ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY

SOME WEAK CONVERGENCE ANALYSIS RESULTS OF THE SEMI-IMPLICIT SPLIT-STEP METHODS FOR THE NON-LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

M.Sc. THESIS

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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

LİNEER OLMAYAN STOKASTİK DİFERANSİYEL DENKLEMLER İÇİN YARI-KAPALI BÖLÜNMÜŞ-ADIM METOTLARININ BAZI ZAYIF YAKINSAKLIK ANALİZ SONUÇLARI

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Berivan ARI, a M.Sc. student of ITU Graduate School of Science Engineering and Technology 509171202 successfully defended the thesis entitled “SOME WEAK CONVERGENCE ANALYSIS RESULTS OF THE SEMI-IMPLICIT SPLIT-STEP METHODS FOR THE NON-LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS”, which she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

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FOREWORD

I have come to the end of Mathematical Engineering graduate education with this thesis study. I would like to point out that this training process provides many benefits to my personal development and I am also aware of that there will be many steps standing right front of me to become a qualified. In this thesis, I would like to sincerely thank my advisor Asst. Prof. Dr. Burhaneddin İzgi for all his supports. His experience shared with me and the opportunities he provided me paved my way. Finally, I would like to thank my dear family for all their support in my life.

June 2019 Berivan ARI

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ABBREVIATIONS

SDE	**:** Stochastic Differential Equation
SISS	**:** Semi-Implicit Split-Step
SSBE	**:** Semi-Implicit Backward Euler
MSISS	**:** Milstein-Type of Semi-Implicit Split-Step

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SOME WEAK CONVERGENCE ANALYSIS RESULTS OF THE SEMI-IMPLICIT SPLIT-STEP METHODS FOR THE NON-LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

SUMMARY

The stochastic differential equation is defined as a differential equation including a stochastic or random process. The analytical solutions of the stochastic differential equations usually are not obtained. Therefore, the studies on the numerical solutions of the nonlinear stochastic differential equations have recently increased in the literature. There are different methods such as Euler-Maruyama, Milstein, Tamed Euler, truncated Euler, split-step backward Euler (SSBE), semi-implicit split-step (SISS) methods. The semi-implicit split-step methods among these methods have recently introduced to solve a class of nonlinear stochastic differential equations with non or locally lipschitz drift term.

This thesis is intended for obtaining some theoretical and numerical results for the weak convergence analysis of the SISS methods since there is no enough study for the weak convergence analysis of the SISS methods according to our literature review yet. We especially focus on the SISS1 and SISS3, among others, based on the stochastic Ginzburg-Landau differential equation.

First, we obtain the first moment boundaries of the numerical solutions of the stochastic Ginzburg-Landau differential equation with the SISS1 and SISS3 methods. We also find the first moment boundaries of the actual solution of the equation. Moreover, we observe that the lower and upper boundaries of the first moments for the numerical solutions of the equation behave almost same as the actual solution of the Ginzburg-Landau SDE by our repeated simulation results for the sufficiently small step size.

Then, we exhibit the second moment boundaries of the numerical methods and the actual solution of the stochastic Ginzburg-Landau differential equation. In addition, we illustrate our results by performing simulations for the various model parameters by the graphs. These figures also confirm that the boundaries act the behavior of the solution for the equation.

Finally, we extend our moment boundaries estimates for the pth moments of the SISS1 and SISS3 methods. Furthermore, we obtain pth moment boundaries of the actual solution of the Ginzburg-Landau SDE. Then, we obtain similar simulations results above for the pth moment boundaries by the repeated simulations. Additionally, we create a table with the different p values for the moment boundaries of the both actual and numerical solutions of the equation. Thus, the comparisons between the theoretical results and the numerical results for the SISS methods based on the Ginzburg-Landau SDE show that the theoretical results are consistent with the numerical results for all p according to our analysis results. Moreover, we discuss their empirical rates of weak convergence and show that the weak convergence rate of the SISS1 and SISS3 methods is almost 1 by the log-log graphs.

LİNEER OLMAYAN STOKASTİK DİFERANSİYEL DENKLEMLER İÇİN YARI-KAPALI BÖLÜNMÜŞ-ADIM METOTLARININ BAZI ZAYIF YAKINSAKLIK ANALİZ SONUÇLARI

ÖZET

Stokastik diferansiyel denklemler, bir diferansiyel denkleme genel olarak rassal bir sürecin eklenmesiyle oluşur. Matematik, fizik, finans, ekonomi, meteoroloji gibi birçok disiplinde kullanılmakta olan bu denklemler, finans piyasaları üzerine çalışmalar yapan Lous Bachelier (1900) ve İskoç botanikçi Robert tarafından görülen kolloidal parçacıkların düzensiz hareketinin matematiksel modelini sunan Albert Einstein'nın (1905) çalışmaları ile literatüre girmiştir.

Stokastik diferansiyel denklemler, adi diferansiyel denklemler gibi lineer ve lineer olmayan denklemler olarak ikiye ayrılırken, son zamanlarda lineer olmayan stokastik diferansiyel denklemler üzerine yapılan çalışmalar artış göstermektedir. Bu tez çalışmasında yapılacak olan analizlerde de lineer olmayan stokastik diferansiyel denklemler ele alınmıştır.

Lineer olmayan stokastik diferansiyel denklemlerin gerçek çözümlerinin bulunması genellikle oldukça zordur. Bu nedenle bu gibi denklemlerin yaklaşık çözümlerinin bulunabilmesi için sayısal yöntemler geliştirilmiştir. Bu sayısal yöntemler literatürde açık (explicit), kapalı (implicit), yarı-kapalı (semi-implicit) olmak üzere üç başlık altında toplanmaktadır. Açık sayısal yöntemler olarak adlandırılan metotlar, sürüklenme katsayısı global lipschitz veya lineer büyüme şartı özelliklerini sağlayan stokastik diferansiyel denklemlerin yaklaşık çözümlerinin bulunmasında oldukça elverişli ve kullanışlı metotlardır. Kapalı metotlar diye adlandırılan metotlar her ne kadar bu tarz denklemlerin sayısal çözümleri için de kullanılabilir olsalar da daha çok sürüklenme katsayısı lokal lipschitz olan veya lipschitz olmayan stokastik diferansiyel denklemlerin yaklaşık çözümlerinin bulunması için geliştirilmiştir. Çünkü açık metotların bu gibi katsayılı stokastik diferansiyel denklemlerin sayısal çözümünde oldukça yetersiz kaldığı yapılan çalışmalarla ispatlanmıştır. Öte yandan, kapalı metotlar genel olarak açık metotlara göre daha karmaşık yapıya sahiptirler. Bu yüzden her ne kadar lineer olmayan stokastik diferansiyel denklemlerin çözümleri için kullanışlı metotlar olsalar da, metotların yapısı gereği stokastik diferansiyel denklemlerin sayısal çözümlerini elde etmek açık metotlara göre daha çok zaman harcanmasını gerektirmektedirler. Bu da kapalı metotlar için bir dezavantaj olarak karşımıza çıkmaktadır.

İşte tam da bu noktada, sürüklenme katsayısı lokal lipschitz olan veya lipschitz olmayan stokastik diferansiyel denklemlerin yaklaşık çözümlerini elde etmede açık metotlar kadar pratik, kapalı metotlar kadar da doğru yaklaşık sonuçların elde edilmesinde kullanılan yarı-kapalı metotlar karşımıza çıkmaktadır. Bu tezde yarı-kapalı metotlardan 2017 yılında literatüre B. İzgi ve C. Çetin tarafından kazandırılmış olan yarı-kapalı bölünmüş-adım (semi-implicit split-step) olarak isimlendirilen metot üzerine çalışılmıştır. Özellikle, lineer olmayan stokastik diferansiyel

denklemlerden literatürde oldukça fazla kullanım alanı olan Ginzburg-Landau stokastik diferansiyel denklemi kullanılarak yarı-kapalı bölünmüş-adım metotlarının zayıf yakınsaklık analizleri için bazı sonuçlar üzerine odaklanılmıştır.

Bu hedef doğrultusunda giriş bölümünde:

İlk olarak stokastik diferansiyel denklemlerin kısaca tarihsel sürecinden bahsedilmiştir.

Bir sonraki adımda ise bir sıvıda yüzen veya asılı parçacıkların rastlantısal hareketi olarak tanımlanan Brown hareketinin matematiksel tanımı ve bazı özellikleri kısaca sunulmuştur.

Daha sonra, stokastik diferansiyel denklemin motivasyonu ve genel formu verilmiştir.

Ardından, literatürdeki stokastik diferansiyel denklemlerin yaklaşık çözümünü bulmak üzere geliştirilen metotlardan; Euler-Maruyama, Milstein, Tamed Euler, kesilmiş Euler (Truncated Euler), bölünmüş geri adım Euler (SSBE) ve yarı-kapalı bölünmüş adım (SISS) metotları bazı özellikleri ile birlikte tanıtılmıştır. Bu metotlardan kısaca bahsedecek olursak;

Euler-Maruyama ve Milstein metotları açık sayısal yöntemlerdendir. Euler Maruyama yöntemi ismini Leonhard Euler ve Gisiro Maruyama'dan almıştır. Milstein yöntemi ise ilk olarak Grigori N. Milstein tarafından 1974 yılında tanıtılmıştır. Bu sayısal yöntemler, sürüklenme ve difüzyon katsayısı global lipschitz özelliğini sağlayan stokastik diferansiyel denklemlerin sayısal çözümünü bulmada kullanılan oldukça kullanışlı yöntemlerdir. Aksi taktirde bu koşulları sağlamayan stokastik diferansiyel denklemlerinin yaklaşık çözümlerinin elde edilebilmesi için kullanılabilecek uygun yöntemler arasında yer almamaktadırlar. Ayrıca, Euler-Maruyama ve Milstein metotlarının zayıf yakınsaklık oranları sırasıyla 1 ve 2 olup, güçlü yakınsaklık oranları da sırasıyla 1/2 ve 1 dir.

Diğer taraftan lokal lipschitz olan veya lipschitz olmayan sürüklenme katsayısına sahip stokastik diferansiyel denklemlerin yaklaşık çözümlerinin elde edilişinde kullanılan metotlardan; Tamed Euler ve kesilmiş Euler açık metotları, bölünmüş geri adım Euler kapalı metodu, yarı-kapalı bölünmüş adım veya Milstein tipi yarı-kapalı bölünmüş adım metotları kullanılabilirler.

J.C. Mattingly, A.M. Stuart ve D.J. Higham tarafından sunulan bölünmüş geri adım Euler yöntemi; doğrusal olmayan monoton stokastik diferansiyel denklemlerin ergodiklik özelliklerini korur. Fakat, bölünmüş geri adım Euler yönteminde her adımda/yinelemede bir denklemin çözülmesi gerekmektedir. Bu nedenle, özellikle yüksek boyutlu doğrusal olmayan skaler/vektörel denklemleri içeren problemlerde hesaplama süresi maliyetli olan bir metottur.

Tamed Euler ve kesilmiş Euler yöntemleri Euler yönteminin alternatif versiyonlarındandır. Tamed Euler metotu M. Hutzenthaler, A. Jentzen ve P.E. Kloeden tarafından sunulurken, Mao ve arkadaşları ise kesilmiş Euler yöntemini tanıtmışlardır. Bu yöntemler, stokastik diferansiyel denklemin sürüklenme terimine bazı yaklaşımlar uygulanarak elde edilmiştir. Ayrıca Tamed Euler ve kesilmiş Euler metotları yüksek boyutlu problemler için de uygundur. Her iki metotunda zayıf yakınsaklık ve güçlü yakınsaklık oranları sırasıyla 1 ve 1/2 dir.

2017 yılında, B. İzgi ve C. Çetin tarafından tanıtılan dört adet yarı-kapalı bölünmüş adım metodu (SISS1, SISS2, SISS3 ve SISS4) lineer ve lineer olmayan stokastik diferansiyel denklemlerin lokal lipschitz olan veya lipschitz olmayan sürüklenme terimine bazı yaklaşımlar uygulanarak elde edilmiştir.

