ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE
ENGINEERING AND TECHNOLOGY

LOCAL COHOMOLOGY AND RADICALLY PERFECT IDEALS

Ph.D. THESIS

Tuğba YILDIRIM

Department of Mathematical Engineering

Mathematical Engineering Programme

NOVEMBER 2018

ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE
ENGINEERING AND TECHNOLOGY

LOCAL COHOMOLOGY AND RADICALLY PERFECT IDEALS

Ph.D. THESIS

Tuğba YILDIRIM
(509112078)

Department of Mathematical Engineering

Mathematical Engineering Programme

Thesis Advisor: Prof. Dr. Vahap ERDOĞDU

NOVEMBER 2018

İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

YEREL KOHOMOLOJİ VE RADİKAL OLARAK MÜKEMMEL İDEALLER

DOKTORA TEZİ

Tuğba YILDIRIM
(509112078)

Matematik Mühendisliği Ana Bilim Dalı

Matematik Mühendisliği Programı

Tez Danışmanı: Prof. Dr. Vahap ERDOĞDU

KASIM 2018

Tuğba YILDIRIM, a Ph.D. student of ITU Graduate School of Science Engineering and Technology 509112078 successfully defended the thesis entitled "LOCAL CO-HOMOLOGY AND RADICALLY PERFECT IDEALS", which he/she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

Thesis Advisor :	**Prof. Dr. Vahap ERDOĞDU** Istanbul Technical University	....
Jury Members :	**Prof. Dr. Sefa Feza ARSLAN** Mimar Sinan Fine Arts University	....
	Prof. Dr. A. Muhammed ULUDAĞ Galatasaray University	....
	Assoc. Prof. Dr. Atabey KAYGUN Istanbul Technical University	....
	Assoc. Prof. Dr. Ergün YARANERİ Istanbul Technical University	....

Date of Submission : **1 November 2018**
Date of Defense : **26 November 2018**

To my family, especially to my parents Adnan YILDIRIM and Bakiye YILDIRIM,

FOREWORD

"Success is a journey, not a destination. The doing is often more important than the outcome."
Arthur Ashe

First and foremost I would like to thank my advisor Prof. Dr. Vahap Erdoğdu for his constant insight, support and encouragement. I am grateful to him for his marvelous supervision, inspirations and guiding me toward this research.

I also would like to thank the thesis steering committee members Prof. Dr. Sefa Feza Arslan and Assoc. Prof. Dr. Ergün Yaraneri for their interest.

I would like to express my sincere gratitude to Prof. Gennady Lyubeznik not only for his significant contributions to this thesis but also for his hospitality during my visit at the University of Minnesota. It was my great pleasure to meet and have a chance to study with such a great mathematician.

I gratefully acknowledge the financial support of TUBITAK on the study of this thesis through the grant programs 2211 and 2214/A.

Prof. Dr. Ekrem Savaş and Prof. Dr. Fatma Özdemir have great deal of my gratitude for their meritorious directions and support during all my academic career.

My elementary school teacher, Ayşegül Purtul, has a tremendous impact on my education life. I am indebted to her for believing my success and motivating me in this direction from beginning to end.

I would like to thank my colleagues Dr. Burcu Bektaş Demirci and Dr. Sinem Güler for their sincere friendship since the first day I stepped into my academic life at ITU. We have shared not only the same office but also all our joys and sorrows. I am also very grateful to my bestie Res. Asst. Nazlı Doğan for being close to me from our undergraduate years at İstanbul University, up to now. During these eleven years, she has always supported me in difficult times and shared my exuberance in triumphant times.

Last but not least, my very special thanks to my parents Adnan Yıldırım and Bakiye Yıldırım and to my siblings Gamze, Mert and Yusuf. I always find the necessary power to overcome all kind of obstacles as long as they stand up with me. In particular, i appreciate my parents for their ceaseless love and unconditional support. They are the most precious treasure in my life and this thesis is dedicated to them.

November 2018 Tuğba YILDIRIM
(Research Assistant)

TABLE OF CONTENTS

Page

ABBREVIATIONS

Spec(R)	**:** The set of all prime ideals in a ring R
V(I)	**:** The set of prime ideals containing I
ht(I)	**:** Height of an ideal I
ara(I)	**:** Arithmetic rank of an ideal I
D(M)	**:** Matlis dual of a module M
UFD	**:** Unique factorization domain
$\sqrt{I}$	**:** The radical of an ideal I
$\boldsymbol{H_I^i(M)}$	**:** i^{th} local cohomology of M with support in an ideal I
$\boldsymbol{cd(I,M)}$	**:** The cohomological dimension of M with respect to an ideal I
$\mathscr{C}(\boldsymbol{R})$	**:** Category of R-modules and R-homomorphism
$\mathscr{R}^{\mathbf{i}}$	**:** i^{th} right derived functor

LOCAL COHOMOLOGY AND RADICALLY PERFECT IDEALS

SUMMARY

Local cohomology theory was first introduced by Alexander Grothendieck in 1961 and since then it has been used as a powerful tool to solve many problems in both algebraic geometry and commutative algebra.

Basically, local cohomology functors are defined as the right derived functors of a certain torsion functor: For any module M over a commutative ring R, set

$$\Gamma_I(M) = \{x \in M : there\ exists\ an\ n \in \mathbb{N}\ such\ that\ I^n x = 0\}.$$

The i^{th} local cohomology of M with respect to the ideal I is the i^{th} cohomology module of the sequence obtained by applying the left exact functor $\Gamma_I(-)$, which is defined above, to an injective resolution of M and this module is denoted by $H_I^i(M)$.

Local cohomology theory has been applied to the study of several conjectures in commutative algebra one of which is related to radically perfect ideals.

An ideal I of a commutative (not necessarily Noetherian) ring R is said to be radically perfect if the minimal number of elements of R which generates I up to radical is finite and equals to the height of I. Clearly, when R is Noetherian, the terms radically perfect and set theoretic complete intersection coincide.

One of the classical and long-standing problem in commutative algebra and algebraic geometry is to determine whether each height two prime ideal of the polynomial ring $K[X,Y,Z]$ over the field K is set theoretic complete intersection (radically perfect). Although it is shown by Cowsik and Nori that this conjecture has an affirmative answer when K is of characteristic $p > 0$, it still remains as an open problem in the characteristic zero case. But, based on the observations from his several results, Erdoğdu has a foresight that this problem would fail to be true when K is of characteristic zero. Furthermore he raised another conjecture that "If R is a commutative domain (not necessarily Noetherian) containing a field of characteristic zero, then each prime ideal of $R[X]$ is radically perfect implies R is of Krull dimension one." which was proved to be so in many cases but the exact answer of this conjecture is also not known in general.

The main purpose of this thesis is to understand certain structures of local cohomology modules and determining their relationship with set theoretic complete intersection (radically perfect) ideals and our motivation is suggested by the well-known fact that if I is an ideal of a Noetherian ring R of height n, and if there exists some R-module M such that $H_I^i(M) \neq 0$ for $i > n$, then I is not a set theoretic complete intersection.

Moreover, Hellus showed the relation between set theoretic complete intersection ideals and Matlis duals of local cohomology modules by proving the fact that if

$H_I^i(R) = 0$ for all $i \neq c$ and $\boldsymbol{x}= \{x_1, x_2, ..., x_c\}$ is a regular sequence in I, then $I = \sqrt{x_1, x_2, ..., x_c}$ if and only if x_i form a $D(H_I^c(R))$-regular sequence.

In this thesis, motivated by Hellus' result, we first deal with the set of associated prime ideals of Matlis duals of local cohomology modules and show that over a Noetherian regular local ring of characteristic $p > 0$, for any non-zero ideal I of R and for $i > 0$, zero ideal is in the set of associated prime ideals of Matlis dual of any non-zero local cohomology module $H_I^i(R)$.

We then determine conditions under which a given positive integer t is a lower bound for the cohomological dimension $\mathrm{cd}(I,M) := \sup \{i \in \mathbb{N} \mid H_I^i(M) \neq 0\}$ of any module M with respect to an ideal I of a Noetherian ring R, and use this to conclude that non-catenary Noetherian integral domains contain prime ideals that are not radically perfect (i.e. set theoretic complete intersection). Bearing in mind that non-catenary rings are of Krull dimension > 2, this result is in partial support with Erdoğdu's conjecture. Furthermore if I is any ideal of R and M is any R- module with $\mathrm{cd}(I,M) = c > 0$, we show the existence of a descending chain of ideals $I = I_c \supsetneq I_{c-1} \supsetneq \cdots \supsetneq I_0$ of R such that for each $0 \leq i \leq c$, $\mathrm{cd}(I_i, M) = i$.

In the last chapter of this thesis, we examine the structures of local cohomology modules and show that over a Noetherian unique factorization domain of dimension at most three, top local cohomology module $H_I^{\mathrm{cd}(I,R)}(R)$ is Artinian only in the trivial case when $\mathrm{cd}(I,R) = \dim R$. We then obtain several results on the Artinianness of top local cohomology modules in more general cases. Finally, our study is concerned around the modules of finite length and, in this regard, we first present necessary and sufficient conditions for various modules to be of finite length. We then use our results to give an alternative proof of the well-known result that if R is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and M is a finitely generated R- module of dimension d, then $\mathrm{H}_{\mathfrak{m}}^d(M)$ is finitely generated if and only if $d = 0$.

Throughout, R will always denote a commutative ring with identity, the dimension of a ring R will always mean its Krull dimension.

YEREL KOHOMOLOJİ VE RADİKAL OLARAK MÜKEMMEL İDEALLER

ÖZET

Yerel kohomoloji teorisi ilk olarak 1961 yılında Alexander Grothendieck tarafından tanımlanmış olup tanımlandığı zamandan bu yana cebirsel geometri, cebirsel topoloji ve değişmeli cebir alanlarında çalışan pek çok araştırmacının ilgisini çekmiş, bu alanlardaki bir çok problemin çözümünde kullanılmıştır. Ayrıca, bu teori günümüzde hala tam olarak doğrulanamayan önemli homolojik sanılarla alakalı çalışmalara da uygulanmıştır.

Yerel kohomoloji modüllerinin Grothendieck tarafından verilen orjinal tanımı cebirsel geometrideki kavramlar kullanılarak ifade edilmiş olsa da bazı özel varsayımlar altında bu tanım değişmeli cebir kavramlarıyla aşağıdaki şekilde ifade edilebilir:

R bir halka, I da R'nin bir ideali olsun. R üzerindeki herhangi bir M modülü için

$$\Gamma_I(M) = \{x \in M : I^n x = 0 : n \in \mathbb{N}\},$$

I-torsiyon fonktörü tanımlanabilir. Bu şekilde tanımlanan $\Gamma_I(-)$ toplamsal, kovaryant, sol tam fonktördür ve dolayısıyla bu fonktörün sağ türetilmiş fonktörleri mevcuttur. Bu sağ türetilmiş fonktörlere M nin I idealine göre yerel kohomoloji modülleri denir ve $H^i_I(M) = \mathscr{R}^i\Gamma_I(M)$ ile gösterilir.

Yerel kohomoloji modülleri ile değişmeli cebirdeki birçok önemli kavramın (aritmetik rank, yükseklik vb.) hesaplanmasında kullanılan bazı yeni değişmezler tanımlanmıştır. Bu değişmezlerin en önemlilerinden biri de kohomolojik boyuttur. Herhangi bir M modülünün I idealine göre kohomolojik boyutu, $\mathrm{cd}(I,M)$, aşağıdaki şekilde tanımlanabilir:

$$\mathrm{cd}(I,M) = \sup\{i \in \mathbb{N} : H^i_I(M) \neq 0\}.$$

Özel olarak $M = R$ olması durumunda I idealinin yüksekliğinin, $\mathrm{ht}(I)$, kohomolojik boyut için bir alt sınır, aritmetik rankının, $\mathrm{ara}(I)$, ise bir üst sınır olduğu bilinmektedir. Bir idealin kümesel tam arakesit ideali olması da ancak $\mathrm{cd}(I,R) = \mathrm{ht}(I) = \mathrm{ara}(I)$ eşitliğinin sağlanması ile mümkün olduğundan kümesel tam arakesit ideal kavramı ile kohomolojik boyut dolayısıyla yerel kohomoloji kavramları arasında çok yakın bir ilişki olduğu söylenebilir.

Bir idealin kümesel tam arakesit ideali olup olmadığının belirlenmesi değişmeli cebirin ve cebirsel geometrinin temel araştırma konularından biridir. Bu alanda önemli pek çok sonuç elde edilmiş olmasına rağmen, klasik problemlerden biri olan "Karakteristiği sıfır olan bir K cismi üzerindeki $K[X,Y,Z]$ polinom halkasının, yüksekliği iki olan tüm asal idealleri kümesel tam arakesit midir?" sorusuna halen tam olarak bir cevap verilebilmiş değildir.

