by (4.337), (4.338), (4.339) and (4.340), respectively. □

Now, with the help of parallel canonical structures we obtain some results.

Theorem 38. *Let π be a generic submersion from a Kaehler manifold (M, g, J) onto a Riemannian manifold (N, g_N). Then, we have the following:*

$$\text{If } t \text{ is parallel, then } T_{ker\pi_*}\underline{D}^0 \perp D^1, \tag{4.341}$$

$$\text{If } f \text{ is parallel, then } T_{ker\pi_*}\underline{D}^1 \perp D^0, \tag{4.342}$$

$$\text{If } P \text{ is parallel, then } T_{ker\pi_*}D^1 \perp \underline{D}^0, \tag{4.343}$$

$$\text{If } F \text{ is parallel, then } T_{ker\pi_*}D^0 \perp \underline{D}^1, \tag{4.344}$$

Proof. Let t be parallel. Then, for $\xi \in \Gamma(\underline{D}^0)$ and $E_1 \in \Gamma(ker\pi_*)$ from (4.339) we have $PT_{E_1}\xi = T_{E_1}f\xi$. Since, for $\xi \in \Gamma(\underline{D}^0)$, $f\xi = 0$, we obtain $PT_{E_1}\xi = 0$. Therefore, for $X \in \Gamma(D^1)$ by (3.1), (3.2) and (4.273), we get

$$\begin{aligned} g(T_{E_1}\xi, X) &= g(JT_{E_1}\xi, JX) \\ &= g(PT_{E_1}\xi, PX) = 0. \end{aligned} \tag{4.345}$$

So, we get (4.341).

Let f be parallel. Then, for $\xi \in \Gamma(\underline{D}^1)$ and $E_1 \in \Gamma(ker\pi_*)$ from (4.340) we have $FT_{E_1}\xi = T_{E_1}t\xi$. Since, for $\xi \in \Gamma(\underline{D}^1)$, $t\xi = 0$, we get $FT_{E_1}\xi = 0$. Thus, for $V \in \Gamma(D^0)$ by (3.1), (3.2) and (4.270), we obtain

$$\begin{aligned} g(T_{E_1}\xi, V) &= g(JT_{E_1}\xi, JV) \\ &= g(FT_{E_1}\xi, FV) = 0. \end{aligned} \tag{4.346}$$

That means $T_{ker\pi_*}\underline{D}^1 \perp D^0$.

Assume that P is parallel. Then, for $E_1 \in \gamma(ker\pi_*)$ and $X \in \Gamma(D^1)$ from (4.337) we have $tT_{E_1}X = T_{E_1}FX$. Since, for $X \in \Gamma(D^1)$, $FX = 0$, we obtain $tT_{E_1}X = 0$. Therefore, for $JU \in \Gamma(\underline{D}^0)$ by (3.1) and (3.2), we get

$$\begin{aligned} g(T_{E_1}X, JU) &= -g(JT_{E_1}X, U) \\ &= -g(tT_{E_1}X, U) = 0 \end{aligned} \tag{4.347}$$

Hence, (4.343) is obtained.

Assume that F is parallel. Then, for $E_1 \in \Gamma(ker\pi_*)$ and $U \in \Gamma(D^0)$ from (4.338) we get $fT_{E_1}U = T_{E_1}PU$. Since, for $U \in \Gamma(D^0)$, $PU = 0$, we have $fT_{E_1}U = 0$. Thus, for $\xi \in \Gamma(\underline{D}^1)$ by (3.1) and (3.2), we have

$$\begin{aligned} g(T_{E_1}U, \xi) &= g(JT_{E_1}U, J\xi) \\ &= g(fT_{E_1}U, J\xi) = 0. \end{aligned} \tag{4.348}$$

So, $T_{ker\pi_*}D^0 \perp \underline{D}^1$. □

We observe that there is a relation between paralellism of F and t. The following lemma establishes that relationship.

Theorem 39. *Let π be a generic submersion from a Kaehler manifold (M, g, J) onto a Riemannian manifold (N, g_N). Then, F is parallel if and only if t is parallel.*

Proof. Let $E_1, E_2 \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma(ker\pi_*^\perp)$. Assume that F is parallel. Then, from (4.338) we have $fT_{E_1}E_2 = T_{E_1}PE_2$. By using (3.1), (3.2) and the properties of tensor field T, we get

$$\begin{aligned} g(PT_{E_1}\xi, E_2) &= g(JT_{E_1}\xi, E_2) = -g(T_{E_1}\xi, JE_2) \\ &= -g(T_{E_1}\xi, PE_2) = g(T_{E_1}PE_2, \xi). \end{aligned} \tag{4.349}$$

Since $fT_{E_1}E_2 = T_{E_1}PE_2$, with the help of (3.1), (3.2) and the properties of tensor field T, we have

$$\begin{aligned} \Rightarrow g(PT_{E_1}\xi, E_2) &= g(fT_{E_1}E_2, \xi) = g(JT_{E_1}E_2, \xi) \\ &= -g(T_{E_1}E_2, J\xi) = -g(T_{E_1}E_2, f\xi) \\ &= g(T_{E_1}f\xi, E_2). \end{aligned} \tag{4.350}$$

$$\Rightarrow g(PT_{E_1}\xi, E_2) = g(T_{E_1}f\xi, E_2) \text{ for any } E_2 \in \Gamma(ker\pi_*). \tag{4.351}$$

Thus, $PT_{E_1}\xi = T_{E_1}f\xi$. From (4.339) that means t is parallel. Similarly, the converse follows. □

In the view of Theorem 35, we obtain the following result.