Bölünmüş geri adım Euler metodunun aksine, yarı-kapalı bölünmüş adım metodu her adımda/yinelemede herhangi bir denklem çözülmesini gerektirmemektedir. Bu da zaman maliyeti açısından kazanç sağlamaktadır. Ayrıca yarı-kapalı bölünmüş adım yöntemi, yüksek boyutlu problemlere de kolaylıkla uygulanabilmektedir. Bu tarz problemlerin çözümlerinin elde edilmesinde hesaplama süresi açısından da ciddi bir avantaj sağlamaktadır.

Metotların tanıtılmasının ardından, bu çalışmada sürüklenme katsayısı lokal lipschitz şartını sağlayan stokastik Ginzburg-Landau diferansiyel denkleminin genel formunun ve açık çözümünün tanıtımını gerçekleştirdik.

Ardından giriş bölümünün son kısmında, yarı-kapalı bölünmüş adım (SISS1, SISS2, SISS3 ve SISS4) yöntemlerinin stokastik Ginzburg-Landau diferansiyel denklemine uygulanışları sunulmuştur.

İkinci bölümde ise:

İlk olarak Ginzburg-Landau stokastik diferansiyel denklemi ele alınarak SISS1 ve SISS3 metotlarının uygun koşullar altında birinci momentlerinin alt ve üst sınırları için teoremler sunulmuş olup, gerekli ispatlar yapılmıştır. Benzer şekilde Ginzburg-Landau stokastik diferansiyel denkleminin çözümünün birinci momentlerinin de alt ve üst sınırları ile ilgili teorem sunulup, ispatlanmıştır. MATLAB yardımıyla sunmuş olduğumuz teorik sonuçların ilgili sümülasyonları yapılarak, elde ettiğimiz sonuçlar ayrıntıları ile sunulmuştur. Ayrıca birinci momentler için elde edilmiş olan log-log grafiği ile metotların zayıf yakınsaklık oranının beklendiği gibi 1 olduğu nümerik olarak gösterilmiştir.

İkinci olarak SISS1 ve SISS3 metotları ile Ginzburg-Landau stokastik diferansiyel denkleminin ve bu denklemin gerçek çözümünün yine bazı koşullar altında ikinci momentlerinin alt ve üst sınırlarıyla ilgili teoremler sunulmuştur. Bir önceki adımda yaptığımız işlemlere benzer olarak, bulunan teorik sonuçlar ile elde edilen simülasyon sonuçlarından yararlanılarak bazı grafikler oluşturulmuştur. Ayrıca ikinci momentler için elde edilmiş olan log-log grafiğiyle, SISS yöntemlerinin zayıf yakınsaklık oranının yine beklenildiği gibi 1 olduğu gösterilmiştir.

Son olarak, SISS1 ve SISS3 metotları ile stokastik Ginzburg-Landau diferansiyel denkleminin birinci ve ikinci momentlerinin alt ve üst sınırlarından yararlanılarak; bu metotların yüksek momentleri için alt ve üst sınırlar sunulmuş ve böylece yaklaşımlarımız genelleştirilmiştir. Diğer adımlarda olduğu gibi Ginzburg-Landau diferansiyel denkleminin gerçek çözümünün sınırları için de bu genelleştirme işlemi gerçekleştirilmiştir. Ardından, metotların ve denklemin gerçek çözümünün örnek olarak ele alınan dokuzuncu momenti için simülasyonlar yapılmış ve bazı grafikler oluşturulmuş olup, zayıf yakınsaklık oranının neredeyse 1 olduğu sonucuna tekrardan ulaşılmıştır. Ayrıca elde edilen teorik sonuçların farklı momentleri için analizler yapılmış ve bu analiz sonuçları tablo yardımıyla sunulmuştur.

Sonuç olarak, teorik gösterimi başka bir çalışmada ele alınmak üzere daha sonraya bırakılan yarı-kapalı geri-adım yönteminin zayıf yakınsaklık oranının 1 olduğunun gösterilmesinde, bu tezin bir çıktısı olarak bulunan sonuçların önemli bir rol oynayacağı öngörülmektedir.

1. INTRODUCTION

The history of stochastic differential equations started with Louis Bachelier (1900) who worked on financial markets for the modeling of price fluctuations and Albert Einstein (1905), giving a mathematical model which is called Brownian motion for the irregular movement of colloidal particles seen by Scottish botanist Robert.

Brownian Motion

Brownian motion; a mathematical model used to describe the random motion of floating or suspended particles in a liquid. This model is also called the Wiener process. The limit of the discrete version of the Wiener process which is denoted W(t) first emerged as follows:

$$W(t_{k+1}) = W(t_k) + z(t_k)\sqrt{\Delta t}, \quad W(0) = 0 \tag{1.1}$$

where $z(t_k)$ is the standard normal distribution independent of each other. Thus, expected value and variance of $W(t_{k+1}) - W(t_k)$ are zero and Δt, respectively.

Definition 1.0.1 *[1] Let* (Ω, F, P) *be a probability space with filtration* $\{F_t\}_{t\geq 0}$. *A (standard) one-dimensional Brownian motion is a real-valued continuous* F_t *adapted process* $\{W_t\}_{t\geq 0}$ *with the following properties:*

- $W_0 = 0$;
- *For* $0 \leq s < t$, *the increment* $W_t - W_s$ *is normally distributed with mean zero and variance* $t - s$;
- *For* $0 \leq s < t$, *the increment* $W_t - W_s$ *is independent of* F_s.

The stochastic differential equation is defined as a differential equation including a stochastic or random process which consists of random variables indexed by some mathematical variables that are generally time. Also, the stochastic process (for more details see [2]) is usually denoted by X or $X(t)$.

The general form of stochastic differential equations (in short, SDEs) is

$$dX(t) = a(X(t))dt + \sigma(X(t))dW(t) \text{ with } X(0) = X_0 \tag{1.2}$$

where a and σ are called respectively drift and diffusion term, W is a standart Brownian motion and also X is a stochastic process and satisfies Markov properties [3]. Here, the Markov process (see [4]) says that the future situation depends on only the present situation and also in the present case, it is independent of its past situations.

These are examined as linear and nonlinear stochastic differential equations. It is usually hard to find the actual solution of SDEs. Thus, numerical methods have been developed to find an approximated solution of the SDEs. The numerical methods can be classified under three headings as explicit, implicit and semi-implicit methods.

We briefly introduce the application of these methods for equation (1.2) in this section as following:

Euler-Maruyama Method: The explicit Euler-Maruyama method (see [5, 6]) is

$$X_{k+1} = X_k + a(X_k)\Delta + \sigma(X_k)\Delta W_k \text{ with } X(0) = X_0 \tag{1.3}$$

where $\Delta = \Delta t = T/n$ and $\Delta W_k = W_{k\Delta t} - W_{(k-1)\Delta t}$ for each $k = 1, 2, ...n$. If the coefficients a, σ of equation (1.2) satısfy global lipschitz or linear growth condition, then this method converges to the solution of the SDEs. Otherwise, it is not suitable to find the approximated solution of this equation by this method. The weak convergence rate of the Euler-Maruyama method equals 1 while the strong convergence rate of this method is 1/2.

Milstein Method: The Milstein (see [7]) is an explicit method and its application to equation (1.2) is

$$X_{k+1} = X_k + a(X_k)\Delta + \sigma(X_k)\Delta W_k + \frac{1}{2}\sigma(X_k)\sigma_X(X_k)((\Delta W_k)^2 - \Delta) \tag{1.4}$$

where $\Delta = \Delta t = T/n$ and $\Delta W_k = W_{k\Delta t} - W_{(k-1)\Delta t}$ for each $k = 1, 2, ...n$. As like as the Euler Maruyama method, if the drift and diffusion term of the given stochastic differential equation provide the global lipschitz property or linear growth condition, it can be used to find the approximated solution of the equations. Otherwise, this method is not convenient to find approximated solution of the equations. The weak convergence rate of Milstein method equals 2 and the strong convergence rate of this method is 1. On the other hand, the Milstein method will be identical to the Euler Maruyama method when the diffusion term of the (1.2) does not depend on X.

In addition to these, the scientific studies in the literature on the solution of the stochastic differential equations with the non or locally lipschitz coefficients have recently increased (see [8]). For example; Tamed Euler, truncated Euler, split-step backward Euler, Milstein type of semi-implicit split-step and semi-implicit split-step methods are used to find the approximate solution of these SDEs.

Tamed Euler Method: Hutzenthaler et al. introduced Tamed Euler method (see [9]) which is modified version of Euler-Maruyama method to the nonlinear SDEs with non or locally lipschitz coefficients. This method is as follows

$$X_{k+1} = X_k + \Delta \bar{a}(X_k) + \sigma(X_k)\Delta W_n \tag{1.5}$$

where $\bar{a}(X_k) = \frac{a(X_k)}{1+\Delta\|a(X_k)\|}$ and $\Delta = T/n$, $\Delta W_k = W_{k\Delta t} - W_{(k-1)\Delta t}$ for each $k = 1, 2, ...n$.

Thereafter, they also developed alternative versions of Tamed Euler method in [10] and obtained the results for the convergence and stability of the methods. The strong and weak convergence rates of Tamed Euler method are 1/2 and 1, respectively.

After that, Mao presented truncated Euler Method as follows:

Truncated Euler Method: Truncated Euler method (see [11]) is improved to obtaining the numerical solution of the stochastic differential equation whose coefficients satisfy locally lipschitz but without linear growth condition. The existence of the global solution of the equation is guaranteed by Khasminskii-type condition, which is $2X^T a(X) + |\sigma(X)|^2 \leq K(1+|X|^2)$ for $K > 0$, on the drift and diffusion terms. The following steps are used to define the truncated Euler method. First, Mao chooses any strictly increasing continuous function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies $\rho(r) \to \infty$ as $r \to \infty$ and $\sup_{|X|\leq r}(|a(X)| \vee |\sigma(X)|) \leq \rho(r)$ for all $r \geq 0$. After that, he also chooses any strictly decreasing function $h : (0,1] \to (0,\infty)$ such that $\lim_{\Delta\to\infty} h(\Delta) = \infty$ and $\Delta^{1/4}h(\Delta) \leq 1$ for all $\Delta \in (0,1]$.

Then, the discrete-time truncated Euler Maruyama method is defined with $a_\Delta(x) = a\left((|x| \wedge \rho^{-1}(h(\Delta)))\frac{x}{|x|}\right)$ and $\sigma_\Delta(x) = \sigma\left((|x| \wedge \rho^{-1}(h(\Delta)))\frac{x}{|x|}\right)$ as follows:

$$X_{k+1} = X_k + a_\Delta(X_k) + \sigma_\Delta(X_k)\Delta W_k \tag{1.6}$$

for all k=0,1,...,n and $\Delta = T/n$ where $\Delta W_k = W_{k\Delta t} - W_{(k-1)\Delta t}$.

Moreover, the continuous-time truncated Euler Maruyama has two forms. The first form of these method is

$$\bar{x}_{k+1} = \sum_{k=0}^{\infty} X_k I_{[t_k, t_{k+1})}(t) \; with \; t \geq 0 \tag{1.7}$$

and the second form of these method is

$$x_{k+1} = x_0 + \int_0^t a_\Delta(\bar{x}(s))\, ds + \int_0^t \sigma_\Delta(\bar{x}(s))\, dW(s). \tag{1.8}$$

with $t \geq 0$. Here $x_\Delta(t_k) = \bar{x}_\Delta(t_k) = X_\Delta(t_k) = X_k$. The strong convergence rate of truncated Euler is proved by Mao in [12] that it equals 1/2.