K cisminin karakteristiğinin pozitif olduğu durumda, bu sorunun cevabının olumlu olduğu, 1978 yılında Cowsik ve Nori tarafından ispatlanmıştır. Ancak 1994 yılında Erdoğdu ve McAdam tarafından yapılan ortak çalışmada karakteristiğin pozitif olduğu durumun, karakteristiğin sıfır olduğu durumdan farklı davrandığı gösterilmiştir. Söz konusu çalışmada elde edilen sonuçlara paralel olarak Erdoğdu, ele alınan sanının karakteristiğin sıfır olduğu durumda doğru olmayacağı tezini savunmuş ve sonraki çalışmalarında bu tezi desteleyen çok sayıda önemli sonuç elde etmiştir. Hatta kümesel tam arakesit olma tanımını Noether olmayan halkalara da genişleterek "radikal olarak mükemmel ideal" tanımını literatüre kazandırmış ve böylelikle konuyu daha geniş bir perspektifle ele alabilmiştir. Yaptığı çalışmalar sırasında elde ettiği gözlemler neticesinde de " R (Noether olmak zorunda olmayan) karakteristiği sıfır olan bir cisim içeren bir tamlık bölgesi ve $R[X]$ de R üzerinde her asal ideali radikal olarak mükemmel olan bir polinom halkası ise R'nin boyutu bir midir?" sorusunu gündeme getirmiştir. Birçok durumda bu sorunun cevabının olumlu olduğu gösterilmiş olsa da, henüz tüm durumları kapsayan genel bir çözüm bulunabilmiş değildir.

Erdoğdu'nun sorusuna öngörüldüğü gibi olumlu cevap verilebildiği taktirde pek çok araştırmacının yüzyıllardır üzerinde çalıştığı sanının çok daha genel halinin karakteristiğin sıfır olması durumunda olumsuz cevaba sahip olduğu gösterilecek olup bu durumda yapılacak çalışma literatüre geçecek boyutta olacaktır. Fakat var olan metotlar böyle bir sonuca ulaşmada yetersiz kalmaktadır. Bundan dolayıdır ki, bilindiği üzere, şimdiye kadar bu tip alanlarda çalışan araştırmacılar disiplinler arası ilişkilerden yararlanarak yeni metotlar geliştirmiş ve bir takım önemli sonuçlara ancak bu şekilde ulaşabilmişlerdir.

Kümesel tam arakesit idealleri dolayısıyla radikal olarak mükemmel idealler ile ilişkili teorilerden biri de, yukarıda da değinildiği üzere, yerel kohomoloji teorisidir. Hellus, bu alanlar arasındaki ilişkiden yararlanarak Noetheryen yerel halkalar üzerinde bir idealin kümesel tam arakesit ideali olabilmesi için gerekli ve yeterli bir koşul vermiştir. Aynı zamanda bu koşul, kümesel tam arakesit idealleri ile yerel kohomoloji modüllerinin Matlis duallerinin ilgili asal idealleri arasında da kuvvetli bir ilişki olduğunu ortaya koymuştur. Dolayısıyla da bu sonuçtan aldığı motivasyonla Hellus, yerel kohomoloji modüllerinin Matlis duallerinin ilgili asal idealleri üzerinde de çalışmalar yapmış ve pek çok önemli sonuç elde edebilmiştir.

Ancak uzun yıllardır birçok araştırmacı yerel kohomoloji modüllerinin yapıları üzerine çalıştığı halde yine de bu yapılar halen tam olarak çözülebilmiş değildir. Dolayısıyla bu modüllerin Artin modüller olup olmadığı, ne zaman sıfırlandığı (diğer bir ifadeyle, kohomolojik boyut için aşikar olmayan alt-üst sınırlar belirlenip belirlenemediği), bu modüller üzerindeki sonluluk özelliklerinin belirlenmesi (örneğin; ilgili asal idealler kümesinin veya desteğinin(support) sonlu elemana sahip olup olmadığı; Bass sayıları, injektif boyut gibi değişmezlerin sonlu sayı olup olmadığı vb. belirlenmesi) yerel kohomoloji teorisinin günümüzde halen aktif olarak çalışılan konulardandır.

Yerel kohomoloji modüllerinin yapısının bu denli karmaşık ve anlaşılamaz olmasının en önemli nedenlerinden biri, bu modüllerin çoğu durumda R üzerinde sonlu eleman tarafından üretilememesi yani R- modül olarak Noetheryen olmamasıdır. Bu durum göz önünde bulundurularak, yerel kohomoloji modüllerinin yapısını daha iyi kavramada geliştirilen stratejilerden biri de bu modüllerin "daha küçük" olduğu yani sonlu eleman tarafından üretilebildiği yapılar inşa etmektir. Bu bağlamda Lyubeznik, 1993 yılında yaptığı bir çalışmasında $\mathscr{D}$- Modül teorisini yerel kohomolojiye uygulayarak hem bu teoriyi değişmeli cebire uygulayan ilk kişi olmuş, hem de yerel kohomoloji modüllerinin sonluluk özellikleri ile alakalı pek çok önemli sonuç elde etmiştir. Daha sonra 1997 yılında da F-modül tanımını literatüre kazandırarak karakteristiğin pozitif olduğu durumda da benzer sonuçları elde edebilmiştir. Yerel kohomoloji modülleri $\mathscr{D}$ ve F-modül yapılarına sahip olduğundan, ve bu yapılar üzerinde sonlu eleman tarafından üretilebildiğinden, sonuçları elde etmek nispeten daha kolay olmaktadır.

Bu çalışmanın temel amacı, yerel kohomoloji teorisini kullanarak radikal olarak mükemmel idealler ile ilgili sonuçlar elde etmektir. Bu bağlamda ilk olarak Hellus'un çalışmalarından elde ettiği sonuçlardan alınan motivasyonla, Lyubeznik ve Yıldırım tarafından "Noetheryen regüler yerel halkalar üzerinde sıfırdan farklı herhangi bir ideal için tüm yerel kohomoloji modüllerinin, $H_I^i(R)$ $i > 0$, Matlis duallerinin ilgili asal idealler kümesinde sıfır ideali daima bulunmakta mıdır, yani daima $0 \in \mathrm{Ass}(D(H_I^i(R)))$ olmak zorunda mıdır?" sorusu ortaya atılmış ve bu sorunun halkanın karakteristiğinin pozitif olması durumunda olumlu cevaba sahip olduğu ispatlanmıştır. Bu sonucun ispatında F-modül teorisindeki tekniklerden yararlanılmıştır.

Daha sonra kohomolojik boyut kavramı ele alınmış ve kohomolojik boyut için aşikar olmayan alt-üst sınırlar belirlenmiştir. Ayrıca elde edilen sonuçlar kullanılarak, eğri (catenary) olmayan Noether tamlık bölgelerinde kümesel tam arakesit olmayan en az bir asal idealin varlığı gösterilmiştir. Eğri olmayan Noether tamlık bölgelerinin Krull boyutunun en az üç olması gerektiği gerçeği $R[X]$ de her idealin radikal olarak mükemmel olması için R nin boyutunun en fazla bir olması gerektiğini perçinleyen bir sonuçtur.

Tüm bunların yanısıra "Verilen bir halka üzerinde radikal olarak mükemmel asal idealler zinciri bulunabilir mi?" sorusundan hareketle $\mathrm{cd}(I,M) = c > 0$ koşulunu sağlayan herhangi bir I ideali ve herhangi bir M modülü için $\mathrm{cd}(I_i,M) = i$, $0 \leq i \leq c$, olacak şekilde bir $I = I_c \supsetneq I_{c-1} \supsetneq \cdots \supsetneq I_0$ azalan idealler zincirinin var olduğu kanıtlanmıştır.

Son bölümde ise bir önceki bölümlerde elde edilen sonuçların da yardımıyla yerel kohomoloji modüllerinin yapısal özellikleri ile ilgili sonuçlar elde edilmiştir. Bu bağlamda ilk olarak boyutu en fazla üç olan asal ideallere ayrılış bölgelerinde üst yerel kohomoloji modüllerinin $H_I^{\mathrm{cd}(I,R)}(R)$ Artinyen olabilmesi için gerek ve yeter koşul verilmiştir. Ardından daha yüksek boyutlarda bu modüllerin Artinyenliği incelenmiştir. Son olarak da sonlu uzunluktaki modüller ele alınmış; yerel kohomoloji modüllerinin ne zaman sonlu uzunlukta olabileceği ile ilgili sonuçlar elde edilmiştir ve aynı zamanda "Grothendieck'in Sıfırlanmama Teoremi" olarak da bilinen sonuca alternatif bir ispat verilmiştir.

Bu çalışmada tüm halkalar değişmeli ve birim elemana sahip halkalar olup, boyut ile de her zaman Krull boyutu kastedilmektedir.

1. INTRODUCTION

The main objective of this thesis is to understand certain structures of local cohomology modules and their Matlis duals as well as determining their relationships with radically perfect ideals.

Radically perfect ideals are just the generalization of the notion of set theoretic complete intersection of ideals in Noetherian rings to rings need not be Noetherian. This generalization was raised by Erdoğdu in search of an answer to a long standing conjecture detailed in the following.

Let R be a Noetherian ring and X be a closed subset of $\operatorname{Spec}(R)$ defined by an ideal I. Then X is defined set theoretically by s elements $f_1, \cdots, f_s \in R$ if I can be generated by $f_1, \cdots, f_s$ up to radical, that is, $\sqrt{(f_1, f_2, \cdots, f_s)} = \sqrt{I}$. Now a natural question arises as to how can one determine the least number s. This question leads to the following main definitions:

Definition 1.0.1 *If I is an ideal of R, the* ***arithmetic rank*** *of I, denoted by* $\operatorname{ara}(I)$*, is defined by*

$$ara(I) = min\{n \geq 0 | \text{ there exists } a_1, a_2, \cdots, a_n \text{ such that } \sqrt{I} = \sqrt{(a_1, a_2, \cdots, a_n)}\ \}.$$

By Krull's height theorem , if R is Noetherian, then $\operatorname{ara}(I) \geq \operatorname{ht}(I)$; meanwhile $\operatorname{ara}(I) \leq \dim(R) + 1$ by [1]. Hence the arithmetic rank of an ideal is bounded when $\dim(R) < \infty$. If $\operatorname{ara}(I) = \operatorname{ht}(I)$, then I is called a **set theoretic complete intersection ideal**. Determining set-theoretic complete intersection ideals is a classical and long-standing problem in commutative algebra and algebraic geometry, for a survey see [2]. Among an enormous amount of research, many questions related to an ideal being set-theoretic complete intersection are still open including the following major one:

Conjecture 1 *Is every (irreducible) curve in 3-space the set theoretic intersection of two hypersurfaces, or equivalently is every height two prime ideal of $K[X, Y, Z]$ set theoretic complete intersection?*

It was proven by Cowsik and Nori that this question has an affirmative answer in characteristic $p > 0$ case. However, it still remains open in the case when K is of characteristic zero.

One strategy to approach this conjecture is to use the local cohomology modules and the motivation is suggested by the well-known fact that $\mathrm{ht}(I) \leq \mathrm{cd}(I,R) = \sup\{i \in \mathbb{N} \mid H_I^i(R) \neq 0\} \leq \mathrm{ara}(I)$. However if the relevant local cohomology module vanishes (i.e. if $\mathrm{cd}(I,R) \lneq \mathrm{ara}(I)$), then $H_I^i(R)$ does not give any information to determine $\mathrm{ara}(I)$. But, surprisingly, Hellus showed that the Matlis duals of local cohomology modules, $D(H_I^i(R))$, determine exactly whether or not an ideal is set theoretic complete intersection by proving the following result:

Theorem 1.0.2 ([3], Corollary 1.1.4) *Let $(R,\mathfrak{m})$ be a Noetherian local ring , I a proper ideal of R, $c \in \mathbb{N}$ and $f_1, f_2, \cdots, f_c \in I$ an R-regular sequence. The following statements are equivalent:*

(i) $\sqrt{(f_1, f_2, \cdots, f_c)} = \sqrt{I}$*-up to radical-the set theoretic complete intersection ideal* $(f_1, f_2, \cdots, f_c)$*; in particular it is a set theoretic complete intersection ideal itself.*

(ii) $H_I^l(R) = 0$ *for all* $l > c$ *and the sequence* $f_1, f_2, \cdots, f_c \in I$ *is regular on* $D(H_I^c(R))$.