Corollary 17. *Let π be a generic submersion from a Kaehler manifold (M, g, J) onto a Riemannian manifold (N, g_N) with parallel canonical structure P. Then, D^0 is totally geodesic.*

Proof. Let P be parallel. Then, by (4.287), we have

$$g(T_{PX}FV, U) = g(\hat{\nabla}_U V, X) \tag{4.352}$$

for $U, V \in \Gamma(D^1)$ and $X \in \Gamma(D^0)$. Using (3.1), (3.2), (4.273) and (4.333) , we obtain

$$\Rightarrow g(T_{PX}FV, U) = g(\hat{\nabla}_U PV, PX). \tag{4.353}$$

Since, $PV = 0$ for any $V \in \Gamma(D^0)$, we find

$$g(T_{PX}FV, U) = 0. \tag{4.354}$$

On the other hand, with the help of (4.288), we have

$$g(T_V FPZ_i - A_{FV}FZ_i, U) = \sin^2\theta_i g(\hat{\nabla}_U V, Z_i) \tag{4.355}$$

for $U,V \in \Gamma(D^0)$ and $Z_i \in \Gamma(D^{\theta_i})$ for $i = \overline{1,k}$. Then, by using (3.1), (3.2) and (4.250), we get

$$\Rightarrow \sin^2\theta_i g(\hat{\nabla}_U V, Z_i) = \sin^2\theta_i\{g(P\hat{\nabla}_U V, PZ_i) + g(F\hat{\nabla}_U V, FZ_i)\} \tag{4.356}$$

By (4.333), (4.251) and (4.270) we have

$$\Rightarrow \sin^2\theta_i g(\hat{\nabla}_U V, Z_i) = -\sin^2\theta_i g(\hat{\nabla}_U V, tFZ_i). \tag{4.357}$$

Finally, from Corollary 12-(b), we obtain

$$\Rightarrow \sin^2\theta_i g(\hat{\nabla}_U V, Z_i) = \sin^4\theta_i g(\hat{\nabla}_U V, Z_i) \tag{4.358}$$

$$\Rightarrow (\sin^2\theta_i - \sin^4\theta_i) g(\hat{\nabla}_U V, Z_i) = 0. \tag{4.359}$$

Since, $(\sin^2\theta_i - \sin^4\theta_i) = \sin^2\theta_i \cos^2\theta_i \neq 0$, and we have

$$g(T_V FPZ_i - A_{FV} FZ_i, U) = \sin^2\theta_i g(\hat{\nabla}_U V, Z_i) = 0. \tag{4.360}$$

□

By Theorem 35, we get the following result.

Corollary 18. *Let π be a generic submersion from a Kaehler manifold (M,g,J) onto a Riemannian manifold (N,g_N) with parallel canonical structure F. Then, D^1 is totally geodesic.*

Proof. Let F be parallel. Then, by (4.282), we obtain

$$g(T_{PY} FU, X) = -g(\hat{\nabla}_X Y, U) \tag{4.361}$$

for $X,Y \in \Gamma(D^1)$ and $U \in \Gamma(D^0)$. Using (3.1), (3.2) and (4.250), we get

$$\Rightarrow g(T_{PY} FU, X) = -g(F\hat{\nabla}_X Y, FU). \tag{4.362}$$

Thus, by (4.334), we have

$$g(T_{PY} FU, X) = 0. \tag{4.363}$$

Otherwise, with the help of (4.281), we have

$$g(T_Y FPZ_i - T_{PY} FZ_i, X) = \sin^2\theta_i g(\hat{\nabla}_X Y, Z_i) \tag{4.364}$$

where $X, Y \in \Gamma(D^1)$ and $Z_i \in \Gamma(D^{\theta_i})$ for $i = \overline{1,k}$. By (3.1), (3.2) and (4.250), we get

$$\Rightarrow \sin^2\theta_i g(\hat{\nabla}_X Y, Z_i) = \sin^2\theta_i \{g(P\hat{\nabla}_X Y, PZ_i) + g(F\hat{\nabla}_X Y, FZ_i)\}. \quad (4.365)$$

From (4.334), we obtain

$$\Rightarrow \sin^2\theta_i g(\hat{\nabla}_X Y, Z_i) = \sin^2\theta_i g(J\hat{\nabla}_X Y, PZ_i). \quad (4.366)$$