Split-Step Backward Euler Method: Mattingly et al. presented split-step backward Euler (in short, SSBE [13]) method which is one of the alternative Euler methods. The SSBE method is defined by

$$X_k^* = X_k + \Delta a(X_k^*) \tag{1.9}$$

$$X_{k+1} = X_k^* + \sigma(X_k^*)\Delta W_k \tag{1.10}$$

for $k = 1, ..., n$, $\Delta = T/n$ and $\Delta W_k = W_{k\Delta t} - W_{(k-1)\Delta t}$. They proved strong convergence results under less restrictive conditions for the coefficients of the nonlinear SDEs. Furthermore, Bastani and Tahmasebi proved some results for the strong convergence analysis of SSBE method for the nonlinear SDEs with non-smooth drift term [14].

On the other hand, Schurtz achieved some results on implicit and partial implicit methods for nonlinear stochastic differential equations in [15]. Moreover, Szpruch and Mao studied and presented the convergence of some implicit numerical methods for nonlinear SDEs in [16].

Another method is semi-implicit split-step methods as follows:

1.1 Semi-Implicit Split-Step Methods

First, in [17], İzgi and Çetin presented semi-implicit split-step (in short, SISS) methods for the nonlinear SDEs with non or locally lipschitz drift term and then they improved SISS methods for the high dimensional cases in 2018 (see [3]). They generated the SISS methods by using some approximations for the drift term of the nonlinear stochastic differential equations. The general form of SISS method for the equation (1.2) is

$$X_k^* = X_k + X_k^* h(X_k)\Delta \tag{1.11}$$

$$X_{k+1} = X_k^* + \sigma(X_k)\Delta W_{k+1} \tag{1.12}$$

where $a(X_k^*) = X_k^* h(X_k)$, $\Delta = T/n$, and $\Delta W_k = W_{k\Delta t} - W_{(k-1)\Delta t}$ for all $k = 1, ..., n$.

The first and second SISS methods (for more details see [3]) are follows.

The first SISS method (SISS1): While $f^{\Delta}(x) = \frac{x}{1-\Delta h(X_k)}$ SISS1 methods is

$$X_{k+1} = f^{\Delta}(X_k) + \sigma(X_k)\Delta W_{k+1}, \tag{1.13}$$

The second SISS method (SISS2): While $f^{\Delta}(x) = \frac{x}{1-\Delta h(X_k)}$ and $\sigma^{\Delta}(x) = \sigma(f^{\Delta}(x))$, SISS2 methods is

$$X_{k+1} = f^{\Delta}(X_k) + \sigma^{\Delta}(X_k)\Delta W_{k+1}. \tag{1.14}$$

for all $k = 1, ..., n$ $\Delta = T/n$, and $\Delta W_k = W_{k\Delta t} - W_{(k-1)\Delta t}$.

On the other hand, it is known that the SSBE method [13] is costly since the first step of SSBE is needed to be solved in each iteration. In addition to this, the computational cost of this method will be increased in the high dimensional cases. Because of this approximation, we do not need to solve any equations at each step. Furthermore, the SISS methods will offer a great advantage when working at high dimensions [3] with respect to the SSBE method and it decreased the computational cost. You can see more information about the SISS methods and their details in the articles of İzgi and Çetin in [3, 17]. Thereafter, Milstein type versions of the SISS methods are also introduced by them in [18]. Furthermore, some results of the strong convergence analysis for the MSISS and SISS methods are presented by İzgi and Çetin in [19, 20].

In this thesis, the weak convergence analysis of the SISS method will be carried out by focusing on the stochastic Ginzburg-Landau differential equation. So, brief information about the equation will be given below.

1.2 Ginzburg-Landau Stochastic Differential Equation

In [5], general form of SDE with polynomial drift term of degree n is

$$dX(t) = (aX^n(t) + bX(t))dt + cX(t)dWt \text{ with } X(0) = X_0 \tag{1.15}$$

where $X(t)$ is stochastic process, W is a standard Brownian motion, a,b, and c are constant.

The explicit solution of this equation (1.15) satisfies the following process:

$$X(t) = F^{-1}(t)\left(X_0^{1-n} + a(1-n)\int_0^t F^{1-n}(s)ds\right)^{1/(1-n)} \tag{1.16}$$

with $F^{-1}(t) = e^{(b-\frac{1}{2}c^2)t+cW_t}$.

For n=2, the equation which is represented in (1.15) is called the stochastic Verhults equation (see [5])

Moreover, for n=3, the equation in (1.16) is called as stochastic Ginzburg-Landau differential equation which is

$$dX(t) = (AX(t) - \delta X^3(t))dt + \sigma X(t)dW(t),\ \ 0 < t \leq T, \tag{1.17}$$

The explicit and unique solution of the equation (1.17) satisfies the following process

$$X(t) = F^{-1}(t)\{x_0^{-2} + 2\delta\int_0^t F^{-2}(s)ds\}^{-1/2} \tag{1.18}$$

where $F(t) = e^{(\frac{1}{2}\sigma^2-A)t-\sigma W(t)}$ with $F(0) = 1$.

1.3 SISS Methods with Ginzburg-Landau SDE

This section provides partial information on obtaining SISS methods for stochastic differential equation in (1.17). The split-step method for this scalar SDE reduces to

$$y = x + \Delta(Ay - \delta y^3). \tag{1.19}$$

Then, İzgi and Çetin use the approximation of $a(y) = Ay - \delta y^3$ by the $h_1(x,y) = Ay - \delta yx^2$ or $h_2(x,y) = Ax - \delta yx^2$, in [3, 17]. First, we use $h_1(x,y)$ function for an approximation to the drift term and solve $y = x + h_1(x,y)\Delta$ equation for y. Then, we obtain

$$f^{\Delta}(x) \quad = \quad \frac{x}{1+\Delta(\delta x^2 - A)}. \tag{1.20}$$

If we define $a^{\Delta}(x)$ as following

$$a^{\Delta}(x) \quad = \quad \frac{a(x)}{1-\Delta(A-\delta x^2)}, \tag{1.21}$$

then we may redefine $f^{\Delta}(x)$ as $x+\Delta a^{\Delta}(x)$.

Similar to this approach, when we use $h_2(x,y)$ function, then we have

$$g^{\Delta}(x) = \frac{(1+\Delta A)x}{1+\Delta\delta x^2} \tag{1.22}$$

and

$$a^{\Delta}(x) = \frac{a(x)}{1+\Delta\delta x^2} \tag{1.23}$$

where $g^{\Delta}(x) = x+\Delta a^{\Delta}(x)$.

The types of SISS methods for equation (1.17) are presented in [3, 17] as follows:

The first SISS method (SISS1): While $f^{\Delta}(x) = \frac{x}{1+\Delta(\delta x^2-A)}$ and $\sigma(X) = \sigma X$, defined as follows

$$X_{k+1} = f^{\Delta}(X_k) + \sigma(X_k)\Delta W_{k+1}, \tag{1.24}$$

The second SISS method (SISS2): While $f^{\Delta}(x)$ the same as in (1.24) and $\sigma^{\Delta}(x) = \sigma(f^{\Delta}(x)) = \sigma f^{\Delta}(x)$, then

$$X_{k+1} = f^{\Delta}(X_k) + \sigma^{\Delta}(X_k)\Delta W_{k+1}. \tag{1.25}$$

The third SISS method (SISS3):While $g^{\Delta}(x) = \frac{(1+A\Delta)x}{1+\Delta\delta x^2}$, and $\sigma(X) = \sigma X$ SISS3 is defined as follows

$$X_{k+1} = g^{\Delta}(X_k) + \sigma(X_k)\Delta W_{k+1}. \tag{1.26}$$

The fourth SISS method (SISS4): While $g^{\Delta}(x)$ the same as in (1.26) and $\sigma^{\Delta}(x) = \sigma(g^{\Delta}(x)) = \sigma g^{\Delta}(x)$ then SISS4 is

$$X_{k+1} = g^{\Delta}(X_k) + \sigma^{\Delta}(X_k)\Delta W_{k+1}. \tag{1.27}$$

where $\Delta = \Delta t = T/n$ and $\Delta W_{k+1} = W_{(k+1)\Delta t} - W_{k\Delta t}$ for each $k = 0,1,2,...n$.

Moreover, if the diffusion term of SDE is independent from X, then the first and second (likewise the third and fourth) SISS methods are coincide.

According to our literature review, there is no much study for the weak convergence analysis of the SISS methods yet. Therefore, in this thesis, we work on the some results of the weak convergence analysis for SISS1 and SISS3 methods, among others, depending on stochastic Ginzburg-Landau differential equation. First of all, we find the first moment boundaries for the SISS1 and SISS3 methods and also obtain the first moment of the actual solution of the stochastic Ginzburg-Landau differential equation in [21]. Afterthat, we also present the second moment boundaries of the numerical methods and actual solution of the stochastic Ginzburg-Landau differential equation in [22]. Lastly, we extend our results for the pth moments for the methods and actual solution of the equation.

2. SOME MOMENT BOUNDARIES

In this section, we work on obtaining some moment boundaries for the SISS methods based on Ginzburg-Landau stochastic differential equation. In particular, we deal with the some moment boundaries for the SISS1 and SISS3 methods. In addition to these, it is possible to extend our approaches in this thesis for the second and fourth SISS methods, as well. Moreover, we work on moment estimates for the actual solution of the stochastic Ginzburg-Landau differential equation. Either weak or strong convergence analysis of the SISS methods, all moments of the iterations should be exist. Therefore, we start with obtaining the boundaries of the first moments for both iterations and actual solutions of stochastic Ginzburg-Landau differential equation. Then, we present similiar work for the second moment boundaries of them. Finally, we extend our approaches for the boundaries of the pth moments.

2.1 The First Moment Boundaries for the Numerical and Actual Solutions

As the first study in this thesis, we obtain the first moment boundaries of the SISS1 and SISS3 methods depending on the stochastic Ginzburg-Landau differential equation and its explicit solution.

Theorem 2.1.1 *The iterations X_k using the first SISS method (1.24) to solve the equation (1.17) (see [21]) satisfy the following:*

(i) The upper bound for the first moment of the iterations while $\delta > 0, A \in \mathbb{R}$ is

$$E[X_k] \leq \frac{X_0}{(1-A\Delta)^k} \tag{2.1}$$

for all $k = 0,1,2,...,n$ and sufficiently small $0 < \Delta \leq T$.

(ii) There is sufficiently small $\Delta_0 > 0$ such that $X_0 < \sqrt{1+2\sqrt{\frac{1-A\Delta_0}{\delta\Delta_0}}} \cdot (1-A\Delta_0)^n$ while $1-A\Delta_0 > 0$ then the lower bound for the first moment of the iterations

$$E[X_k] \geq \frac{1}{(1-A\Delta)^k}\left(1+\sqrt{\frac{\delta\Delta}{1-A\Delta}}\right)^{-2k} X_0 \tag{2.2}$$

holds for all $k = 0,1,2,...,n$ and $0 < \Delta \leq \Delta_0$ while $\delta > 0, A \in \mathbb{R}$.

Proof 2.1.2 ***(i)*** *If we take the expectation of the iterations for the first SISS method $X_k = f^\Delta(X_{k-1}) + \sigma X_{k-1}\Delta W_k$ then we have $E[X_k] = E\left[f^\Delta(X_{k-1})\right]$ by using the basic stochastic calculus rules. Then, we have the following results after some iterations for $f^\Delta(x) = \frac{x}{1+\Delta(\delta x^2 - A)}$ which is given in (1.24) for equation (1.17).*

$$
\begin{aligned}
E[X_k] &= E\left[\frac{X_{k-1}}{1-(A-\delta X_{k-1}^2)\Delta}\right] \leq \frac{E[X_{k-1}]}{1-A\Delta} \\
E[X_1] &\leq \frac{X_0}{1-A\Delta} \\
E[X_2] &\leq \frac{E[X_1]}{1-A\Delta} \leq \frac{X_0}{(1-A\Delta)^2}. \\
&\vdots \\
E[X_k] &\leq \frac{X_0}{(1-A\Delta)^k}.
\end{aligned}
$$

(ii) *If we use similar approaches in (i), then we have*

$$
\begin{aligned}
E[X_k] = E\left[f^\Delta(X_{k-1})\right] &= E\left[\frac{X_{k-1}}{1-(A-\delta X_k^2)\Delta}\right] \\
&= E\left[\frac{X_{k-1}}{u+mX_k^2}\right]
\end{aligned}
\tag{2.3}
$$

where $u = 1 - A\Delta$ and $m = \delta\Delta$.