Motivated by this result, Hellus studied the associated primes of Matlis duals of the top local cohomology modules and conjectured the following equality:

$$\mathrm{Ass}_R(D(H^c_{(x_1,x_2,\cdots,x_c)}(R))) = \{\mathfrak{p} \in Spec(R) \mid H^c_{(x_1,x_2,\cdots,x_c)}(R/\mathfrak{p}) \neq 0\}$$

It has been shown that this conjecture holds true in many cases; see eg. [4–7]. Furthermore, he proved that the above conjecture is equivalent to the following condition [[3], Theorem 1.2.3]:

- If $(R,\mathfrak{m})$ is a Noetherian local domain, $c \geq 1$ and $x_1, x_2, \cdots, x_c \in R$, then the implication
  $H^c_{(x_1,x_2,\cdots,x_c)}(R) \neq 0 \implies 0 \in \mathrm{Ass}_R(D(H^c_{(x_1,x_2,\cdots,x_c)}(R))$
  holds.

With this in mind, Lyubeznik and Yıldırım, [8], conjectured that if R is regular, then the above implication holds for all non-zero ideals independently of the number of generators:

Conjecture 2 *Let* $(R,\mathfrak{m})$ *be a Noetherian regular local ring, I be a non-zero ideal of R and* $i \geq 1$. *If* $H_I^i(R) \neq 0$, *then* $0 \in \operatorname{Ass}_R(D(H_I^i(R)))$.

Note that Conjecture 2 is not true for non-regular rings. For a concrete example of a Noetherian local ring $(A,\mathfrak{m})$ of dimension > 1 such that $H_{\mathfrak{m}}^1(A) = A/\mathfrak{m}$, hence $0 \notin \operatorname{Ass}_R(D(H_{\mathfrak{m}}^1(A)))$, see [[9], Example 2.4].

One of the main result in this thesis lend credence to Conjecture 2 in equicharacteristic $p > 0$ case.

On the other hand, Erdoğdu approached Conjecture 1 from an original and a broader perspective. In his joint work with McAdam, [10], they gave an example which showed that the radical of ideals in characteristic zero behaves differently than in characteristic positive case. Afterwards, Erdoğdu's several results supported this fact and made him have an inkling that Conjecture 1 is not true in more general case and define the radically perfect ideals.

Call an ideal I of a commutative ring R radically perfect if among the ideals of R whose radical is equal to the radical of I the one with the least number of generators has this number of generators equal to the height of I. Clearly when R is Noetherian, the terms radically perfect and set theoretic complete intersection are synonymous. In non-Noetherian cases, examples of radically perfect ideals include all prime ideals of a finite character UFD R of Krull dimension ≤ 2.(A ring R is of finite character if each nonzero element of it is contained in only finitely many maximal ideals.) This is because if $\mathfrak{p}$ is any prime ideal of R, then either $\mathfrak{p}$ is of height one in which case $\mathfrak{p} = (u)$ for some irreducible $u \in \mathfrak{p}$ and hence $\operatorname{grade}(\mathfrak{p}) = \operatorname{ara}(\mathfrak{p})$ and so $\mathfrak{p}$ is radically perfect, or $\mathfrak{p}$ is of height two in that case we may choose an irreducible element u in $\mathfrak{p}$. Then u is contained in only finitely many maximal ideals. Let $\mathfrak{p} = \mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_n$ be the only maximal ideals of R containing u, then clearly there are elements v in $\mathfrak{p} = \mathfrak{p}_1$ and w in $\mathfrak{p}_2\mathfrak{p}_3 \cdots \mathfrak{p}_n$ such that $v + w = 1$ and that $\mathfrak{p} = \sqrt{(u,v)}$. It is also clear that $\{u,v\}$ is a regular sequence in $\mathfrak{p}$ and therefore $\operatorname{grade}(\mathfrak{p}) = \operatorname{ara}(\mathfrak{p})$. (For a more general statement, see Theorem 4.1 of [11]).

Now let S be an integrally closed strong S-domain of Krull dimension one having the property that each prime ideal of it is the radical of a principal ideal (e.g. S could be either a semi-local Noetherian, or a PID, or a Dedekind normal domain with torsion ideal class group, or more generally a Prüfer domain with torsion Picard group, see [[12], Theorem 2.1] and [[13], Theorem 3.3], respectively) and let $R = S[X]$, then each prime ideal $\mathfrak{p}$ of R has the property that $\mathrm{grade}(\mathfrak{p}) = \mathrm{ara}(\mathfrak{p})$.

Facts of these types led Erdoğdu to state the following conjecture [[14], Question 3.3]:

Conjecture 3 *For any commutative domain R containing a field of characteristic zero, each prime ideal of $R[X]$ is radically perfect implies R is of Krull dimension at most one.*

which was proved to be so by A. Mimouni in [15] in the case when R is a Prüfer domain but still remains open in the general case. Another major motivation of this study is to find some related results which support Conjecture 3.

The outline of this thesis is as follows:

In chapter 2, we collect some preliminary materials on local coholomogy, Matlis duality and spectral sequences.

In chapter 3, we concentrate on the Matlis duals of local cohomology modules when the underlying ring is of characteristic $p > 0$. The main result of this chapter is that over a complete Noetherian regular local ring of characteristic $p > 0$, for an F-finite F module $\mathscr{M}$ with $0 \notin \mathrm{Ass}(\mathscr{M})$, $0 \in \mathrm{Ass}(D(\mathscr{M}))$. As an immediate consequence of this result, we establish Conjecture 2 in the equicharacteristic $p > 0$ case.

In chapter 4, we examine the relation between radically perfect ideals and local cohomology modules. In this regard in Section 1, we first prove a theorem which gives a sufficient condition for an integer t to be a lower bound for the cohomological dimension $\mathrm{cd}(I,M)$, and then use this to prove the main result of this section which states that non-catenary Noetherian integral domains contain prime ideals that are not radically perfect. In Section 2, we show the existence of a descending chain of ideals $I = I_c \supsetneq I_{c-1} \supsetneq \cdots \supsetneq I_0$ of R with succesive cohomological dimensions $\mathrm{cd}(I_i,M) = i$, $0 \leq i \leq c$.

Chapter 5 constitutes the results on some structures of local cohomology modules. In Section 1, we determine the Artinianness of top local cohomology modules $H_I^{\mathrm{cd}(I,M)}(M)$, and we first prove that over a local unique factorization domain R of dimension at most three, the top local cohomology module $H_I^{\mathrm{cd}(I,R)}(R)$ is Artinian if and only if $\mathrm{cd}(I,R) = \dim R$. On the other hand, it is known that if R is of dimension ≥ 4, then there are cases where $H_I^{\mathrm{cd}(I,R)}(R)$ is Artinian when $\mathrm{cd}(I,R) \neq \dim R$. With this in mind, we then investigate conditions on I which guarantees the existence of a sub-ideal J of I with $\mathrm{cd}(J,R) = \mathrm{cd}(I,R) = c$ and $H_J^c(R)$ being always non-Artinian (regardless of $H_I^c(R)$ being Artinian or not). Finally, we use the results of Chapter 4, among other things, to prove that over a Noetherian local ring $(R,\mathfrak{m})$, for a finitely generated R-module M of dimension n and for an ideal I of R with $\dim(M/IM) = 1$, $H_I^{\mathrm{cd}(I,M)}(M)$ is Artinian if and only if $\mathrm{cd}(I,M) = n$. In Section 2, our study is concerned around modules of finite length. We first provide conditions equivalent to M and all its Koszul cohomology modules $\mathrm{H}^i(\mathbf{x}^\infty M)$ to be of length at most two, where $\mathbf{x} = \{x_1, x_2, \cdots, x_n\}$ is any sequence of elements in R. We then consider the case for divisible modules and show that over a reduced Noetherian ring, finitely generated divisible modules are of finite length and that a reduced local ring R with finitely many prime ideals possesses a nonzero finitely generated divisible module implies that R is of Krull dimension zero. We use these results to give an alternative proof of the well-known fact that if R is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and M a finitely generated R-module with dimension d, then $\mathrm{H}_\mathfrak{m}^d(M)$ is finitely generated if and only if $d = 0$.

Chapter 6 is the final chapter which contains a brief summary of the contributions of this thesis, along with some suggestions for future study.

2. PRELIMINARIES

In this thesis, we assume the background knowledge of commutative algebra and homological algebra for which we suggest the references [16] and [17], respectively.

In this chapter, we give a very brief introduction to local cohomology modules, Matlis duality and spectral sequences which we will need them in this thesis.

2.1 Local Cohomology Modules

Local cohomology theory was first recognized by Alexander Grothendieck in his 1961 Harvard seminar, the notes of which was later written out by Robin Harthorne in [18]. Since then, with its widespread applications in commutative algebra, algebraic geometry and algebraic topology, this theory has become an important and interesting research area of its own and sparks numerous algebraists' interest.

Here we collect some basic definitions and theorems on local cohomology modules and our main reference is [19].

Definition 2.1.1 *Let R be a ring and I be an ideal of R. For an R-module M, set*

$$\Gamma_I(M) = \{x \in M : \text{there exists an } n \in \mathbb{N} \text{ such that } I^n x = 0\}.$$

Then $\Gamma_I(M)$ is defined as the I-torsion submodule of M.
It is not difficult to see that $\Gamma_I(-) : \mathscr{C}(R) \longrightarrow \mathscr{C}(R)$ defines an additive, left exact and covariant functor and this functor is referred as an "I-torsion functor".

Definition 2.1.2 *The local cohomology functor, denoted by $H^i_I(-)$, is defined as the i^{th}-right derived functors of $\Gamma_I(-)$.*
Hence the i^{th} local cohomology module of any R-module M with support in an ideal I is the i^{th} cohomology module of the sequence obtained by applying $\Gamma_I(-)$ to an injective resolution of M. But then since $\Gamma_I(-)$ is left exact, $H^0_I(M) = \Gamma_I(M)$.

Henceforth in this section, let R denote a Noetherian commutative ring with unity, I and J be ideals of R and M be an R-module.

Now we list some basic properties of local cohomology modules which we need them in the following parts of this thesis.

Theorem 2.1.3 *1. If $\sqrt{I}=\sqrt{J}$, then $H^i_I(-)\cong H^i_J(-)$, for all i. [[19], Remark 1.2.3]*

2. *For any multiplicatively closed set S, $S^{-1}(H^i_I(M))\cong H^i_{IS^{-1}R}(S^{-1}M)$[[19], Exercise 1.2.7]*

3. *If M is a J-torsion R-module, then $H^i_I(M)\cong H^i_{I+J}(M)$ for all i. [[19], Exercise 2.1.9]*

4. *If I is generated by n-elements, then $H^i_I(M)=0$ for all $i>n$. [[19], Theorem 3.3.1]*

5. *Grothendieck's vanishing theorem: $H^i_I(M)=0$ for all $i>\dim\operatorname{Supp}(M)$. [[19], Theorem 6.1.2]*

6. *Grothendieck's non-vanishing theorem: If $(R,\mathfrak{m})$ is a local ring and M is a finitely generated R-module of dimension d, then $H^d_{\mathfrak{m}}(M)\neq 0$. [[19], Theorem 6.1.4]*

7. *If $(R,\mathfrak{m})$ is a local ring and M is a finitely generated R-module, then $H^i_{\mathfrak{m}}(M)$ is Artinian for all i. [[19], Theorem 7.1.3]*

8. *If M is a finitely generated R-module of dimension d, then $H^d_I(M)$ is Artinian. [[19], Theorem 7.1.7]*

9. *If $(R,\mathfrak{m})$ is a Gorenstein local ring of dimension n, then $H^i_{\mathfrak{m}}(R)=0$ for all $i\neq n$ and $H^n_{\mathfrak{m}}(R)$ is isomorphic to the injective hull of $R/\mathfrak{m}$. [[19], Lemma 11.2.3]*

Here we give some exact sequences related to local cohomology modules:

Theorem 2.1.4 ([19], Theorem 3.2.3) *(Mayer-Vietoris Sequence) There is a long exact sequence of R-modules*

$$\cdots\longrightarrow H^{i-1}_{I\cap J}(M)\longrightarrow H^i_{I+J}(M)\longrightarrow H^i_I(M)\bigoplus H^i_J(M)\longrightarrow H^i_{I\cap J}(M)\longrightarrow H^{i+1}_{I+J}(M)\longrightarrow\cdots$$