And from (3.1) and (3.2), we get

$$\Rightarrow \sin^2\theta_i g(\hat{\nabla}_X Y, Z_i) = -\sin^2\theta_i g(\hat{\nabla}_X Y, P^2 Z_i). \quad (4.367)$$

So, with the help of Corollary 12-(h), we have

$$\Rightarrow \sin^2\theta_i g(\hat{\nabla}_X Y, Z_i) = \sin^2\theta_i \cos^2\theta_i g(\hat{\nabla}_X Y, Z_i) \quad (4.368)$$

$$\Rightarrow (\sin^2\theta_i - \sin^2\theta_i \cos^2\theta_i) g(\hat{\nabla}_X Y, Z_i) = 0. \quad (4.369)$$

Since, $(\sin^2\theta_i - \sin^2\theta_i \cos^2\theta_i) = \sin^4\theta_i \neq 0$, and we have

$$g(T_Y FPZ_i - T_{PY} FZ_i, X) = \sin^2\theta_i g(\hat{\nabla}_X Y, Z_i) = 0. \quad (4.370)$$

□

In the case of parallelism of F, we obtain the following results.

Theorem 40. *Let π be a generic submersion from a Kaehler manifold (M, g, J) onto a Riemannian manifold (N, g_N) with parallel canonical structure F. Then, for $i \neq j$, the followings hold:*

$$\textit{the fibers are } D^0 - D^1 \textit{ mixed geodesic} \quad (4.371)$$

and

$$\textit{the fibers are } D^{\theta_i} - D^{\theta_j} \textit{ mixed geodesic}. \quad (4.372)$$

Proof. Let F be parallel. Then, for $X \in \Gamma(D^1)$ and $U \in \Gamma(D^0)$ from (4.338), we have

$$fT_U X = T_U PX. \quad (4.373)$$

By (4.373) and Corollary 12-(a), we get

$$f^2 T_U X = f(T_U PX) = T_U P^2 X = -T_U X. \quad (4.374)$$

On the other hand, from (4.270) and (4.373), we have

$$f^2 T_U X = f^2 T_X U = f(T_X PU) = 0. \tag{4.375}$$

Then, with the help of (4.374) and (4.375), we obtain $T_U X = 0$. It means that the fibers are $D^0 - D^1$ mixed geodesic.
At this time, for $Z_i \in \Gamma(D^{\theta_i})$ and $Z_j \in \Gamma(D^{\theta_j})$ $(i \neq j)$ $i, j = \overline{1,k}$, from (4.338), we have

$$f T_{Z_i} Z_j = T_{Z_i} P Z_j. \tag{4.376}$$

By (4.376) and Corollary 12-(h), we get

$$f^2 T_{Z_i} Z_j = f(T_{Z_i} P Z_j) = T_{Z_i} P^2 Z_j = -\cos^2\theta_j T_{Z_i} Z_j. \tag{4.377}$$

Otherwise, from (4.376) and Corollary 12-(h), we obtain

$$f^2 T_{Z_i} Z_j = f^2 T_{Z_j} Z_i = f(T_{Z_j} P Z_i) = T_{Z_j} P^2 Z_i = -\cos^2\theta_i T_{Z_j} Z_i. \tag{4.378}$$

Thus, with the help of (4.377) and (4.378), we have

$$f^2 T_{Z_i} Z_j = -\cos^2\theta_j T_{Z_i} Z_j = -\cos^2\theta_i T_{Z_j} Z_i \tag{4.379}$$

$$\Rightarrow (\cos^2\theta_j - \cos^2\theta_i) T_{Z_j} Z_i = 0. \tag{4.380}$$

Since, for $(i \neq j), i, j = \overline{1,k}$, $\cos^2\theta_j \neq \cos^2\theta_i$, we have $T_{Z_j} Z_i = 0$. It means that the fibers are $D^{\theta_i} - D^{\theta_j}$ mixed geodesic. □

5. CONCLUSIONS

This thesis is based on to giving a generalization for Riemannian submersions whose total manifolds are Kaehlerian and base manifolds are Riemannian. Thus, as a generalization we define the generic submersion. In future, the curvature relations between total manifold, base manifold and fibers of a generic submersion can be investigated. It is an open area to study. Furthermore, this type of submersion can be studied for Weyl manifolds and a new concept, which can be called as "Weyl submersion", can be defined. Moreover, in the contact geometry the generic submersion can be studied.

Also, for the total manifold of a generic submersion, the following problem can be studied: "under what conditions the total manifold can be Einstein space". On the other hand, it is known that all these theory of submersion have a relation with Mathematical Physics. Especially, the following question can be answered: "What is the relation of a generic submersion with Mathematical Physics?". Finally, the theory of submersion has a relation with statistical machine learning processes, which is popular area in the world. Generic submersion and statistical machine learning process relation can be investigated.

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