Now, let's define $g(x)$ as $\frac{x}{u+mx^2}$, then

$$
\begin{aligned}
g(x) &= \frac{1}{u}\left(\frac{x}{1+\frac{m}{u}x^2}\right);\ let\ N = \frac{m}{u} \\
&\geq \frac{1}{u}\left(\frac{x}{(1+\sqrt{N})^2}\right) while\ x < \sqrt{1+2\sqrt{\frac{1-A\Delta}{\delta\Delta}}}.
\end{aligned}
$$

Thus, we have $g(x) \geq h(x)$ where $h(x) = \frac{1}{u}\left(\frac{x}{(1+\sqrt{N})^2}\right)$. By the condition on X_0 and the result in (i) above it is clear that the monotonicity $E[g(X_{k-1})] \geq E[h(X_{k-1})]$ holds for all $k \in \mathbb{Z}^+$. Now if we use this fact in (2.3), then we have the following after some iterations:

$$
\begin{aligned}
E[X_k] &\geq \frac{1}{u}\left(\frac{1}{(1+\sqrt{N})^2}\right)E[X_{k-1}] \\
&\geq \frac{1}{u^2}\left(\frac{1}{(1+\sqrt{N})^4}\right)E[X_{k-2}] \\
&\vdots \\
&\geq \frac{1}{u^k}\left(\frac{1}{(1+\sqrt{N})^{2k}}\right)X_0 = \left(\frac{1}{(1-A\Delta)^k}\right)\left(1+\sqrt{\frac{\delta\Delta}{1-A\Delta}}\right)^{-2k} X_0 \ \blacksquare.
\end{aligned}
$$

Theorem 2.1.3 *The iterations X_k using the third SISS method (1.26) to solve the equation (1.17) (see [21]) satisfy the following:*

(i) The upper bound for the first moment of the iterations while $\delta > 0, A \in \mathbb{R}$ is

$$E[X_k] \leq \frac{(1+A\Delta)^k}{1+\delta\Delta X_0^2} X_0 \tag{2.4}$$

for all $k = 0, 1, 2, ..., n$ and sufficiently small $0 < \Delta \leq T$.

(ii) For sufficiently small $\Delta_0 > 0$, if $X_0 < \sqrt{1+\frac{2}{\sqrt{\delta\Delta_0}}} \cdot (1-A\Delta_0)^{-n}$ then the lower bound for the first moment of the iterations

$$E[X_k] \geq \frac{(1+A\Delta)^k}{(\sqrt{\delta\Delta}+1)^{2k}} X_0 \tag{2.5}$$

holds for all $k = 0, 1, 2, ..., n$ and $0 < \Delta \leq \Delta_0$ while $\delta > 0$, $A \in \mathbb{R}$.

Proof 2.1.4 *(i) The proof can be done by using the similar steps that are used in the proof of Theorem 2.1.1 for the third SISS method $X_k = g^{\Delta}(X_{k-1}) + \sigma X_{k-1}\Delta W_k$ while $g^{\Delta}(x) = \frac{(1+A\Delta)x}{1+\Delta\delta x^2}$. Then we have $E[X_k] = E[g^{\Delta}(X_{k-1})] = E\left[\frac{(1+A\Delta)X_{k-1}}{1+\Delta\delta X_{k-1}^2}\right]$. After some iterations, we have*

$$\begin{aligned}
E[X_1] &= \frac{X_0(1+A\Delta)}{1+\delta\Delta X_0^2}. \\
E[X_2] &\leq (1+A\Delta)E[X_1] = (1+A\Delta)^2 \frac{X_0}{1+\delta\Delta X_0^2} \\
&\vdots \\
E[X_k] &\leq (1+A\Delta)^k \frac{X_0}{1+\delta\Delta X_0^2}.
\end{aligned}$$

(ii) It is clear that we have the following for SISS3 method

$$\begin{aligned}
E[X_k] &= E\left[g^{\Delta}(X_{k-1})\right] = E\left[\frac{X_{k-1}}{1+\delta\Delta X_{k-1}^2}\right](1+A\Delta) \\
&= E\left[\frac{X_{k-1}}{1+mX_{k-1}^2}\right](1+A\Delta).
\end{aligned} \tag{2.6}$$

where $m = \delta\Delta$.

In this step, if we define $h(x)$ as $\frac{x}{1+mx^2}$ then

$$\begin{aligned}
h(x) &= \frac{x}{(\sqrt{m}x+1)^2 - 2\sqrt{m}x} \\
&\geq \frac{x}{(\sqrt{m}+1)^2} = g(x) \text{ while } x < \sqrt{1+\frac{2}{\sqrt{\delta\Delta}}}.
\end{aligned}$$

Use the monotonicity of $h(x) \geq g(x)$ in (2.6) and iterate it, then we obtain

$$\begin{aligned} E[X_k] &\geq (1+A\Delta)\frac{E[X_{k-1}]}{(\sqrt{m}+1)^2} \\ &\geq \frac{(1+A\Delta)^k}{(\sqrt{m}+1)^{2k}}X_0 \\ &= \frac{(1+A\Delta)^k}{(\sqrt{\delta\Delta}+1)^{2k}}X_0 \blacksquare. \end{aligned}$$

Corollary 2.1.4.1 *In [21], the terminal value, X_n, satisfies $E[X_n] \leq X_0 e^{AT}$ while using the SISS1 and SISS3 methods to solve the equation (1.17) when $\delta > 0, A \in \mathbb{R}$ and sufficiently small $0 < \Delta \leq T$.*

Theorem 2.1.5 *The actual solution of equation (1.17) is given in equation (1.18) has the following upper and lower boundaries:*

$$E[X(t)] \leq X_0 e^{At} \tag{2.7}$$

$$E[X(t)] \geq \frac{X_0 e^{(A-\frac{3}{2}\sigma^2)t}}{\sqrt{1+\frac{2\delta X_0^2(e^{(2A+\sigma^2)t}-1)}{2A+\sigma^2}}} \tag{2.8}$$

while the model parameters $\delta > 0$, $A \in \mathbb{R}$ with $X(0) = X_0$.

Proof 2.1.6 *If we take the expected value of the actual solution where $F(t) = e^{(\frac{1}{2}\sigma^2-A)t-\sigma W(t)}$ then we have the following for the upper bound of the actual solution:*

$$\begin{aligned} E[X(t)] &= E[F^{-1}(t)\{x_0^{-2}+2\delta\int_0^t F^{-2}(s)ds\}^{-1/2}] \\ &\leq E[F^{-1}(t)]X_0, \textit{ by the expectation of the geometric Brownian motions,} \\ &\leq X_0 e^{At} \end{aligned}$$

Similarly, we start with

$$E[X(t)] = E\left[\frac{F^{-1}(t)}{\sqrt{x_0^{-2}+2\delta\int_0^t F^{-2}(s)ds}}\right]$$

then, by Jensen's inequality, we obtain

$$E[X(t)] \geq \frac{X_0 e^{(A-\frac{1}{2}\sigma^2)t}}{E\left[e^{-\sigma W(t)}\sqrt{1+2\delta X_0^2\int_0^t F^{-2}(s)ds}\right]}.$$

Now, if we use Cauchy-Schwarz inequality with Fubini's theorem then we have

$$\begin{aligned} E[X(t)] &\geq \frac{X_0 e^{(A-\frac{1}{2}\sigma^2)t}}{\sqrt{e^{\frac{4\sigma^2}{2}t}\{1+2\delta X_0^2\int\limits_0^t E[e^{(2A-\sigma^2)s+2\sigma W(s)}]ds\}}} \\ &= \frac{X_0 e^{(A-\frac{3}{2}\sigma^2)t}}{\sqrt{1+\frac{2\delta X_0^2(e^{(2A+\sigma^2)t}-1)}{2A+\sigma^2}}}\blacksquare. \end{aligned}$$

after some calculations.

Corollary 2.1.6.1 *The expected value of the actual solution of equation (1.17) at the terminal time $E[X(T)]$ is bounded above by X_0e^{AT}*

2.2 The Second Moment Boundaries for the Numerical and Actual Solutions

In this section, we state and prove some theorems (see [22]) for the second moment boundaries of the numerical and actual solutions of Ginzburg-Landau stochastic differential equation.

Theorem 2.2.1 *The iterations X_k using the first SISS methods (1.24) to solve the equation (1.17) satisfy the following:*
(i)The upper bound for the second moment of the iterations is

$$E[X_k^2] \leq \Big(\frac{1}{1-2A\Delta}+\sigma^2\Delta\Big)^k X_0^2. \tag{2.9}$$

for all $k=0,1,2,...,n$ and sufficiently small $0<\Delta\leq T$ where $\delta>0, A\in\mathbb{R}$.
(ii) The lower bound for the second moment of the iterations, while $\Delta_0>0$ is sufficiently small such that $X_0 \leq \sqrt{\Big(1+2\sqrt{\frac{1-A\Delta_0}{\delta\Delta_0}}\Big)\Big(\frac{1}{1-2A\Delta_0}+\sigma^2\Delta_0\Big)^{-n}}$ when $\sigma^2\Delta_0>\frac{1}{2A\Delta_0-1}$, is

$$E[X_k^2] \geq \frac{1}{(1-A\Delta)^{2k}}\Big(1+\sqrt{\frac{\delta\Delta}{1-A\Delta}}\Big)^{-4k}X_0^2 \tag{2.10}$$

for all $k=0,1,2,....,n$ and $0<\Delta<\Delta_0$ while $\delta>0, A\in\mathbb{R}$

Proof 2.2.2 *(i) We start with the calculate square of the expression in (1.24), and later on evaluate the conditional expectation of the result. We have*

$$X_k^2 = \left(\frac{X_{k-1}}{1+\Delta(\delta X_{k-1}^2-A)}\right)^2 + \frac{2X_{k-1}\sigma X_{k-1}\Delta W_k}{1+\Delta(\delta X_{k-1}^2-A)} + \sigma^2 X_{k-1}^2\Delta^2 W_k$$

$$\begin{aligned} E_{k-1}[X_k^2] &= \left(\frac{X_{k-1}}{1+\Delta(\delta X_{k-1}^2 - A)}\right)^2 + \sigma^2 X_{k-1}^2 \Delta \\ &\leq \frac{X_{k-1}^2}{1+2\Delta(\delta X_{k-1}^2 - A)} + \sigma^2 X_{k-1}^2 \Delta, \ \ \textit{since } \Delta \textit{ is sufficiently small} \\ &\leq \left(\frac{1}{1-2\Delta A} + \sigma^2 \Delta\right) X_{k-1}^2 \end{aligned}$$

Now, we take the expected values of both sides of the above inequality and use basic probability rules.

$$\begin{aligned} E[E_{k-1}[X_k^2]] = E[X_k^2] &\leq \left(\frac{1}{1-2\Delta A} + \sigma^2 \Delta\right) E[X_{k-1}^2] \\ &\vdots \\ &\leq \left(\frac{1}{1-2\Delta A} + \sigma^2 \Delta\right)^k X_0^2 \end{aligned}$$

(ii) If we use the same approaches as (i) to prove this for the sufficiently small $\Delta > 0$, then we obtain

$$X_k^2 = \left(\frac{X_{k-1}}{1+\Delta(\delta X_{k-1}^2 - A)}\right)^2 + \frac{2X_{k-1}\sigma X_{k-1}\Delta W_k}{1+\Delta(\delta X_{k-1}^2 - A)} + \sigma^2 X_{k-1}^2 \Delta^2 W_k$$

$$\begin{aligned} E_{k-1}[X_k^2] &= X_{k-1}^2 \left(\frac{1}{(1 - A\Delta + \Delta\delta X_{k-1}^2)^2} + \sigma^2 \Delta\right) \\ &\geq \left(\frac{X_{k-1}}{1 - A\Delta + \Delta\delta X_{k-1}^2}\right)^2 \end{aligned}$$