Theorem 2.1.5 ([20], Corollary 3.5) *Let $x\in R$ be any element of R. Then there is a short exact sequence*

$$0\longrightarrow H^1_{Rx}(H^i_I(M))\longrightarrow H^{i+1}_{I+Rx}(M)\longrightarrow H^0_{Rx}(H^{i+1}_I(M))\longrightarrow 0.$$

The definition of local cohomology modules given in terms of right derived functor is sometimes inconvenient. But there are several equivalent definitions for local cohomology modules which make the calculations easier. In the following, we list two of them:

Theorem 2.1.6 *Let R be a Noetherian ring, I be an ideal of R and M be an R-module. Then*

$$H_I^i(M) \cong \varinjlim \operatorname{Ext}_R^i\left(R/I^n, M\right)$$

Our next objective is to give a quite different definition of local cohomology; using either a direct limit of Koszul cohomology or a certain kind of Čech cohomology:

Definition 2.1.7 *For any $x \in R$, the Čech complex of R with respect to x is the complex given by $C^\bullet(x;R): \ 0 \longrightarrow R \longrightarrow R_x \longrightarrow 0.$ graded so that the degree 0 piece of the complex is R, and the degree 1 is R_x where the differential is the natural localization map. Let now $\boldsymbol{x} = \{x_1, x_2, \cdots, x_n\}$ be a sequence of elements in R. Then the Čech complex of R with respect to the sequence $\boldsymbol{x}$ is defined as in the following:*

$$C^\bullet(\boldsymbol{x};R) = C^\bullet(x_1;R) \otimes_R C^\bullet(x_2;R) \otimes_R \cdots \otimes_R C^\bullet(x_n;R).$$

For any R-module M, the Čech complex of M with respect to the sequence $\boldsymbol{x}$, denoted by $C^\bullet(\boldsymbol{x};M)$, is the tensor product $C^\bullet(\boldsymbol{x};M) = C^\bullet(\boldsymbol{x};R) \otimes_R M$. Then the modules in $C^\bullet(\boldsymbol{x};M)$ are

$$0 \longrightarrow M \longrightarrow \oplus_i M_i \longrightarrow \oplus_{i<j} M_{x_i x_j} \longrightarrow \cdots \longrightarrow M_{x_1 x_2 \cdots x_n} \longrightarrow 0,$$

where the differentials are the natural maps induced from localization, but with suitable signs attached.

Definition 2.1.8 *Given any $x \in R$, the Koszul complex of R with respect to x is given by $K^\bullet(x;R): \ 0 \longrightarrow R \xrightarrow{x} R \longrightarrow 0,$ where the differential is just multiplication by x. It is not difficult to see that $C^\bullet(x;R) = \varinjlim K^\bullet(x^i;R)$ and so for any sequence $\boldsymbol{x} = \{x_1, x_2, \cdots, x_n\}$, we have the isomorphism $C^\bullet(\boldsymbol{x};R) \cong \varinjlim K^\bullet(x_1^j, x_2^j, \cdots, x_n^j;R)$.*

Theorem 2.1.9 *Let M be any R-module and $I = (x_1, x_2, \cdots, x_n)$ be an ideal of R. Then*

$$H_I^i(M) \cong H^i(C^\bullet(x_1, x_2, \cdots, x_n;M)) \cong H^i(\varinjlim K^\bullet(x_1^j, x_2^j, \cdots, x_n^j;M))$$

We end this section with the following basic example:

Example 2.1.10 *Let $R = k[x]$ be a polynomial ring over the field k and $I = (x)$ an ideal of R. Then the exact sequence*

$$0 \longrightarrow R \longrightarrow K \longrightarrow K/R \longrightarrow 0,$$

where $K = k(x)$ is the fraction field of R, is the injective resolution of R. Then the local cohomology modules, $H_I^i(R)$, can be computed by taking the cohomology of the sequence

$$0 \longrightarrow \Gamma_I(K) \longrightarrow \Gamma_I(K/R) \longrightarrow 0.$$

Since K is torsion-free, $\Gamma_I(K) = 0$ and so $H_I^0(R) = 0$ and $H_I^1(R) = \Gamma_I(K/R)$. Hence $H_I^1(R)$ is the set of x-torsion elements in K/R which can be identified as $R_x/R = k[x,x^{-1}]/k[x]$. Moreover it is obvious from the above exact sequence that $H_I^i(R) = 0$ for all $i > 0$.
One could obtain the same result by using Čech complex in which there is no need to determine the injective resolution of R.

2.2 Matlis Duality

In his Ph.D. thesis, Eben Matlis studied the theory of injective modules and a special kind of duality, which was later referred as "Matlis duality", [21]. In this section, we recall some basic definitions and theorems about Matlis duality. All results listed here and more can be found in the Appendix A of [22].

Definition 2.2.1 *Let R be a commutative ring, M and N be R-modules, and $f : M \hookrightarrow N$ be an injective R-module homomorphism. If every nonzero R-submodule of N has nonzero intersection with $f(M)$, then $f : M \hookrightarrow N$ is called as an essential extension.*

It is a well-known fact that an R-module is injective if and only if it has no proper essential extension. Moreover, any R-module M has an essential extension $f : M \hookrightarrow \mathscr{I}$ with $\mathscr{I}$ is injective.

Definition 2.2.2 *The injective hull or injective envelope of M, which is denoted by $E_R(M)$, is an injective module containing M, and has the property that any injective module containing M contains an isomorphic copy of $E_R(M)$.*

Definition 2.2.3 *Let $(R,\mathfrak{m})$ be a Noetherian local ring, $E := E_R(R/\mathfrak{m})$ be the injective hull of the residue field $R/\mathfrak{m}$. Then Matlis dual of any R-module M is denoted by $D(M)$ and defined as $D(M) := \mathrm{Hom}_R(M,E)$.*

It is clear by definition that $D(-)$ is a contravariant exact functor.

Theorem 2.2.4 *Let $(R,\mathfrak{m})$ be a Noetherian complete local ring. Then*

- *There is one-to-one correspondence between Noetherian and Artinian R-modules given as follows: If M is Artinian (resp. Noetherian), then $D(M)$ is Noetherian (resp. Artinian).*
- *If M is either Noetherian or Artinian, then $D(D(M)) \cong M$.*
- *R and E are Matlis duals of each others.*

2.3 Spectral Sequences

We use spectral sequences in the proofs of our many results and so we give some definitions and basic facts on them. Our reference in this section is [17]. Throughout, let $\mathscr{C}$ be an abelian category.

Definition 2.3.1 *For all integers p,q and r with $r \geq 1$, a cohomological spectral sequence in $\mathscr{C}$ consists of*

1. *a family of objects $\{E_r^{p,q}\}$ in $\mathscr{C}$,*
2. *$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$ such that $d_r^2 = 0$ (i.e. $d_r^{p,q} \circ d_r^{p+r,q-r+1} = 0$), and*
3. *isomorphisms $\alpha : \ker(d_r^{p,q})/\mathrm{im}(d_r^{p+r,q-r+1}) \longrightarrow E_{r+1}^{p,q}$*

Definition 2.3.2 *A spectral sequence $\{E_r^{p,q}\}$ is bounded if for all $r,n \in \mathbb{N}$, the number of non-zero objects of the form $\{E_r^{k,n-k}\}$ is finite.*

Example 2.3.3

- *If $E_r^{p,q} = 0$ unless $p \geq 0$ and $q \geq 0$, $\{E_r^{p,q}\}$ is first quadrant spectral sequences.*

- *If $E_r^{p,q} = 0$ unless $p \leq 0$ and $q \leq 0$, $\{E_r^{p,q}\}$ is third quadrant spectral sequences.*

Such spectral sequences are bounded.

It is worth noting that if $\{E_r^{p,q}\}$ is a bounded spectral sequence, then for each p, q, there is an r_0 such that $E_r^{p,q} = E_{r+1}^{p,q}$ for all $r \geq r_0$. We write $E_\infty^{p,q}$ to this stable value of $E_r^{p,q}$.

Definition 2.3.4 *A bounded spectral sequence $\{E_r^{p,q}\}$ convergences to $\mathrm{H}^\star$ if there exists a finite filtration*

$$0 = \Phi^t \mathrm{H}^n \subseteq \Phi^{t-1} \mathrm{H}^n \subseteq \cdots \subseteq \Phi^1 \mathrm{H}^n \subseteq \Phi^0 \mathrm{H}^n = \mathrm{H}^n$$

of H^n such that $E_\infty^{p,q} = \Phi^p \mathrm{H}^n / \Phi^{p+1} \mathrm{H}^n$ for all $p + q = n$.

Recall that for any additive functor $F : \mathscr{A} \longrightarrow \mathscr{B}$ between abelian categories, an object $A \in \mathscr{A}$ is F acyclic if all right derived functors $\mathscr{R}^i F(A) = 0$ for all $i > 0$. As an example, take any ideal I of a Noetherian ring R, then any injective R-module can be viewed as a $\Gamma_I(-) : \mathscr{C}(R) \longrightarrow \mathscr{C}(R)$ acyclic, where $\Gamma_I(-)$ is an I-torsion functor defined in 2.1.

In this thesis, we will need the following special type of spectral sequence; which is known as the Grothendieck composite-functor spectral sequence:

Theorem 2.3.5 ([17], Theorem 5.8.3) *Let $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be Abelian categories, and suppose $\mathscr{A}$ and $\mathscr{B}$ have enough injective objects. Let $F : \mathscr{A} \longrightarrow \mathscr{B}$ and $G : \mathscr{B} \longrightarrow \mathscr{C}$ be left-exact additive functors. Suppose that for every injective object I of $\mathscr{A}$, the object $F(I)$ of $\mathscr{B}$ is acyclic for G. Then for every object A of $\mathscr{A}$, there is a spectral sequence such that*

$$E_2^{p,q} = \mathscr{R}^p F \mathscr{R}^q G(A) \Longrightarrow \mathscr{R}^{p+q}(FG)(A)$$

3. MATLIS DUALS OF LOCAL COHOMOLOGY MODULES IN CHARACTERISTIC $p > 0$

This chapter consists of results from the joint work with Gennady Lyubeznik, [8].

Our main purpose here is to prove the following result which establish Conjecture 2 in equicharacteristic $p > 0$:

Theorem 3.0.1 *Let $(R, \mathfrak{m})$ be a complete Noetherian regular local ring containing a field of characteristic $p > 0$ and I a non-zero ideal of R. If $H^i_I(R) \neq 0$, then $0 \in \mathrm{Ass}_R(D(H^i_I(R)))$.*

We need F-module theory for the proof of Theorem 3.0.1 and so we first collect some basic definitions and results about this theory and our main reference is [23].

3.1 Preliminaries on Lyubeznik's F-Modules

Throughout, R is a commutative Noetherian regular ring of characteristic $p > 0$.

Definition 3.1.1 *Let $R^{'}$ be the additive group of R regarded as an R- bi-module with the usual left action and with the right R- action defined by $r^{'}r = r^p r^{'}$ for all $r \in R$ and $r^{'} \in R^{'}$. The Frobenius functor*

$$F : R - mod \longrightarrow R - mod$$

of Peskine-Szpiro [24] is defined by

$$F(M) = R^{'} \otimes_R M$$

$$F(M \xrightarrow{h} N) = (R^{'} \otimes_R M \xrightarrow{id \otimes_R h} R^{'} \otimes_R N)$$

for all R-modules M and all R-module homomorphisms h, where $F(M)$ acquires its R-module structure via the left R-module structure on $R^{'}$.

The iteration of a Frobenius functor on R leads one to the iterated Frobenius functors $F^i(-)$ which are defined for all $i \geq 1$ recursively by $F^1(-) = F(-)$ and $F^{i+1} = F \circ$

$F^i(-)$ for all $i \geq 1$.

Note that the Frobenius functor $F(-)$ is exact [[25], Theorem 2.1]; $F(R) \cong R$ and for any ideal I of R, $F(R/I) = R/I^{[p]}$, where $I^{[p]}$ is the ideal of R generated by p-th powers of all elements of I [[24], I.1.3d].

Note also that if R is a complete local ring, then for any Artinian R-module N, $F(D(N)) = D(F(N))$ [[23], Lemma 4.1] and so $R = F(R) = F(D(E)) = D(F(E))$ implies $F(E) = E$. Then it follows from Remark 1.0.(f) of [23] that for any finitely generated R-module M, $F(D(M)) = D(F(M))$.

Now, for an R-module M, define a Frobenius map $\psi_M : M \longrightarrow F(M)$ on M by $\psi_M(m) := 1 \otimes m \in F(M)$ for all $m \in M$. It is worth pointing out that if $\mathrm{ann}(m) = I \subseteq R$, then $\mathrm{ann}(\psi_M(m)) = I^{[p]}$.

An F-module $\mathscr{M}$ is an R- module equipped with R-module isomorphism $\theta : \mathscr{M} \longrightarrow F(\mathscr{M})$ which we call the structure morphism.

A generating morphism of an F module $\mathscr{M}$ is an R-module homomorphism $\beta : M \longrightarrow F(M)$, where M is some R-module, such that $\mathscr{M}$ is the limit of the inductive system in the top row of the commutative diagram

$$\begin{array}{ccccccc} M & \xrightarrow{\beta} & F(M) & \xrightarrow{F(\beta)} & F^2(M) & \xrightarrow{F^2(\beta)} & \cdots \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle F(\beta)} & & \downarrow{\scriptstyle F^2(\beta)} & & \\ F(M) & \xrightarrow{F(\beta)} & F^2(M) & \xrightarrow{F^2(\beta)} & F^3(M) & \xrightarrow{F^3(\beta)} & \cdots \end{array}$$

and $\theta : \mathscr{M} \longrightarrow F(\mathscr{M})$, the structure isomorphism of $\mathscr{M}$, is induced by the vertical arrows in this diagram.

If β is an injective map, then the exactness of F implies that all maps in the direct limit system are injective, so that M injects into $\mathscr{M}$. In this case, we shall refer to β as a root morphism of $\mathscr{M}$, and M as a root of $\mathscr{M}$. If $\mathscr{M}$ is an F-module possessing a root morphism $\beta : M \longrightarrow \mathscr{M}$ with M finitely generated, then we say that $\mathscr{M}$ is F-finite. In particular, R, with any F-module structure, is an F-finite module.

3.2 Main Results

In this section, we first prove the following result as a consequence of which we lend credence to Conjecture 2 in equicharacteristic $p > 0$:

Theorem 3.2.1 *Let $(R,\mathfrak{m})$ be a complete Noetherian regular local ring of characteristic $p > 0$ and $\mathscr{M}$ be an F-finite F module such that $0 \notin \mathrm{Ass}(\mathscr{M})$. Then $0 \in \mathrm{Ass}(D(\mathscr{M}))$.*

We would like to point out that $0 \notin \mathrm{Ass}(\mathscr{M})$ is a necessary condition of Theorem 3.2.1. Indeed, R itself is an F-finite F module and $0 \in \mathrm{Ass}(R)$ but $0 \notin \mathrm{Ass}(D(R)) = \mathrm{Ass}(E) = \{\mathfrak{m}\}$.

We need a series of lemmas to give the proof of Theorem 3.2.1.

Lemma 3.2.2 *Let $(R,\mathfrak{m})$ be a complete Noetherian regular local ring containing a field of characteristic $p > 0$ and $\mathscr{M}$ be an F-finite F-module such that $0 \notin \mathrm{Ass}(\mathscr{M})$. Then the Matlis dual of $\mathscr{M}$, $D(\mathscr{M})$, can be expressed as*

$$D(\mathscr{M}) = \varprojlim(N \xleftarrow{\alpha} F(N) \xleftarrow{F(\alpha)} F^2(N) \xleftarrow{F^2(\alpha)} \cdots),$$

where N is an Artinian R-module and $\alpha : F(N) \longrightarrow N$ is a surjective map such that $\mathrm{Ker}(\alpha : F(N) \to N) \neq 0$.

Proof. Since $\mathscr{M}$ is an F-finite F-module, there exists a root morphism $\beta : M \to F(M)$ with a finitely generated R-module M such that

$$\mathscr{M} = \varinjlim(M \xrightarrow{\beta} F(M) \xrightarrow{F(\beta)} F^2(M) \xrightarrow{F^2(\beta)} \cdots).$$

Then applying Matlis dual functor $D(-) = \mathrm{Hom}_R(-, E_R(R/\mathfrak{m}))$ to $\mathscr{M}$, we obtain

$$D(\mathscr{M}) = \varprojlim(D(M) \xleftarrow{D(\beta)} D(F(M)) \xleftarrow{D(F(\beta))} D(F^2(M)) \xleftarrow{D(F^2(\beta))} \cdots).$$

But then since Frobenius functor commutes with $D(-)$, we can write $D(\mathscr{M})$ as

$$D(\mathscr{M}) = \varprojlim(N \xleftarrow{\alpha} F(N) \xleftarrow{F(\alpha)} F^2(N) \xleftarrow{F^2(\alpha)} \cdots),$$

where $N = D(M)$ and $\alpha = D(\beta)$. Then since β is injective and M is a finitely generated, $\alpha = D(\beta)$ is surjective and $N = D(M)$ is Artinian.

On the other hand, since $0 \notin \mathrm{Ass}(\mathscr{M})$, $I = \mathrm{Ann}(M) = \mathrm{Ann}(N)$ is a nonzero ideal of R. Then it follows that $\mathrm{Ann}(F(N)) = I^{[p]}$ and so $\mathrm{Ker}(\alpha : F(N) \to N) \neq 0$, as desired. $\square$

Lemma 3.2.3 *Let the notations be as in Lemma 3.2.2. Then, for each* $k \geq 1$*, there exists* $b_k \in \operatorname{Ker}(F^{k-1}(\alpha))$ *such that* $\operatorname{ann}(b_k) = \mathfrak{m}^{[p^{k-1}]}$.

Proof. Since $\operatorname{Ker}(\alpha : F(N) \to N) \neq 0$ is a non-zero Artinian R-module, there exists an element $b_1 \in \operatorname{Soc}(\operatorname{Ker}(\alpha)) \subseteq F(N)$, where $\operatorname{Soc}(\operatorname{Ker}(\alpha)) := \operatorname{Ann}_{\operatorname{Ker}(\alpha)}(\mathfrak{m})$ denotes the socle of $\operatorname{Ker}(\alpha)$ and define b_k, for all $k \geq 2$, inductively as the image of b_{k-1} under the Frobenius map (defined in the preceding section) on $F^{k-1}(N)$, that is $b_k := \psi_{F^{k-1}(N)}(b_{k-1}) = 1 \otimes b_{k-1} \in F^k(N)$. Then by induction on k (considering that $\operatorname{ann}(b_1) = \mathfrak{m}$ and $\operatorname{ann}(x) = I$ implies $\operatorname{ann}(\psi(x)) = I^{[p]}$), we have $\operatorname{ann}(b_k) = \mathfrak{m}^{[p^{k-1}]}$. On the other hand, since $b_1 \in \operatorname{Ker}(\alpha) := \operatorname{Ker}(F^0(\alpha))$, an easy induction argument shows that $b_k \in \operatorname{Ker}(F^{k-1}(\alpha))$ for all $k \geq 0$. For if $b_{k-1} \in \operatorname{Ker}(F^{k-2}(\alpha))$, then $F^{k-1}(\alpha)(b_k) = F^{k-1}(\alpha)(1 \otimes b_{k-1}) = 1 \otimes F^{k-2}(\alpha)(b_{k-1}) = 0$. □

Lemma 3.2.4 *Let the notations be as in Lemma 3.2.2 and let* b_k *be defined as in Lemma 3.2.3 and* $y \in \mathfrak{m} \setminus \mathfrak{m}^k$*. Then* $\operatorname{ann}(yb_k) \subseteq \mathfrak{m}^{p^{k-1}-k}$*. In particular, if* $k \geq 4$*,* $\operatorname{ann}(yb_k) \subseteq \mathfrak{m}^k$.

Proof. To prove the fact that $\operatorname{ann}(yb_k) \subseteq \mathfrak{m}^{p^{k-1}-k}$, suppose on the contrary that there exists an element $z \in \operatorname{ann}(yb_k)$ such that $z \notin \mathfrak{m}^{p^{k-1}-k}$. Then clearly, $yz \in \operatorname{ann} b_k$. On the other hand as $R \cong \kappa[[X_1,...,X_n]]$, $\kappa \cong R/\mathfrak{m}$ a field of characteristic $p > 0$, and $y \notin \mathfrak{m}^k$ and $z \notin \mathfrak{m}^{p^{k-1}-k}$, we may write

$$y = f + f'$$
$$z = g + g'$$

where f (resp. g) is a nonzero polynomial in $\kappa[[X_1,X_2,...,X_n]]$ of degree at most $k-1$(resp. $p^{k-1}-k-1$) and f'(resp. g') is either zero or a formal power series in $\kappa[[X_1,X_2,...,X_n]]$ in which each summand has degree at least k (resp. $p^{k-1}-k$). Then $yz = fg + fg' + gf' + g'f'$. Note that since $\kappa[[X_1,...,X_n]]$ is an integral domain and f and g are non-zero elements in $\kappa[[X_1,...,X_n]]$, so is fg. Note also that since fg', gf' and $g'f'$ are either zero or contain terms of degrees strictly bigger than the smallest degree of fg, they cannot cancel any terms of smallest degree. But then since the degree of the smallest term of fg is less than or equal to $0 \neq \deg(fg) \leq p^{k-1}-k-1+k-1 = p^{k-1}-2$, $yz \notin \mathfrak{m}^{p^{k-1}}$ which contradicts the fact that $yz \in \operatorname{ann}(b_k) = \mathfrak{m}^{[p^{k-1}]}$. Hence

$\mathrm{ann}(yb_k) \subseteq \mathfrak{m}^{p^{k-1}-k}$, as desired.

If, in particular $k \geq 4$, then $p^{k-1} - k \geq k$ and so $\mathrm{ann}(yb_k) \subseteq \mathfrak{m}^{p^{k-1}-k} \subseteq \mathfrak{m}^k$. □

Now we are ready to give the proof of Theorem 3.2.1:

Proof of Theorem 3.2.1. Since $\mathscr{M}$ is an F-finite F- module such that $0 \notin \mathrm{Ass}(\mathscr{M})$, it follows from Lemma 3.2.2 that

$$D(\mathscr{M}) = \varprojlim(N \xleftarrow{\alpha} F(N) \xleftarrow{F(\alpha)} F^2(N) \xleftarrow{F^2(\alpha)} \cdots),$$

for some Artinian R-module N and surjective map $\alpha : F(N) \longrightarrow N$. It is worth noting that, the exactness of the functor $F^k(-)$ implies that $F^k(\alpha)$ is surjective for all $k \geq 0$.

Now we claim that there exists a nonzero element $n' = (n'_0, n'_1, \cdots, n'_k, \cdots) \in D(\mathscr{M})$ such that $\mathrm{ann}(n'_k) \subseteq \mathfrak{m}^k$ for all $k \geq 4$, where n'_k is the image of n' in $F^k(N)$.

To construct such an element, let n'_0 be an element of N and, for every $1 \leq k \leq 3$, choose $n'_k \in F^k(N)$ such that $n'_{k-1} = F^{k-1}(\alpha)(n'_k)$. For $k \geq 4$, let $b_k \in \mathrm{Ker}(F^{k-1}(\alpha))$ be as defined in Lemma 3.2.3 and define n_k in such a way that $F^{k-1}(\alpha)(n_k) = n'_{k-1}$. Then, either $\mathrm{ann}(n_k) \subseteq \mathfrak{m}^k$ or $\mathrm{ann}(n_k + b_k) \subseteq \mathfrak{m}^k$. Indeed, if $\mathrm{ann}(n_k + b_k) \nsubseteq \mathfrak{m}^k$, there exists an element $y \in \mathfrak{m} \setminus \mathfrak{m}^k$ such that $y(n_k + b_k) = 0$ and so $\mathrm{ann}(n_k) \subseteq \mathrm{ann}(yn_k) = \mathrm{ann}(yb_k)$. But then it follows from Lemma 3.2.4 that $\mathrm{ann}(n_k) \subseteq \mathrm{ann}(yb_k) \subseteq \mathfrak{m}^k$.

Now, for $k \geq 4$, define

$$n'_k = \begin{cases} n_k, & \text{if } \mathrm{ann}(n_k) \subseteq \mathfrak{m}^k, \\ n_k + b_k, & \text{otherwise.} \end{cases}$$

Clearly, $n' = (n'_0, n'_1, \cdots, n'_k, \cdots) \in \mathrm{D}(\mathscr{M})$ and $\mathrm{ann}(n'_k) \subseteq \mathfrak{m}^k$ for all $k \geq 4$. This proves the claim.

Finally, $\mathrm{ann}(n') = 0$ for if $z \in \mathrm{ann}(n')$, then $z \in \mathrm{ann}(n'_k) \subseteq \mathfrak{m}^k$ for all $k \geq 4$ which then implies that $z \in \bigcap_{n \in \mathbb{N}} \mathfrak{m}^n = \{0\}$. This completes the proof. □

We conclude this chapter with the proof of Theorem 3.0.1.