Then, we take the expected value of both sides of the above inequality.

$$E[X_k^2] \ \geq \ E\left[\left(\frac{X_{k-1}}{1 - A\Delta + \Delta\delta X_{k-1}^2}\right)^2\right]$$

Now, let $u = 1 - A\Delta$ and $m = \delta\Delta$ and define $h(x) = \frac{x}{u+mx^2}$. Then, we have $h(x) \geq \frac{1}{u}\left(\frac{x}{(\sqrt{N}+1)^2}\right) = g(x)$ while $x < \sqrt{1+2\sqrt{\frac{1-A\Delta}{\delta\Delta}}}$ and $N = m/u$ as in [22]. According to the result in (i) and the condition on X_0 above it appears that $E[h(X_{k-1})] \geq E[g(X_{k-1})]$ monotonicity is valid for all $k \geq 1$. Therefore, we have

$$E[X_k^2] \ \geq \ E\left[\left(\frac{1}{u}\left(\frac{X_{k-1}}{(\sqrt{N}+1)^2}\right)\right)^2\right] = \frac{1}{u^2}\frac{E[X_{k-1}^2]}{(\sqrt{N}+1)^4}.$$

After some iterations, we reach the result as following

$$\begin{aligned} E[X_k^2] &\geq \frac{1}{u^2}\frac{E(X_{k-1}^2)}{(\sqrt{N}+1)^4} \\ &\vdots \\ &\geq \frac{1}{u^{2k}}\frac{X_0^2}{(\sqrt{N}+1)^{4k}}\blacksquare. \end{aligned}$$

Theorem 2.2.3 *The iterations X_k using the third SISS methods (1.26) to solve the equation (1.17) satisfy the following:*

(i)The upper bound for the second moment of the iterations is

$$E[X_k^2] \leq X_0^2\left((1+\Delta A)^2+\sigma^2\Delta\right)^k \tag{2.11}$$

for all $k = 1,2,...,n$ and small enough $0 < \Delta \leq T$ where $\delta > 0, A \in \mathbb{R}$.

(ii) For sufficiently small $\Delta_0 > 0$, if $X_0 < \sqrt{\left(1+\frac{2}{\sqrt{\delta\Delta_0}}\right)((1+A\Delta_0)^2+\sigma^2\Delta_0)^{-n}}$ then lower bound for the second moment of the iterations is

$$E[X_k^2] \geq \frac{X_0^2(1+A\Delta)^{2k}}{(\sqrt{\delta\Delta}+1)^{4k}} \tag{2.12}$$

for all $k = 1,2,,....,n$ and $0 < \Delta < \Delta_0$ while $\delta > 0$, $A \in \mathbb{R}$

Proof 2.2.4 *(i) We prove it by using the similar steps in the proof of Theorem (2.2.1) for the third SISS method $X_k = g^\Delta(X_{k-1}) + \sigma X_{k-1}\Delta W_k$ for $g^\Delta(x) = \dfrac{(1+A\Delta)x}{1+\Delta\delta x^2}$. Namely,*

$$X_k = \frac{X_{k-1}(1+A\Delta)}{1+\Delta\delta X_{k-1}^2} + \sigma X_{k-1}\Delta W_k$$

Now we get the square of this equation,

$$X_k^2 = \frac{X_{k-1}^2(1+A\Delta)^2}{(1+\Delta\delta X_{k-1}^2)^2} + 2\frac{X_{k-1}(1+A\Delta)\sigma X_{k-1}\Delta W_k}{1+\Delta\delta X_{k-1}^2} + \sigma^2 X_{k-1}^2\Delta^2 W_k$$

After that taking expected value of both sides of this equation, we obtain the following inequalities.

$$E_{k-1}[X_k^2] = \frac{X_{k-1}^2(1+\Delta A)^2}{(1+\Delta\delta X_{k-1}^2)^2} + \sigma^2 X_{k-1}^2\Delta$$

$$\begin{aligned} E[E_{k-1}[X_k^2]] = E[X_k^2] &= E\left[\frac{X_{k-1}^2(1+\Delta A)^2}{(1+\Delta\delta X_{k-1}^2)^2} + \sigma^2 X_{k-1}^2\Delta\right] \\ &\leq E[X_{k-1}^2]\left((1+\Delta A)^2+\sigma^2\Delta\right) \end{aligned}$$

This inequality holds for all $k \geq 1$ and sufficiently small $\Delta > 0$. After some iterations, we have

$$\begin{aligned} E[X_k^2] &\leq E[X_{k-1}^2]\left((1+\Delta A)^2+\sigma^2\Delta\right) \\ &\vdots \\ &\leq X_0^2\left((1+\Delta A)^2+\sigma^2\Delta\right)^k. \end{aligned}$$

(ii) If we use same approaches in (i) above, then we have

$$\begin{aligned} E_{k-1}[X_k^2] &= X_{k-1}^2\left(\frac{(1+\Delta A)^2}{(1+\Delta\delta X_{k-1}^2)^2}+\sigma^2\Delta\right) \quad \textit{since } \sigma^2 > 0 \\ &\geq X_{k-1}^2\left(\frac{(1+\Delta A)^2}{(1+\Delta\delta X_{k-1}^2)^2}\right) \\ &= \left(\frac{X_{k-1}}{1+\Delta\delta X_{k-1}^2}\right)^2(1+\Delta A)^2. \end{aligned}$$

Take an expectation of both sides of the inequality above, we have

$$\begin{aligned} E[X_k^2] &= E[E_{k-1}[X_k^2]] \\ &\geq E\left[\left(\frac{X_{k-1}}{1+\Delta\delta X_{k-1}^2}\right)^2\right](1+\Delta A)^2 \end{aligned}$$

In this step, if we define $h(x)$ as $\frac{x}{1+mx^2}$ while $m = \delta\Delta$ then $h(x) \geq \dfrac{x}{(\sqrt{m}+1)^2} = g(x)$ for $x < \sqrt{1+\dfrac{2}{\sqrt{\delta\Delta}}}$. Use the monotonicity of $h(x) \geq g(x)$ same as proof of Theorem 2.1.1 and iterate it, then we obtain

$$\begin{aligned} E[X_k^2] &\geq \frac{(E[X_{k-1}])^2}{(\sqrt{m}+1)^4}(1+A\Delta)^2 = \frac{(E[X_{k-1}])^2}{(\sqrt{\delta\Delta}+1)^4}(1+A\Delta)^2 \\ &\vdots \\ &\geq \frac{X_0^2(1+A\Delta)^{2k}}{(\sqrt{\delta\Delta}+1)^{4k}}\blacksquare. \end{aligned}$$

Corollary 2.2.4.1 *The terminal value, X_n, satisfies $E[X_n^2] \leq X_0^2 e^{(2A+\sigma^2)T}$ while using the SISS1 and SISS3 methods to solve the equation (1.17) when $\delta > 0, A \in \mathbb{R}$ and sufficiently small $0 < \Delta \leq T$.*

Theorem 2.2.5 *The actual solution of equation (1.17) is given in equation (1.18) has the following upper and lower boundaries:*

$$E[X^2(t)] \leq X_0^2 e^{(2A+\sigma^2)t} \tag{2.13}$$

$$E[X^2(t)] \geq \frac{X_0^2 e^{(2A-5\sigma^2)t}}{\sqrt{1+\frac{4\delta(e^{(2A+\sigma^2)t}-1)}{(2A+\sigma^2)}X_0^2+\frac{4\delta^2 t(e^{(4A+6\sigma^2)t}-1)}{(4A+6\sigma^2)}X_0^4}} \tag{2.14}$$

,respectively, while $\delta > 0$, $A \in \mathbb{R}$ *with* $X(0) = X_0$

Proof 2.2.6 *The actual solution of the stochastic Ginzburg-Landau equation is* $X(t) = F^{-1}(t)\left(X_0^{-2}+2\delta\int_0^t F^{-2}(s)ds\right)^{-1/2}$ *where* $F(t) = e^{(\frac{1}{2}\sigma^2-A)t-\sigma W(t)}$ *with* $F(0) = 1$. *We take the square of this actual solution, then we have*

$$X(t)^2 = F^{-2}(t)\left(X_0^{-2}+2\delta\int_0^t F^{-2}(s)ds\right)^{-1}$$

Now, if we get expected value of both sides of the above equation, then we obtain the follows

$$\begin{aligned} E[X^2(t)] &= E\left[F^{-2}(t)\left(X_0^{-2}+2\delta\int_0^t \left|F(s)^{-2}\right| ds\right)^{-1}\right] \\ &\leq E[X_0^2]E[F^{-2}(t)] \\ &\leq X_0^2 e^{(2A+\sigma^2)t} \end{aligned}$$

by the basic stochastic calculus tools. In a similar way, we start with

$$\begin{aligned} E[X^2(t)] &= E\left[\frac{F^{-2}(t)}{\left(X_0^{-2}+2\delta\int_0^t F^{-2}(s)ds\right)}\right] \\ &= E\left[\frac{X_0^2 e^{(2A-\sigma^2)t}}{e^{-2\sigma W(t)}\left(1+2\delta X_0^2\int_0^t F^{-2}(s)ds\right)}\right] \end{aligned}$$

then, we use Jensen's inequality and apply Cauchy-Schwarz inequality, then obtain

$$\begin{aligned} E[X^2(t)] &\geq \frac{X_0^2 e^{(2A-\sigma^2)t}}{E[e^{-2\sigma W(t)}(1+2\delta X_0^2\int_0^t F^{-2}(s)ds)]} \\ &\geq \frac{X_0^2 e^{(2A-\sigma^2)t}}{\sqrt{E[e^{-4\sigma W(t)}]E[(1+2\delta X_0^2\int_0^t F^{-2}(s)ds)^2]}} \end{aligned}$$

Now, if we use Holder inequality then use Fubini's theroem, then we have

$$E[X^2(t)] \geq \frac{X_0^2 e^{(2A-5\sigma^2)t}}{\sqrt{1+\frac{4\delta X_0^2(e^{(2A+\sigma^2)t}-1)}{2A+\sigma^2}+\frac{4\delta^2 X_0^4 t(e^{(4A+6\sigma^2)t}-1)}{4A+6\sigma^2}}} \blacksquare.$$

after some calculations.

Corollary 2.2.6.1 *The expected value of the actual solution of equation (1.17) at the terminal time $E[X^2(T)]$ is bounded above by $X_0^2 e^{(2A+\sigma^2)T}$.*

2.3 The pth Moment Boundaries for the Numerical and Actual Solutions

In this section, we extent the theoretical results, which is obtained in Section 2.1 and Section 2.2, pth moments for the SISS methods depend on stochastic Ginzburg-Landau equation and its explicit solution. We start with introducing the following lemma to use in our proofs.

Lemma 2.3.1 *If $E[X^p] \leq M^p$ satisfies integer $p \geq 1$ then $E[X^{2p}] \leq M^{2p}$ holds for all p where X is stochastic process and M is constant.*

Theorem 2.3.2 *The iterations X_k using the first SISS methods to solve the stochastic Ginzburg-Landau differential equation satisfy the following:*
(i)*The upper bound for the pth moment of the iterations is*

$$E[X_k^p] \ \leq \ \left(\frac{1}{1-2A\Delta} + \sigma^2\Delta\right)^{\frac{pk}{2}} X_0^p, \ \ p \in \mathbb{Z}^+ \tag{2.15}$$

for all $k = 0,1,2,...,n$ and sufficiently small $0 < \Delta \leq T$ where $\delta > 0, A \in \mathbb{R}$
(ii) *The lower bound for the pth moment of the iterations while $\Delta_0 > 0$ is sufficiently small such that $X_0 \leq \sqrt{\left(1+2\sqrt{\frac{1-A\Delta_0}{\delta\Delta_0}}\right)\left(\frac{1}{1-2A\Delta_0} + \sigma^2\Delta_0\right)^{-n}}$ when $\sigma^2\Delta_0 > \frac{1}{2A\Delta_0 - 1}$, is*

$$E[X_k^p] \ \geq \ \frac{1}{(1-A\Delta)^{pk}}\left(1+\sqrt{\frac{\delta\Delta}{1-A\Delta}}\right)^{-2pk} X_0^p, \ \ p \in \mathbb{Z}^+ \tag{2.16}$$

for all $k = 0,1,2,....,n$ and $0 < \Delta < T$ is sufficiently small while $\delta > 0$, $A \in \mathbb{R}$.