Proof of Theorem 3.0.1. Without loss of generality, we may, and do, assume that R is complete [[3], Remark 4.1.1]. Since R is an F-finite F- module, so are its all local cohomology modules and since $0 \notin \mathrm{Ass}_R(H^i_I(R))$ for any nonzero ideal I of R, the result follows from Theorem 3.2.1. □

4. RADICALLY PERFECT IDEALS AND LOCAL COHOMOLOGY MODULES

This chapter consists of some results from our joint work with Vahap Erdoğdu, [26].

4.1 The Relation Between Radically Perfect Ideals and Local Cohomology

The aim of this section is to prove the following result:

Theorem 4.1.1 *Over a non-catenary Noetherian domain, there exists a prime ideal that is not a set-theoretic complete intersection.*

Although Theorem 4.1.1 is stated for Noetherian rings, the motivation behind it comes from a more general setting as indicated in Chapter 1 and bearing in mind that non-catenary rings are of Krull dimension > 2, this theorem is in partial support of Conjecture 3.

We need the following theorem for the proof of Theorem 4.1.1.

Theorem 4.1.2 *Let R be a Noetherian ring, M a finitely generated R-module and I an ideal of R with $\dim \operatorname{Supp}_R(M/IM) = d$. Let $t \geq 0$ be an integer. If there exists an ideal J of R such that $H^{d+t}_{I+J}(M) \neq 0$, then t is a lower bound for $cd(I,M)$. Moreover, if* $\operatorname{cd}(I,M) = t$*, then*

$$H^d_J(H^t_I(M)) \cong H^{d+t}_{I+J}(M)$$

and $\dim \operatorname{Supp} R(H^t_I(M)) = d$*.*

Proof. Consider the Grothendieck's spectral sequence

$$E_2^{p,q} = H^p_J(H^q_I(M)) \Longrightarrow H^{p+q}_{I+J}(M)$$

and look at the stage $p+q = d+t$. Since $\operatorname{Supp}(H^q_I(M)) \subseteq V(I) \cap \operatorname{Supp}(M) \subseteq V(I + \operatorname{Ann} M)$, $\dim \operatorname{Supp}(H^q_I(M)) \leq d$ for all q. Therefore it follows from Grothendieck's vanishing theorem that for all $p > d$, $E_2^{p,d+t-p} = 0$. But then since the limit term

$E_\infty^{d+t} = H_{I+J}^{d+t}(M)$ does not vanish, there is at least one $p \le d$ such that

$$E_2^{p,d+t-p} = H_J^p(H_I^{d+t-p}(M)) \neq 0.$$

Hence $H_I^{d+t-p}(M) \neq 0$ and so $\operatorname{cd}(I,M) \ge d+t-p \ge t$.

If, in particular, $\operatorname{cd}(I,M) = t$, then the above spectral sequence degenerates to an isomorphism $H_J^d(H_I^t(M)) \cong H_{I+J}^{d+t}(M)$.

Since $H_J^d(H_I^t(M)) \neq 0$, it follows from Grothendieck's vanishing theorem that $\dim \operatorname{Supp}_R(H_I^t(M)) \ge d$. On the other hand, since $\dim \operatorname{Supp}_R(\mathrm{H}_I^t(M)) \le \dim(R/I + \operatorname{Ann} M) = d$, we conclude that $\dim \operatorname{Supp}_R(H_I^t(M)) = d$. □

So far, for a finitely generated R- module M, the best known lower bound for $\operatorname{cd}(I,M)$ is $\operatorname{ht}_M(I) = \operatorname{ht} I(R/\mathrm{AnnM})$. As an immediate consequence of Theorem 4.1.2, we sharpen this bound to $\dim(M) - \dim(M/IM) \ge \operatorname{ht}_M(I)$.

Corollary 4.1.3 *Let* $(R,\mathfrak{m})$ *be a Noetherian local ring,* M *a finitely generated* R*-module of dimension* n *and* I *an ideal of* R *such that* $\dim(M/IM) = d$*. Then* $n-d$ *is a lower bound for* $\operatorname{cd}(I,M)$*. Moreover, if* $\operatorname{cd}(I,M) = n-d$*, then*

$$\mathrm{H}_\mathfrak{m}^d(\mathrm{H}_I^{n-d}(M)) \cong \mathrm{H}_\mathfrak{m}^n(M)$$

and $\dim \operatorname{Supp}(\mathrm{H}_I^{n-d}(M)) = d$.

Proof. This follows from Theorem 4.1.2 and the fact that $H_\mathfrak{m}^n(M) \neq 0$. □

Corollary 4.1.4 *Let* $(R,\mathfrak{m})$ *be a Noetherian local ring of dimension* n *and* I *an ideal of* R *with* $d = dim(R/I)$ *such that* $\operatorname{cd}(I,R) = \operatorname{ht}(I) = h$*. Then* $\dim(R) = \operatorname{ht}(I) + \dim(R/I)$ *and*

$$H_\mathfrak{m}^{n-h}(H_I^h(R)) \cong H_\mathfrak{m}^n(R).$$

Proof. It follows from Corollary 4.1.3 that $\dim(R) - \dim(R/I) \le \operatorname{cd}(I,R) = \operatorname{ht}(I)$, while the other side of the inequality always holds. Therefore $\dim(R) = \operatorname{ht}(I) + \dim(R/I)$. Now the required isomorphim follows from Corollary 4.1.3. □

With the help of the above results, we now prove Theorem 4.1.1:

Proof of Theorem 4.1.1. Let R be a non-catenary Noetherian domain. Then as being catenary is a local property, $R_\mathfrak{q}$ is a non-catenary local domain for some prime ideal $\mathfrak{q}$ of R. Hence there exists a prime ideal $\mathfrak{p}_\mathfrak{q}$ of $R_\mathfrak{q}$ such that $\operatorname{ht}(\mathfrak{p}_\mathfrak{q}) \lneq \dim(R_\mathfrak{q}) - \dim(R_\mathfrak{q}/\mathfrak{p}_\mathfrak{q})$. But then it follows from Corollary 4.1.4 that $\operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(\mathfrak{p}_\mathfrak{q}) \lneq \operatorname{cd}(\mathfrak{p}_\mathfrak{q}, R_\mathfrak{q}) \leq \operatorname{cd}(\mathfrak{p}, R)$ and therefore $\mathfrak{p}$ can not be a radically perfect(set theoretic complete intersection) ideal. □

On the other hand, being catenary is not sufficient to conclude that each prime ideal of a ring R is radically perfect. As an example take $(R, \mathfrak{m})$ to be a valuation ring of Krull dimension ≥ 2, then R is catenary and yet $\operatorname{ara}(\mathfrak{m}) = 1 < 2 \leq \operatorname{ht}(\mathfrak{m})$. Hence the maximal ideal $\mathfrak{m}$ of R is not radically perfect.

4.2 Descending Chain With Successive Cohomological Dimensions

Let $R = S[X]$ be any Noetherian polynomial ring of dimension n over a ring S. Then it follows from Theorem 1 of [27] that every maximal ideal $\mathfrak{m}$ of maximal height in R is radically perfect. The question we are interested in is whether there is a descending chain $\mathfrak{m} = \mathfrak{p}_n \supsetneq \mathfrak{p}_{n-1} \supsetneq \cdots \supsetneq \mathfrak{p}_0 = 0$ of (prime) ideals of R such that each $\mathfrak{p}_i$ is radically perfect for all i, $0 \leq i \leq n$.

The quest to an answer to this question led us to the following result:

Theorem 4.2.1 *Let R be a Noetherian ring, I an ideal of R and M any nonzero R-module with cohomological dimension $\operatorname{cd}(I, M) = c > 0$. Then there is a descending chain of ideals*

$$I = I_c \supsetneq I_{c-1} \supsetneq \cdots \supsetneq I_0$$

such that $\operatorname{cd}(I_i, M) = i$ for all i, $0 \leq i \leq c$.

Proof. Consider the set

$$\mathbb{S} = \{ J \subsetneq I \mid \operatorname{cd}(J, M) < c \}.$$

Clearly, the zero ideal belongs to $\mathbb{S}$ and so $\mathbb{S}$ is a non-empty subset of ideals of R. Since R is Noetherian, $\mathbb{S}$ has a maximal element, say I_{c-1}. We claim that $\operatorname{cd}(I_{c-1}, M) = c - 1$. To prove this, let $x \in I \setminus I_{c-1}$ and so $I_{c-1} + Rx \subseteq I$. But then it follows from the maximality of I_{c-1} in $\mathbb{S}$ and Remark 8.1.3 of [19] that

$$c \leq \mathrm{cd}(I_{c-1}+Rx, M) \leq \mathrm{cd}(I_{c-1}, M)+1 < c+1.$$

Hence $\mathrm{cd}(I_{c-1}+Rx, M) = \mathrm{cd}(I_{c-1}, M)+1 = c$ and so the claim follows.

Iterating this argument, one can obtain a descending chain of ideals, as desired. □

Recall that a subspace Z of a topological space X is said to be *locally closed*, if it is the intersection of an open and a closed set. Let X be a topological space, $Z \subseteq X$ be a locally closed subset of X and let F be an abelian sheaf on X. Then the i^{th} local cohomology group of F with support in Z is denoted by $H^i_Z(X,F)$. For its definition and details, see [18] and [28].

If, in particular, $X = Spec(R)$ is an affine scheme, where R is a commutative Noetherian ring, and $F = M^{\sim}$ is the quasi coherent sheaf on X associated to an R-module M, we write $H^i_Z(M)$ instead of $H^i_Z(X, M^{\sim})$.

The following corollary may be considered as an easy application of our result above.

Corollary 4.2.2 *Let R be a Noetherian ring, M an R-module and I an ideal of R such that $cd(I,M) = c > 1$. Then there is a descending chain of locally closed sets*

$$T_{c-1} \supsetneq T_{c-2} \supsetneq \cdots \supsetneq T_1$$

in Spec(R) such that $cd(T_i, M) = i$ for all $1 \leq i \leq c-1$.

Proof. Let I be an ideal of R with $cd(I,M) = c > 1$. Then it follows from Theorem 4.2.1 that there is a descending chain of ideals

$$I = I_c \supsetneq I_{c-1} \supsetneq \cdots \supsetneq I_1 \supsetneq I_0$$

such that $cd(I_i, M) = i$ for all $0 \leq i \leq c$. Let now $U_i = V(I_i)$ and define the locally closed sets $T_i := U_1 \setminus U_{i+1}$. Then it is easy to see that

$$T_{c-1} \supsetneq T_{c-2} \supsetneq \cdots \supsetneq T_1.$$

On the other hand, it follows from Proposition 1.2 of [28] that there is a long exact sequence,

$$\cdots \longrightarrow H^j_{U_1}(M) \longrightarrow H^j_{T_i}(M) \longrightarrow H^{j+1}_{U_{i+1}}(M) \longrightarrow H^{j+1}_{U_1}(M) \longrightarrow \cdots$$

As $H^j_{U_i}(M) \cong H^j_{I_i}(M)$ for all $1 \leq i \leq c-1$ and for all $j \geq 0$, it follows from the above long exact sequence that $cd(T_i, M) = i$. □

5. SOME APPLICATIONS ON THE STRUCTURES OF LOCAL COHOMOLOGY MODULES

One of the important problems in the theory of local cohomology modules is to determine whether a given local cohomology module is Artinian, or not [Third Problem, [29]], which was studied by several authors, see eg. [30–32]. In the first section of this chapter, we obtain some related results to this problem, particularly for the top local cohomology modules, $H_I^{\mathrm{cd}(I,M)}(M)$, where $\mathrm{cd}(I,M) = \sup\{i \in \mathbb{N} : H_I^i(M) \neq 0\}$.

In the second section, we first present necessary and sufficient conditions for various modules to be of finite length. We then use our results to give an alternative proof of the well-known result that if R is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and M is a finitely generated R- module of dimension d, then $\mathrm{H}_{\mathfrak{m}}^d(M)$ is finitely generated if and only if $d = 0$.

5.1 Artinianness of Top Local Cohomology Modules

This section consists of some results from our joint work with Vahap Erdoğdu, [26].

One of the main results of this section is the following theorem which resolves the Artinianness of top local cohomology modules, $H_I^{\mathrm{cd}(I,R)}(R)$ over local unique factorization domains of dimension at most three:

Theorem 5.1.1 *Let R be a Noetherian local unique factorization domain of dimension at most three and I an ideal of R. Then $H_I^{\mathrm{cd}(I,R)}(R)$ is Artinian if and only if $\mathrm{cd}(I,R) = \dim R$.*

To prove this, we need the following lemma:

Lemma 5.1.2 *Let $(R,\mathfrak{m})$ be a Noetherian local ring, I an ideal of R and M an R-module (not necessarily finitely generated) with $\mathrm{cd}(I,M) = c$. If there exists an element $x \in \mathfrak{m} \setminus I$ such that $\mathrm{cd}(I+Rx,M) \neq c$, then $\mathrm{H}_I^c(M)$ is not Artinian.*

Proof. Let $x \in \mathfrak{m} \setminus I$ such that $\operatorname{cd}(I+Rx,M) \neq c$. Then it follows from the fact $\operatorname{cd}(I+Rx,M) \leq \operatorname{cd}(I,M)+1$ that either $\operatorname{cd}(I+Rx,M) = c+1$ or $\operatorname{cd}(I+Rx,M) < c$. If $\operatorname{cd}(I+Rx,M) = c+1$, then the result follows from Corollary 4.1 of [31]. Now suppose that $\operatorname{cd}(I+Rx,M) < c$. Then it follows from the following exact sequence

$$\cdots \longrightarrow \underbrace{H^c_{I+Rx}(M)}_{=0} \longrightarrow H^c_I(M) \longrightarrow H^c_I(M)_x \longrightarrow \underbrace{H^{c+1}_{I+Rx}(M)}_{=0} \cdots$$

that $H^c_I(M)_x \cong H^c_I(M) \neq 0$ and so $\dim \operatorname{Supp}(H^c_I(M)) \not\subset \{\mathfrak{m}\}$. Therefore $H^c_I(M)$ is not Artinian. $\square$

It is worth noting that (as used in the above proof) there exist ideals such that $\operatorname{cd}(I+Rx,R) \lneq \operatorname{cd}(I,R)$. As an example, let $R = k[[x_1,x_2,x_3]]$ and $I = (x_1) \cap (x_2,x_3)$. Then it follows from Mayer Vietoris sequence that $cd(I,R) = 2$ but $I + Rx_1 = Rx_1$ and so $cd(I+Rx_1,R) = 1 < cd(I,R) = 2$.

Remark 5.1.3 *Let R be a Noetherian local ring of dimension $n > 0$, I be an ideal of R with $h = \operatorname{ht}(I) \lneq n$ and let $\mathfrak{p}$ be a minimal prime ideal of I such that $\operatorname{ht}(\mathfrak{p}) = h$. Then consider the local cohomology module $H^h_I(R)$ and localize it at $\mathfrak{p}$ to obtain the isomorphism $(H^h_I(R))_\mathfrak{p} \cong H^h_{\mathfrak{p}_\mathfrak{p}}(R_\mathfrak{p})$. Now since $R_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}_\mathfrak{p}$ of dimension $\operatorname{ht}(\mathfrak{p}) = h$, it follows from Grothendieck's non-vanishing theorem that $H^h_{\mathfrak{p}_\mathfrak{p}}(R_\mathfrak{p}) \neq 0$ and so $\mathfrak{p} \in \operatorname{Supp}_R(H^h_I(R))$. Hence $H^h_I(R)$ is non-Artinian (as $\mathfrak{p}$ is not maximal).*

We use this remark together with Lemma 5.1.2 to give the proof of Theorem 5.1.1:

Proof of Theorem 5.1.1 . We give the proof only when $\dim(R) = 3$. For smaller dimensions, the argument would be the same.

Let now R be a local UFD of dimension three, I an ideal of R and $c := \operatorname{cd}(I,R)$ and $h := \operatorname{ht}(I)$. Then keeping in mind that $c \in \{0,1,2,3\}$ and $h \leq c$, we have the following cases:

If $c = 0$, then I is necessarily a zero ideal (as R is domain) and so $H^c_I(R) = R$ is non-Artinian. On the other hand, if $c = 1$, then $h = c = 1$ and therefore it follows from Remark 5.1.3 that $H^c_I(R)$ is again non-Artinian.

The case $c = 2$ implies either $h = 1$ or $h = 2$. In particular, if $h = c = 2$, then again by Remark 5.1.3 we obtain $H_I^c(R)$ is non-Artinian. Suppose now $h = 1 \lneq 2 = c$. Then since all height one prime ideals over UFDs have cohomological dimension one, we have that I is not prime. But clearly it is contained in some height one prime ideal $\mathfrak{p} = (x)$ properly and so $I + Rx = Rx$. Therefore $\mathrm{cd}(I + Rx, R) = 1 < 2 = \mathrm{cd}(I, R)$ and so it follows from Lemma 5.1.2 that $H_I^c(R)$ is non-Artinian.

In the final case when $c = 3$, $H_I^c(R)$ is Artinian by Exercise 7.1.7 of [19].

Hence one conclude from the above arguments that the only case for the top local cohomology module $H_I^c(R)$ to be Artinian is when $c = 3 = \dim(R)$. □

Remark 5.1.4 *Note that Theorem 5.1.1 is not valid for UFDs of dimension greater than three. For a concrete example, let $R = k[[x_1, x_2, x_3, x_4]]$ be a formal power series and $I = (x_1, x_2) \cap (x_3, x_4)$. Then it follows from Mayer-Vietoris sequence that $\mathrm{cd}(I, R) = 3$ and $H_I^3(R) = E_R(k)$, where $E_R(k)$ is the injective hull of k.*

In the following, we examine the Artinianness and non-Artinianness of top local cohomology modules in more general cases.

Our following result shows the existence of a chain of ideals with Artinian top local cohomology modules over any Noetherian local ring of dimension ≥ 4:

Proposition 5.1.5 *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $n \geq 4$. Then there exists a descending chain of ideals*

$$J_n \supsetneq J_{n-1} \supsetneq \cdots \supsetneq J_3$$

of R such that $\mathrm{cd}(J_i, R) = i$ for each $3 \leq i \leq n$. Moreover each top local cohomology module, $H_{J_i}^i(R)$, is Artinian.

Proof. Let $x_1, x_2, \cdots, x_n$ be a system of parameters for R that is $\mathfrak{m} = \sqrt{(x_1, x_2, \cdots, x_n)}$ and let $J_n = (x_1, x_2, \cdots, x_n)$. Clearly, $\mathrm{cd}(J_n, R) = n$ and $H_{J_n}^n(R)$ is Artinian. Then it follows from Corollary 5.2 of [33] that there exists an $(n-1)$-generated ideal, say J_{n-1}, such that $H_{J_{n-1}}^{n-1}(R) \cong H_{J_n}^n(R)$. We may proceed in this way and apply Corollary 5.2 of [33] till we reach up to three generated ideal I_3. Then clearly $J_n \supsetneq J_{n-1} \supsetneq \cdots \supsetneq J_3$ is a descending chain of ideals with successive cohomological dimensions, and isomorphic top local cohomology modules $H^3_{J_3}(R) \cong \cdots \cong H^{n-1}_{J_{n-1}}(R) \cong H^n_{J_n}(R)$ which are Artinian, as desired. □

On the other hand, under some mild conditions on an ideal I of a Noetherian local ring R, there exists a sub-ideal K of I such that $\mathrm{cd}(K,R) = \mathrm{cd}(I,R) = c$ and $H^c_K(R)$ is non-Artinian (regardless whether $H^c_I(R)$ is Artinian or not) as the following result shows:

Theorem 5.1.6 *Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension n and I a non-zero ideal of R with $\mathrm{depth}(I,R) > 0$ and $c = \mathrm{cd}(I,R) \leq n-2$. Then there exists an ideal $K \subsetneq I$ of R such that $\mathrm{cd}(K,R) = \mathrm{cd}(I,R) = c$ and $H^c_K(R)$ is non-Artinian.*

To prove Theorem 5.1.6, we need the following two lemmas:

Lemma 5.1.7 *Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension n and I a non-zero ideal of R such that $\mathrm{depth}(I,R) > 0$ and $c = \mathrm{cd}(I,R) \leq n-1$. Then there exists an ideal $J \subseteq I$ of R with $\mathrm{cd}(J,R) = n-1$. Furthermore, if $\mathrm{depth}(I,R) > 1$ and $c = \mathrm{cd}(I,R) \leq n-1$, then $H^{n-1}_J(R)$ is Artinian.*

Proof. Let I be a non-zero ideal of R such that $t = \mathrm{depth}(I,R) > 0$ and $x_1, x_2, \cdots, x_t$ be an R-regular sequence contained in I. Then there exists $x_{t+1}, x_{t+2}, \cdots, x_n$ such that $x_1, x_2, \cdots, x_n$ forms a system of parameters for R. Let now $K := (x_{t+1}, x_{t+2}, \cdots, x_n)$. Clearly, $\sqrt{I+K} = \mathfrak{m}$ and $\mathrm{cd}(K,R) \leq n-t \leq n-1$. Then it follows from the following Mayer-Vietoris sequence

$$\cdots \longrightarrow H^{n-1}_I(R) \bigoplus H^{n-1}_K(R) \longrightarrow H^{n-1}_{I\cap K}(R) \longrightarrow \underbrace{H^n_{\mathfrak{m}}(R)}_{\neq 0}$$

$$\longrightarrow \underbrace{H^n_I(R)}_{=0} \bigoplus \underbrace{H^n_K(R)}_{=0} \longrightarrow H^n_{I\cap K}(R) \longrightarrow H^{n+1}_{\mathfrak{m}}(R) = 0$$

that $\mathrm{cd}(I \cap K, R) = n-1$ and in particular, if $\mathrm{depth}(I,R) > 1$ and $c = \mathrm{cd}(I,R) < n-1$, then the above exact sequence yields an isomorphism $H^{n-1}_{I\cap K}(R) \cong H^n_{\mathfrak{m}}(R)$. Hence $J := I \cap K$ is the desired ideal. □

Lemma 5.1.8 *Let R be a Noetherian ring, I an ideal of R, M an R-module with $cd(I,M) = c > 0$ and*

$$I = I_c \supsetneq I_{c-1} \supsetneq \cdots \supsetneq I_0$$

be the filtration of I as described in Theorem 4.2.1. Then for each $0 \leq i \leq c-1$, the local cohomology module $H^i_{I_i}(M)$ is not Artinian .

Proof. Let $0 \leq i \leq c-1$ and $x \in I_{i+1} \setminus I_i$ and consider the ideal $I_i + Rx$. Then as constructed in the proof of Theorem 4.2.1, $\mathrm{cd}(I_i + Rx, M) = i+1$. Now from Corollary 3.5 of [20], we have the following short exact sequence

$$0 \longrightarrow H^1_{Rx}(H^i_{I_i}(M)) \longrightarrow H^{i+1}_{I_i+Rx}(M) \longrightarrow \underbrace{H^0_{Rx}(H^{i+1}_{I_i}(M))}_{=0} \longrightarrow 0.$$

Hence $H^1_{Rx}(H^i_{I_i}(M)) \cong H^{i+1}_{I_i+Rx}(M) \neq 0$. But then it follows from Grothendieck's vanishing theorem that $\dim \mathrm{Supp}_R(H^i_{I_i}(M)) \geq 1$, and so $H^i_{I_i}(M)$ is not Artinian. □

We now give the proof of Theorem 5.1.6:

Proof of Theorem 5.1.6 . Let I be a non-zero ideal of R with $\mathrm{depth}(I,R) > 0$ and $c = \mathrm{cd}(I,R) \leq n-2$. Then it follows from Lemma 5.1.7 that there exists an ideal $J_{n-1} \subseteq I$ of R such that $\mathrm{cd}(J_{n-1},R) = n-1$. But then by Lemma 5.1.8 we have a descending chain of sub-ideals $J_{n-1} \supsetneq J_{n-2} \supsetneq \cdots \supsetneq J_0$ of J_{n-1} such that $\mathrm{cd}(J_i,R) = i$ and $H^i_{I_i}(R)$ is non-Artinian for each $0 \leq i \leq n-2$. Hence $K := J_c \subsetneq I$ is the desired ideal. □

In the remaining part of this section, we use the notion of Serre subcategory and Corollary 4.1.3 of Section 2 to obtain some further results on the Artinianness of top local cohomology modules.