Proof 2.3.3 ***(i)*** *We use mathematical induction with $p \in \mathbb{Z}^+$ for the proof of this theorem.*
Let's start with assuming M is equal to $\left(\frac{1}{1-2A\Delta} + \sigma^2\Delta\right)^{\frac{k}{2}} X_0$.

- *For p=1, we have $E[X_k] \leq M$.*
- *Assume that it holds for p-1, $E[X_k^{p-1}] \leq M^{p-1}$.*

- *Then consider $E[X_k^p]$. Now taking advantage of cauchy-schwarz inequality, we obtain*

$$\begin{aligned} E[X_k^p] &\leq E[X_k^2]E[X_k^{2(p-1)}] \\ &\leq M^2M^{2(p-1)}, \textit{by Lemma 2.3.1 and Theorem (2.2.1)} \\ &= M^{2p} \end{aligned}$$

Therefore, $E[X_k^p] \leq \left(\frac{1}{1-2A\Delta} + \sigma^2\Delta\right)^{\frac{pk}{2}} X_0^p$.

(ii) *If we take p power of iterations for the first SISS methods $X_k = f^{\Delta}(X_{k-1}) + \sigma X_{k-1}\Delta W_k$, then we have $X_k^p = \left(\frac{X_{k-1}}{1+\Delta(\delta X_{k-1}^2 - A)} + \sigma(X_{k-1}\Delta)W_k\right)^p$ for all $p \in \mathbb{Z}^+$. Now we use Jensen's inequality and second moment lower bounds for SISS1. Thus, we obtain*

$$\begin{aligned} E[X_k^p] &= E[(X_k^2)^{p/2}] \\ &\geq E[X_k^2]^{p/2} \\ &\geq \left(\frac{1}{(1-A\Delta)^{2k}}\left(1+\sqrt{\frac{\delta\Delta}{1-A\Delta}}\right)^{-4k} X_0^2\right)^{p/2} \quad \textit{while } x \leq \sqrt{1+2\sqrt{\frac{1-A\Delta}{\delta\Delta}}} \\ &= \frac{1}{(1-A\Delta)^{pk}}\left(1+\sqrt{\frac{\delta\Delta}{1-A\Delta}}\right)^{-2pk} X_0^p \; \blacksquare. \end{aligned}$$

Theorem 2.3.4 *The iterations X_k using the third SISS methods to solve the Ginzburg-Landau equation in (1.17) satisfy the following:*

(i) *The upper bound for the pth moment of the iterations while $\delta > 0, A \in \mathbb{R}$ is*

$$E[X_k^p] \leq X_0^p((1+A\Delta)^2 + \sigma^2\Delta)^{\frac{kp}{2}}, \quad p \in \mathbb{Z}^+ \tag{2.17}$$

for all $k = 0, 1, .., n$ and small enough $0 < \Delta \leq T$.

(ii) *For sufficiently small $\Delta_0 > 0$, if $X_0 < \sqrt{\left(1+\frac{2}{\sqrt{\delta\Delta_0}}\right)((1+A\Delta_0)^2+\sigma^2\Delta_0)^{-n}}$, the lower bound for the pth moment of the itearitons is*

$$E[X_k^p] \geq \left(\frac{X_0(1+A\Delta)^k}{(\sqrt{\delta\Delta}+1)^{2k}}\right)^p, \quad p \in \mathbb{Z}^+ \tag{2.18}$$

holds for all $k = 0, 1, ..., n$ and $0 < \Delta \leq T$ while $\delta > 0, A \in \mathbb{R}$

Proof 2.3.5 ***(i)*** *We prove it through using the similar approximation in the proof of Theorem 2.3.2 for the third method SISS,* $X_k = g^{\Delta}(X_{k-1}) + \sigma X_{k-1}\Delta W_k$ *while* $g^{\Delta}(X) = \frac{(1+A\Delta)X}{1+\Delta\delta X^2}$. *Namely, we use the mathematical induction that* $E[X_k^p]$ *holds for all* $p \geq 1$.

- *For p=1, we have* $E[X_k] \leq X_0((1+A\Delta)^2+\sigma^2\Delta)^{\frac{k}{2}}$
- *Suppose that* $E[X_k^{p-1}] \leq X_0^{p-1}((1+A\Delta)^2+\sigma^2\Delta)^{\frac{k(p-1)}{2}}$ *for* $p > 1$.
- *Now consider* $E[X_k^p]$. *By using the Cauchy-Schwarz inequality, we get*

$$\begin{aligned} E[X_k^p] &\leq \sqrt{E[X_k^2]E[X_k^{2(p-1)}]} \\ &\leq \sqrt{M^2M^{2(p-1)}}, \text{by Lemma and Theorem (2.2.3)} \\ &= \sqrt{M^{2p}} \end{aligned}$$

where $M = X_0((1+A\Delta)^2+\sigma^2\Delta)^{\frac{k}{2}}$.

Thus, we obtain $E[X_k^p] \leq X_0^p((1+A\Delta)^2+\sigma^2\Delta)^{\frac{kp}{2}}$.

(ii) *If we use same approaches in the proof of Theorem 2.3.2, then* $X_k^p = \left(\frac{X_{k-1}(1+A\Delta)}{1+\Delta\delta X_{k-1}^2} + \sigma X_{k-1}\Delta W_k\right)^p$ *for all* $p \in \mathbb{Z}^+$. *Now we use Jensen's inequality and second moment upper bounds for third SISS. That's way, we obtain*

$$\begin{aligned} E[(X_k^2)^{p/2}] &\geq E[X_k^2]^{p/2} \\ &\geq \left(\frac{X_0^2(1+A\Delta)^{2k}}{(\sqrt{\delta\Delta}+1)^{4k}}\right)^{p/2} \quad \text{while } x \leq \sqrt{1+\frac{2}{\sqrt{\delta\Delta}}} \\ &= \left(\frac{X_0(1+A\Delta)^k}{(\sqrt{\delta\Delta}+1)^{2k}}\right)^p \blacksquare. \end{aligned}$$

Corollary 2.3.5.1 *The terminal value,* X_n, *satisfies* $E[X_n^p] \leq X_0^p e^{(pA+p\frac{1}{2}\sigma^2)T}$ *while using the SISS1 and SISS3 methods to solve the equation (1.17) when* $\delta > 0, A \in \mathbb{R}$ *and sufficiently small* $0 < \Delta \leq T$.

Theorem 2.3.6 *The actual solution of Ginzburg-Landau stochastic differential equation in (1.17) has the following upper and lower boundaries:*

$$E[X^p(t)] \leq X_0^p e^{(pA+\frac{p}{2}\sigma^2)t} \tag{2.19}$$

$$E[X^p(t)] \geq \frac{X_0^p e^{(pA-\frac{5p\sigma^2}{2})t}}{\left(1+\frac{4\delta X_0^2(e^{(2A+\sigma^2)t}-1)}{2A+\sigma^2}+\frac{4\delta^2X_0^4t(e^{(4A+6\sigma^2)t}-1)}{4A+6\sigma^2}\right)^{p/4}} \tag{2.20}$$

for all $p \in \mathbb{Z}^+$ *where* $\delta > 0, A \in \mathbb{R}$ *and* $X(0) = X_0$.

Proof 2.3.7 *By using induction hypothesis, we prove the upper bound for the pth moment of actual solution of Ginzburg-Landau stochastic differential equation. Let* $M = X_0 e^{(A+\frac{1}{2}\sigma^2)t}$.

- *For* $p = 1$, $E[X] \leq M$.
- *Assume that* $E[X^{p-1}] \leq M^{p-1}$ *for* $p > 1$.
- *Then, we want to show that upper bound of* $E[X^p]$.

$$
\begin{aligned}
E[X^p] &\leq \sqrt{E[X^2]E[X^{2(p-1)}]}, \text{ by Cauchy Schwarz} \\
&\leq \sqrt{M^2 M^{2(p-1)}}, \text{by Lemma and Theorem 2.2.5} \\
&= \sqrt{M^{2p}} \\
&= X_0^p e^{(pA+\frac{p}{2}\sigma^2)t}.
\end{aligned}
$$

Afte that, we start with $X^p(t) = F^{-p}(t)\left(X0^{-2} + 2\delta\int_0^t F^{-2}(s)ds\right)^{-p/2}$ *for the proof of **(ii)** where* $p \in \mathbb{Z}^+$. *Now we take an expectation. Then we use Jensen's inequality and the second moment bounds for actual solution of stochastic Ginzburg-Landau equation. Therefore we obtain*

$$
\begin{aligned}
E[(X^2(t))^{p/2}] &\geq (E[X^2(t)])^{p/2} \\
&\geq \left(\frac{X_0^2 e^{(2A-5\sigma^2)t}}{\left(1 + \frac{4\delta X_0^2(e^{(2A+\sigma^2)t}-1)}{2A+\sigma^2} + \frac{4\delta^2 X_0^4 t(e^{(4A+6\sigma^2)t}-1)}{4A+6\sigma^2}\right)^{1/2}} \right)^{p/2} \\
&= \frac{X_0^p e^{(pA-\frac{5p\sigma^2}{2})t}}{\left(1 + \frac{4\delta X_0^2(e^{(2A+\sigma^2)t}-1)}{2A+\sigma^2} + \frac{4\delta^2 X_0^4 t(e^{(4A+6\sigma^2)t}-1)}{4A+6\sigma^2}\right)^{p/4}} \blacksquare.
\end{aligned}
$$

Corollary 2.3.7.1 *The expected value of the actual solution of equation (1.17) at the terminal time* $E[X^p(T)]$ *is bounded above by* $X_0^p e^{(pA+p\frac{1}{2}\sigma^2)T}$.

3. SIMULATION RESULTS

In this section, we conduct simulations (see [6,23]) through SISS1 and SISS3 methods based on the Ginzburg-Landau stocastic differential equation in (1.17) and the actual solution of the equation. First, we perform $N = 10.000$ simulations for the SISS methods and the actual solution of the Ginzburg-Landau SDE. Then, we present our simulation results for the first moment boundaries which are given in chapter 2. Second, we analyze the consistency of the boundaries obtained for the methods by comparing them with the iteration results which are obtained by the simulations. Finally, we obtain the similar comparisons results above for the pth moment boundaries with 100.000 repeated simulations and the different p values. In these analyses (for more details see [24,25]), we enhance the simulation size to the 100.000 since it reflects the sensitvity of the model parameters better for the big moment values.

3.1 Experimental Results for the First Moment Boundaries

We conduct simulations through the first and third SISS methods for the stochastic Ginzburg-Landau differential equation and the actual solution of the equation while we investigate the consistency of the first moment boundaries. For example, we perform $N = 10.000$ simulations [21] by the following parameters $A = -1,\ \delta = 0.1,\ \sigma = 1,$ $x_0 = 5$.

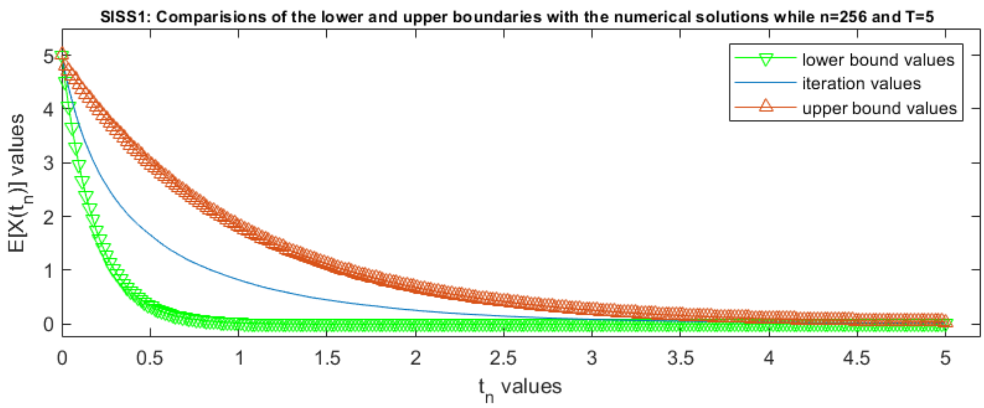

Figure 3.1 : SISS1: Comparisions of the boundaries for the first moment with the numerical solutions.

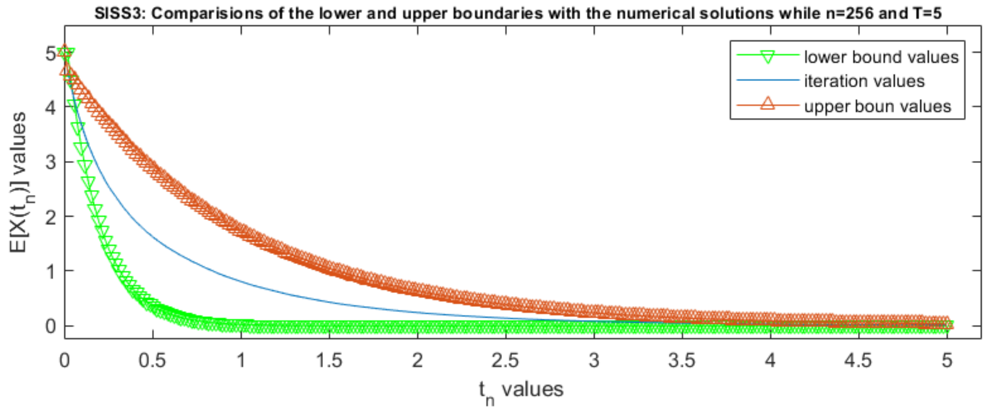

Figure 3.2 : SISS3: Comparisions of the boundaries for the first moment with the numerical solutions.

Figure 3.1 and Figure 3.2 show that the numerical results of the Ginzburg-Landau SDE are consistent with the first moment boundaries of the SISS1 and SISS3 methods.

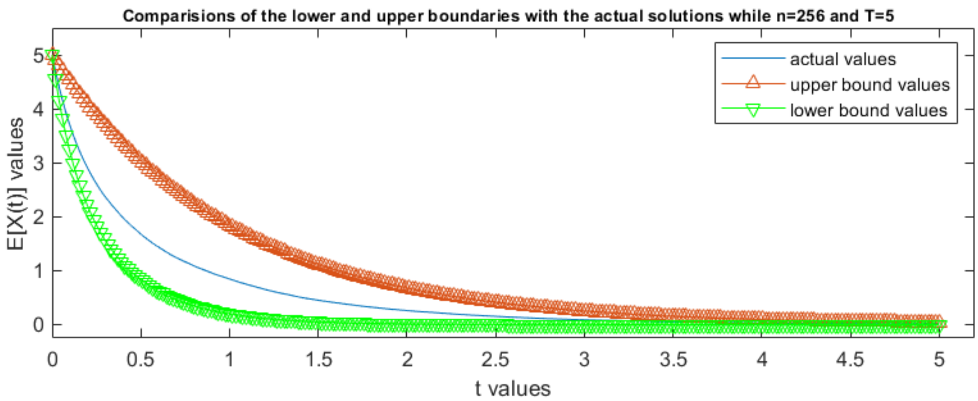

Figure 3.3 : Comparisions of the boundaries for the first moment with the actual solutions.

Moreover, the lower and upper boundaries for the first moment of the actual solution of the equation, which is obtained in Theorem (2.1.5), with the actual solution of the Ginzburg-Landau SDE are shown in Figure 3.3. These figures confirm that the boundaries preserve the behavior of the solution of the stochastic Ginzburg-Landau differential equation.

On the other hand, it is necessary to examine the terminal time values for the weak convergence analysis (see [24]). In addition, we explore the weak convergence rate of the SISS1 and SISS3 methods. For this purpose, we implement $N = 100.000$ simulations for each method with $n = 2^6, 2^7, 2^8$ and 2^9 step size, accordingly. Then, we obtain the log-log graphs [21] and compare the analysis' results with the reference line of slope 1 in Figure 3.4.

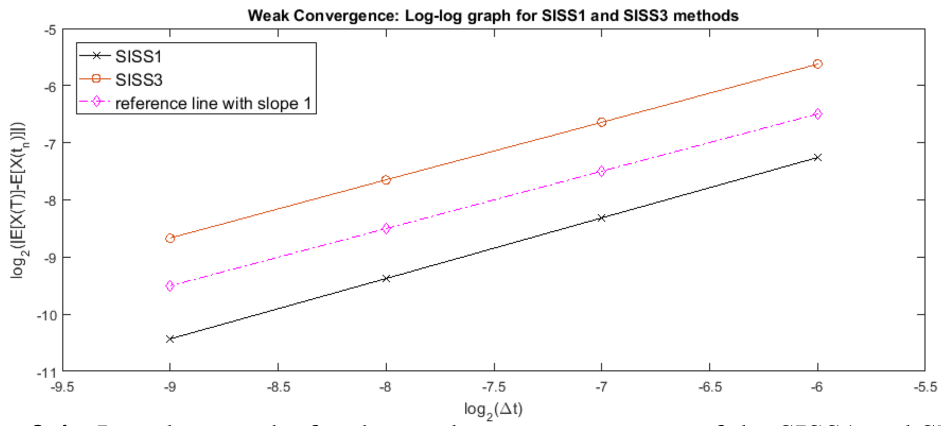

Figure 3.4 : Log–log graphs for the weak convergence rate of the SISS1 and SISS3 methods.

It is clear from Figure 3.4 that the rate of the weak convergence of the SISS1 and SISS3 methods is almost 1.

3.2 Experimental Results for the Second Moment Boundaries

In this section, we conduct simulations via SISS1 and SISS3 methods for Ginzburg-Landau stochastic differential equation while we illustrate the consistency of the second moment boundaries for the methods using the same parameters in Section 3.1.

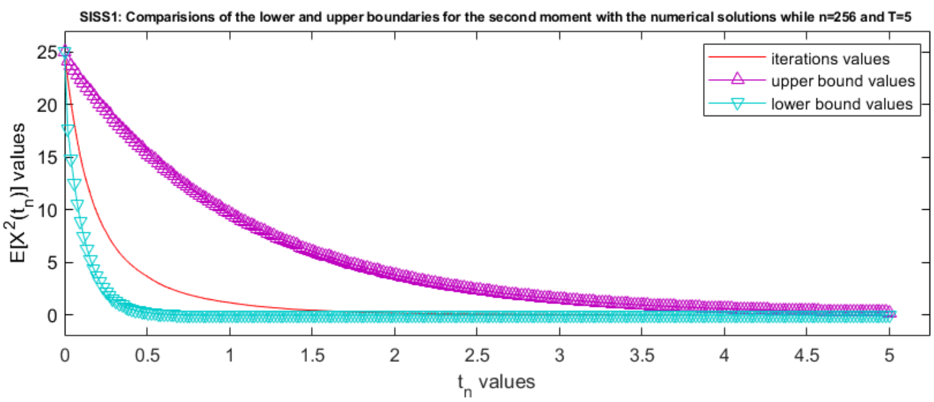

Figure 3.5 : SISS1: Comparisions of the boundaries for the second moment with the numerical solutions.

Figure 3.5 and Figure 3.6 exhibit that the second moment of the numerical solutions of the equation in (1.17) are consistent with the upper and lower boundaries for the second moments of the SISS1 and SISS3 methods.

Moreover, Figure 3.7 displays that the second moment of the actual solution in (1.18) for stochastic Ginzburg-Landau differential equation with the second moment boundaries. It is observed that the second moment boundaries are reflect the behavior

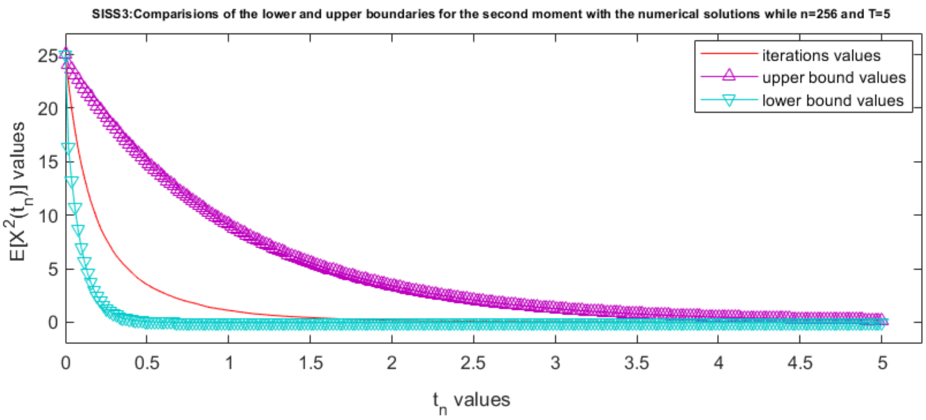

Figure 3.6 : SISS3: Comparisions of the boundaries for the second moment with the numerical solutions.

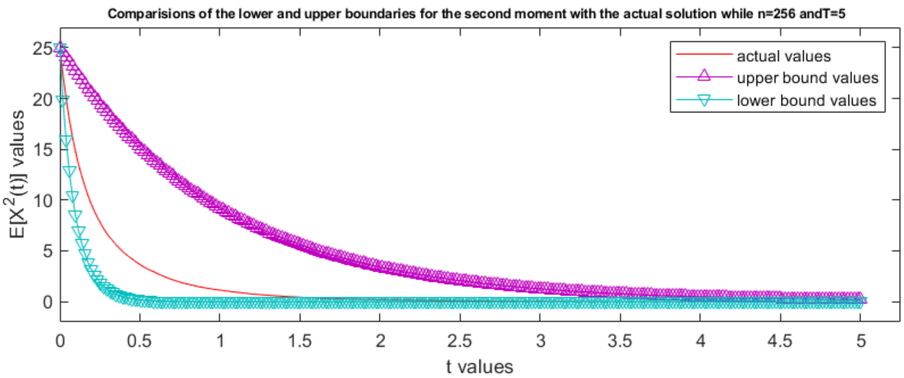

Figure 3.7 : Comparisions of the boundaries for the second moment with the actual solutions.

of the solution of Ginzburg-Landau SDE as in the first moment boundaries of the solutions.

In addition, we perform $N = 100.000$ simulations for each method with the respective step sizes $n = 2^6$, $n = 2^7$, $n = 2^8$, $n = 2^9$. Then, we exhibit the log-log graph in Figure 3.8 and observe that the weak convergence rate of the SISS1 and SISS3 methods is almost 1.

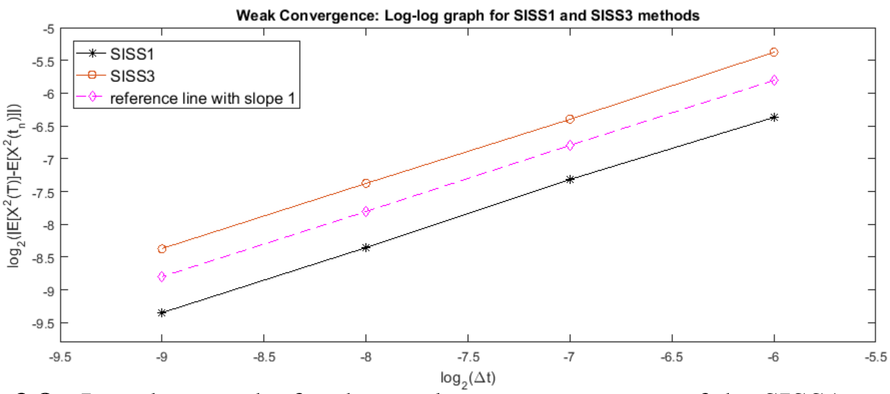

Figure 3.8 : Log–log graphs for the weak convergence rate of the SISS1 and SISS3 methods.

3.3 Experimental Results for the pth Moment Boundaries

The analysis in this section are done with respect to the theoretical results for the pth moments in Section 2.3. In the applications, for instance, we choose the parameters as $A = -0.5$, $X_0 = 5$, $\sigma = 0.2$, $\delta = 0.2$, $n = 2^9$, $T = 5$ when we conduct $N = 100.000$ repeated simulations . Although we perform simulations with the parameters above, someone may conduct similar simulations with different parameters by taking account the theoretical results. First, we present some figures with the parameters above for $p = 9$.

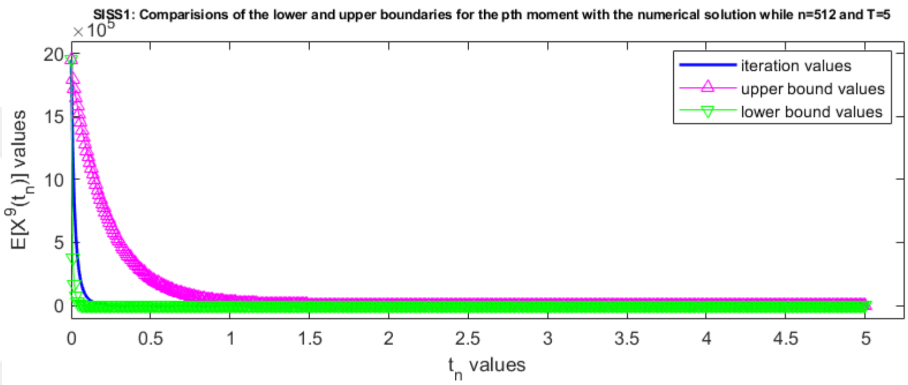

Figure 3.9 : SISS1: Comparisions of the boundaries for the pth moment with the numerical solutions.

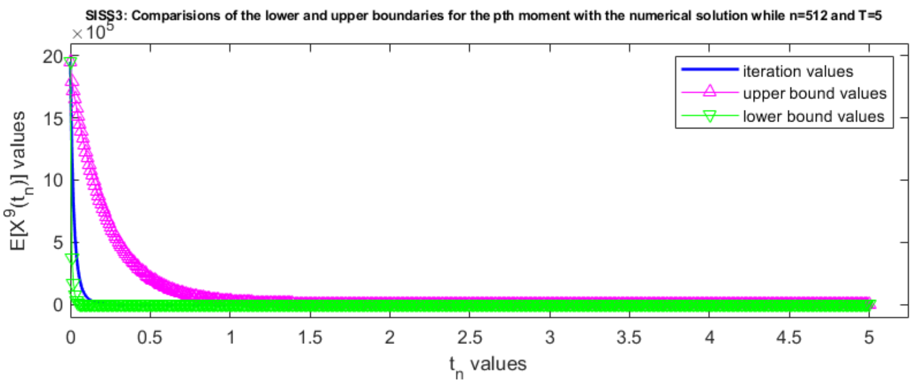

Figure 3.10 : SISS3: Comparisions of the boundaries for the pth moment with the numerical solutions.

Figure 3.9 and Figure 3.10 display that the boundaries found for SISS1 and SISS3 methods are consistent with the respective numerical solutions of the stochastic Ginzburg-Landau differential equation.

Moreover, we obtain Figure 3.11 for the actual solution of the Ginzburg-Landau SDE with the boundaries.

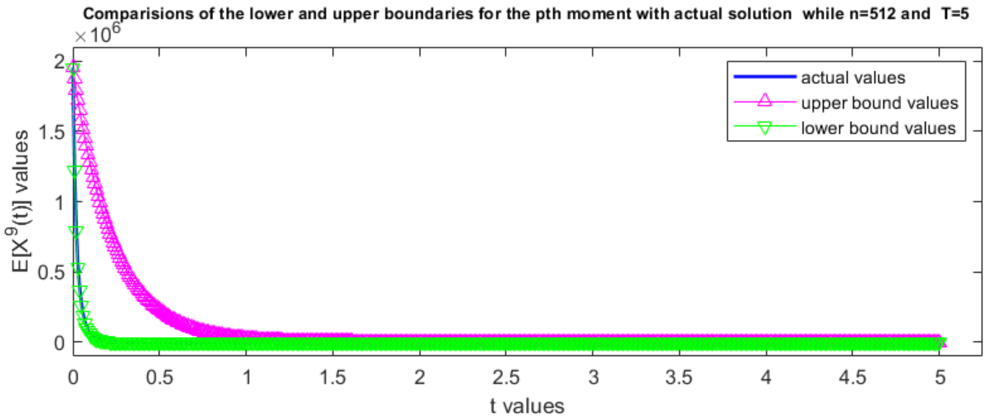

Figure 3.11 : Comparisions of the boundaries for the pth moment with the actual solutions.

Additionally, we perform 100.000 repeated simulations for pth moments of each method when $p = 9$ with $n = 2^6, 2^7, 2^8$ and 2^9 step size.

Then, we obtain the log-log graphs by using these analyses results . Hence, we observe that the weak convergence rate of the methods is nearly 1 in Figure 3.12.

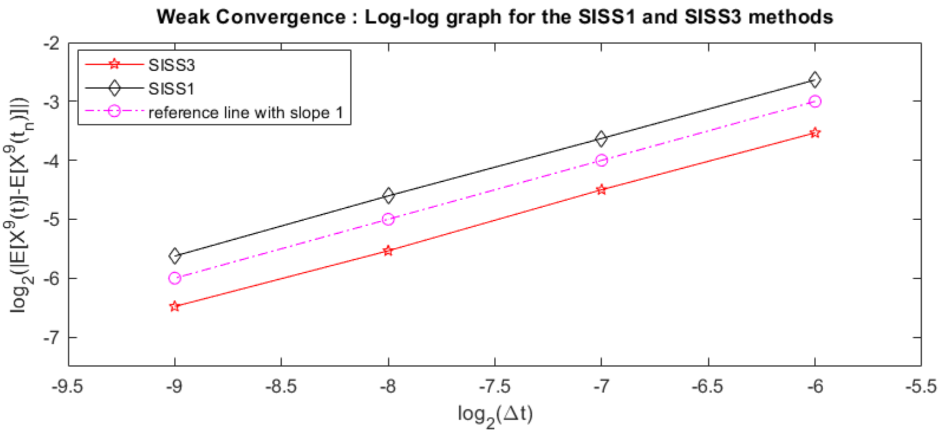

Figure 3.12 : Log–log graphs for the weak convergence rate of the SISS1 and SISS3 methods.

After that, we create Table 3.1 based on the theorems' results by the model parameters for Ginzburg-Landau equation with respect to different p values for the terminal time T.

Consequently, these figures and Table 3.1 show that theoretical and the numerical result are consistent.

Table 3.1 : The pth moments of the SISS1 and SISS3 methods with the Ginzburg-Landau equation at the terminal time for the weak convergence analysis while $N = 100.000$, $A = -0.5$, $X_0 = 5$, $\sigma = 0.2$, $\delta = 0.2$, $n = 2^9$, $T = 5$

p	SISS1 methods			SISS3 methods			Actual Solution of Ginzburg-Landau eq.		
	Lower Values	Iteration Values	Upper Values	Lower Values	Iteration Values	Upper Values	Lower Values	Actual Values	Upper Values
1	2,46E-20	1,21E-01	4,57E-01	2,18E-20	1,19E-01	4,49E-01	5,95E-02	1,20E-01	4,54E-01
2	6,03E-40	1,68E-02	2,09E-01	4,76E-40	1,64E-02	2,02E-01	3,50E-03	1,66E-02	2,06E-01
3	1,48E-45	2,72E-03	9,57E-02	1,04E-45	2,62E-03	9,06E-02	2,11E-04	2,67E-03	9,33E-02
6	2,19E-104	2,46E+09	9,16E-03	1,08E-104	2,29E+09	8,21E-03	4,45E+06	2,34E+09	8,71E-03
9	3,24E-163	6,12E+07	8,77E-04	1,12E-163	5,54E+07	7,44E-04	9,39E+02	5,66E+07	8,13E-04
12	4,79E-222	3,68E+06	8,39E+09	1,17E-222	3,23E+06	6,74E+09	1,98E-01	3,32E+06	7,59E+09
15	7,09E-281	1,47E+06	8,03E+08	1,21E-281	1,26E+06	6,11E+08	4,18E-05	1,16E+06	7,08E+08
18	0,00E+00	6,47E+04	7,69E+07	0,00E+00	5,36E+04	5,53E+07	8,82E-09	5,37E+04	6,61E+07
21	0,00E+00	4,31E+04	7,36E+06	0,00E+00	3,47E+04	5,01E+06	1,86E-12	3,44E+04	6,16E+06
24	0,00E+00	9,07E+02	7,04E+05	0,00E+00	7,17E+02	4,54E+05	3,93E-16	7,37E+02	5,75E+05
27	0,00E+00	3,67E+01	6,74E+04	0,00E+00	2,75E+01	4,11E+04	8,29E-20	2,88E+01	5,37E+04
30	0,00E+00	1,57E+00	6,45E+03	0,00E+00	1,15E+00	3,73E+03	1,75E-23	1,15E+00	5,01E+03
33	0,00E+00	4,06E+00	6,17E+02	0,00E+00	2,88E+00	3,38E+02	3,69E-27	2,61E+00	4,68E+02
36	0,00E+00	1,71E-01	5,91E+01	0,00E+00	1,17E-01	3,06E+01	7,79E-31	1,08E-01	4,36E+01
39	0,00E+00	5,94E-18	5,66E-14	0,00E+00	3,96E-18	2,77E-14	1,64E-48	3,53E-18	4,07E-14
42	0,00E+00	2,26E-03	5,41E-01	0,00E+00	1,45E-03	2,51E-01	3,47E-38	1,90E-03	3,80E-01
45	0,00E+00	1,97E-04	5,18E-02	0,00E+00	1,24E-04	2,28E-02	7,31E-42	1,18E-04	3,55E-02

4. CONCLUSIONS

We present some theoretical and numerical results of the SISS1-3 methods based on Ginzburg-Landau stochastic differential equation for the weak convergence analysis in this thesis which consists of four chapters.

In the first chapter, the definition and historical process of the stochastic differential equations are stated. Then, we briefly introduced Euler-Maruyama, Milstein, Tamed Euler, truncated Euler, split-step backward Euler (SSBE), semi-implicit split-step (SISS) numerical methods. Moreover, we introduce stochastic Ginzburg-Landau differential equation, which is the special case (n=3) of the general form of stochastic differential equation with polynomial degree n, in this section.

Chapter 2 consists of some theoretical results for the moment boundaries of the numerical and actual solutions of Ginzburg-Landau stochastic differential equation. First, we present the first moment boundaries of the SISS1 and SISS3 methods based on the Ginzburg-Landau SDE in Theorem 2.1.1 and Theorem 2.1.3 and prove these theorems. Similarly, we state and prove Theorem 2.1.5 is about the first moment boundaries of the actual solution of the Ginzburg-Landau SDE. Then, the second moment boundaries for the numerical and actual solutions of stochastic Ginzburg-Landau differential equation are obtained by using the similar approaches for the first moments. After that, we extend the results for the pth moments.

In Chapter 3, we conduct simulations and present some figures for the first, second and pth moments boundaries for the numerical methods and the actual solution of the equation. In addition to these, we summary the analyses results for the different p values at the terminal time T for the moment boundaries of the solutions by a table.

Consequently, we believe that these results may take important role to show the usual weak convergence rate of these methods is 1 theoretically which is shown numerically in this thesis.

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