Recall that a class $\mathscr{S}$ of R-modules is a Serre subcategory of the category of R-modules, $\mathscr{C}(R)$, when it is closed under taking submodules, quotients and extensions. To obtain a necessary condition for the non-Artinianness of top local cohomology modules, we need the following lemma:

Lemma 5.1.9 *Let R be a Noetherian ring, M an R-module (not necessarily finitely generated) and let $\mathscr{S}$ be a Serre subcategory of $\mathscr{C}(R)$. Let I and J be two ideals of R such that $H^{t+i}_J(H^{c-i}_I(M)) \in \mathscr{S}$ for all $0 < i \leq c = \mathrm{cd}(I,M)$ and $H^{t+c}_{I+J}(M) \notin \mathscr{S}$ for some positive integer t. Then $H^t_J(H^c_I(M)) \notin \mathscr{S}$.*

Proof. Consider the Grothendieck's spectral sequence

$$E_2^{p,q} = H_J^p(H_I^q(M)) \Longrightarrow H_{I+J}^{p+q}(M)$$

and look at the stage $p+q = c+t$. Let now $0 < i \leq c = \mathrm{cd}(I,M)$. Since $E_\infty^{t+i,c-i} = E_r^{t+i,c-i}$ for sufficiently large r and $E_r^{t+i,c-i}$ is a subquotient of $E_2^{t+i,c-i} \in \mathscr{S}$, $E_\infty^{t+i,c-i} \in \mathscr{S}$ for all $0 < i \leq c = \mathrm{cd}(I,M)$.
On the other hand, since $E_2^{t,c} = \mathrm{H}_J^t(H_I^c(M)) \Longrightarrow H_{I+J}^{t+c}(M)$, there exists a finite filtration

$$0 = \Phi^{t+c+1}\,\mathrm{H}^{t+c} \subseteq \Phi^{t+c}\,\mathrm{H}^{t+c} \subseteq \cdots \subseteq \Phi^1\,\mathrm{H}^{t+c} \subseteq \Phi^0\,\mathrm{H}^{t+c} = \mathrm{H}^{t+c}$$

of $\mathrm{H}^{t+c} = H_{I+J}^{t+c}(M)$ such that $E_\infty^{p,q} = \Phi^p\,\mathrm{H}^{t+c}/\Phi^{p+1}\,\mathrm{H}^{t+c}$ for all $p+q = t+c$. Since for all $p < t$, $E_\infty^{p,q} = 0$, we have that $\Phi^t\,\mathrm{H}^{t+c} = \cdots = \Phi^1\,\mathrm{H}^{t+c} = \Phi^0\,\mathrm{H}^{t+c} = \mathrm{H}^{t+c}$. But then since $E_\infty^{t+i,c-i} = \Phi^{t+i}\,\mathrm{H}^{t+c}/\Phi^{t+i+1}\,\mathrm{H}^{t+c} \in \mathscr{S}$ for all $0 < i \leq c$, $\Phi^{t+1}\,\mathrm{H}^{t+c} \in \mathscr{S}$ and so it follows from the short exact sequence

$$0 \longrightarrow \underbrace{\Phi^{t+1}\,\mathrm{H}^{t+c}}_{\in \mathscr{S}} \longrightarrow \underbrace{H_{I+J}^{t+c}(M)}_{\notin \mathscr{S}} \longrightarrow E_\infty^{t,c} \longrightarrow 0$$

that $E_\infty^{t,c} \notin \mathscr{S}$. Since $E_\infty^{t,c}$ is a subquotient of $E_2^{t,c}$ and $E_\infty^{t,c} \notin \mathscr{S}$, it follows that $E_2^{t,c} = H_J^t(H_I^c(M)) \notin \mathscr{S}$. $\square$

Corollary 5.1.10 *Let $(R,\mathfrak{m})$ be a Noetherian local ring, M a finitely generated R-module of dimension n with $\dim(M/IM) = d$. If $H_I^{\mathrm{cd}(I,M)}(M)$ is Artinian, then either $\mathrm{cd}(I,M) = n$ or $H_\mathfrak{m}^{n-i}(H_I^i(M)) \neq 0$ for some i with $n-d \leq i < \mathrm{cd}(I,M)$.*

Proof. We prove the contrapositive of the statement. Let $\mathscr{S}$ be the category of zero module and suppose that $c = \mathrm{cd}(I,M) < n$ and $H_\mathfrak{m}^{n-i}(H_I^i(M)) = 0 \in \mathscr{S}$ for all $n-d \leq i < c$. But then since $H_\mathfrak{m}^n(M) \notin \mathscr{S}$, it follows from Lemma 5.1.9 that $H_\mathfrak{m}^{n-c}(H_I^c(M)) \neq 0$. Hence $\dim \mathrm{Supp}(H_I^c(M)) > 0$ and so $H_I^c(M)$ is not Artinian. $\square$

We conclude this section with the following results that determine the Artinianness and non-Artinianness of the top local cohomology module, $H_I^{\mathrm{cd}(I,M)}(M)$, for the ideals of small dimension, the first of which is a consequence of Corollary 4.1.3:

Theorem 5.1.11 *Let $(R,\mathfrak{m})$ be a Noetherian local ring, M a finitely generated R-module of dimension n and let I be an ideal of R such that $\dim(M/IM) = 1$. Then $H_I^{\mathrm{cd}(I,M)}(M)$ is Artinian if and only if $\mathrm{cd}(I,M) = n$.*

Proof. Since $\dim(M/IM) = 1$, it follows from Corollary 4.1.3 that either $\operatorname{cd}(I,M) = n-1$ or $\operatorname{cd}(I,M) = n$. If $\operatorname{cd}(I,M) = n$, then clearly $H_I^n(M)$ is Artinian. If, on the other hand, $\operatorname{cd}(I,M) = n-1$, then it again follows from Corollary 4.1.3 that $\dim \operatorname{Supp}(H_I^{n-1}(M)) = 1$ and so $H_I^{n-1}(M)$ is non-Artinian. □

Theorem 5.1.12 *Let $(R,\mathfrak{m})$ be a Noetherian local ring, M a finitely generated R-module of dimension n and let I be an ideal of R with $\dim(M/IM) = 2$. If $H_I^{\operatorname{cd}(I,M)}(M)$ is Artinian, then either $\operatorname{cd}(I,M) = n$, or $\operatorname{cd}(I,M) = n-1$ and $H_{\mathfrak{m}}^2(H_I^{n-2}(M)) \neq 0$.*

Proof. Since $\dim(M/IM) = 2$ and $H_{\mathfrak{m}}^n(M) \neq 0$, it follows from Corollary 4.1.3 that $n-2$ is a lower bound for $\operatorname{cd}(I,M)$. If $\operatorname{cd}(I,M) = n-2$, then again by Corollary 4.1.3, $\dim \operatorname{Supp}(H_I^{n-2}(M)) = 2$ and so $H_I^{\operatorname{cd}(I,M)}(M)$ is non-Artinian. If, on the other hand, $\operatorname{cd}(I,M) = n$, then from Lemma 5.1.9, $H_I^{\operatorname{cd}(I,M)}(M)$ is Artinian. Finally, if $\operatorname{cd}(I,M) = n-1$ and $\mathrm{H}_I^{n-1}(M)$ is Artinian, then the result follows from Corollary 5.1.10. □

5.2 Modules of Finite Length

This section is motivated by the question of "what are the most elementary properties that are required for an R-module M to be of finite length?" and consists of the results from our joint work with Sevgi Harman, [34].

Throughout $Z(M)$ will denote the set of zero divisors of M.

5.2.1 Modules of length at most two

In this subsection we make some simple but somehow interesting observations first of which provide conditions equivalent to M and all its Koszul cohomology modules to be of finite length.

Proposition 5.2.1 *For an R-module M the following statements are equivalent.*

(i) For any two distinct proper submodules K, L of M, $\operatorname{Ann}(K) + \operatorname{Ann}(L) = R$.

(ii) For any two distinct proper submodules K, L of M, $\operatorname{Hom}_R(K,L) = 0$.

(iii) M is a direct sum of at most two non-isomorphic simple submodules.

(iv) M has length at most two with non-isomorphic simple quotient modules.

(v) For all i and for all sequences $\mathbf{x} = \{x_1, x_2, \cdots, x_n\}$ *of elements in R ,* $\mathrm{H}^i(\mathbf{x}^\infty; M)$ *is of length at most two with non-isomorphic simple quotient modules. (In particular if R is Noetherian, then for all i and for all ideals I of R,* $\mathrm{H}^i_I(R)$ *is of length at most two with non-isomorphic simple quotient modules.)*

Proof. (i)⇒(iii). Let K be any non-zero proper submodule of M and x a non-zero element of K. We claim that $K = Rx$. Suppose not, then Rx is a proper submodule of K and hence by the assumption $\mathrm{Ann}(Rx) + \mathrm{Ann}(K) = R$. But this is the same thing as saying that $\mathrm{Ann}(Rx) = R$, a contradiction to the fact that x is non-zero in K. Therefore it follows that every proper submodule of M is simple. If K is different than M, then there is a proper submodule L of M different than K, and for the same reason as above $L = Ry$, for some y in M. Thus $K \cong R/\mathrm{Ann}(x)$ and $L \cong R/\mathrm{Ann}(y)$, and $\mathrm{Ann}(x)$, $\mathrm{Ann}(y)$ are two distinct maximal ideals of R. Next we show that $K \cap L = 0$. But this follows from the fact that if $K \cap L$ has a non-zero element z, then $\mathrm{Ann}(z) + \mathrm{Ann}(K) = \mathrm{Ann}(z) = R$, which is not possible. Therefore $M = K \oplus L$. Since otherwise $K \oplus L$ would be a proper submodule of M which would then contradict the fact that K is a non-zero and yet $\mathrm{Ann}(K) = \mathrm{Ann}(K) + \mathrm{Ann}(K \oplus L) = R$.

(iii)⇒(i). Is clear.

(iii) ⇒ (ii). Is clear.

(ii)⇒(iii). Let K be again a proper submodule of M and x a non-zero element of K. If Rx is different than K, then $\mathrm{Hom}_R(Rx, K) \neq 0$, a contradiction. Therefore each proper submodule of M is simple. If L is another proper submodule of M different from K, then $M = K \oplus L$. Because otherwise $K \oplus L$ would be a proper submodule of M, and that would give $\mathrm{Hom}_R(K, K \oplus L) \neq 0$, contradicting the assumption.

(iii) ⇒ (iv). Follows from the fact that $length(M_1 \oplus M_2) = length(M_1) + length(M_2)$.

(iv)⇒(iii). Is clear.

(iii) ⇒ (v). Let K be one of the simple submodules of M and $\mathbf{x} = \{x_1, x_2, \cdots, x_n\}$ be a sequence of elements in R. Let $I = (x_1, x_2, \cdots, x_n)$. Then by Lemma 7.7 of [35], $\mathrm{H}^i(\mathbf{x}^\infty; K) = 0$ for all $i > 0$. Therefore we only consider the remaining case $\mathrm{H}^0(\mathbf{x}^\infty; K) = \{x \in K : I^t x = 0 \ for\ some\ positive\ integer\ t\}$. Since K is simple, $K \cong R/\mathfrak{M}$ for some maximal ideal $\mathfrak{M}$ of R. If now $I \nsubseteq \mathfrak{M}$ then $I + \mathfrak{M} = I + \mathrm{Ann}\, K = R$ and since $I + \mathrm{Ann}\, K$ annihilates $\mathrm{H}^0(\mathbf{x}^\infty; K)$, $\mathrm{H}^0(\mathbf{x}^\infty; K) = 0$. If however $I \subseteq \mathfrak{M}$, then it follows from the definition that $\mathrm{H}^0(\mathbf{x}^\infty;K)=K$. Since Koszul cohomology commutes with direct sum, it follows that for any sequence $\mathbf{x}=\{x_1,x_2,\cdots,x_n\}$ of elements of R and any i, $\mathrm{H}^i(\mathbf{x}^\infty;M)$ is either zero or one of the factors of M or is M itself. Therefore for all i and for all sequences $\mathbf{x}=\{x_1,x_2,\cdots,x_n\}$ of elements in R, $\mathrm{H}^i(\mathbf{x}^\infty;M)$ has length at most two with non-isomorphic simple quotient submodules.

(v)⇒(i). Let K and L be any two distinct proper submodules of M and $J=\mathrm{Ann}(K+L)$ which is both contained in $\mathrm{Ann}(K)$ and $\mathrm{Ann}(L)$. Let now $\mathbf{y}=\{y\}$ where $y\in J$. Then it follows from the definition of Koszul cohomology that $K=\mathrm{H}^0(\mathbf{y}^\infty;K)\subseteq \mathrm{H}^0(\mathbf{y}^\infty;M)$ and $L=\mathrm{H}^0(\mathbf{y}^\infty;L)\subseteq \mathrm{H}^0(\mathbf{y}^\infty;M)$ and so K and L are also distinct proper submodules of $\mathrm{H}^0(\mathbf{y}^\infty;M)$. Hence by assumption $\mathrm{Ann}(K)+\mathrm{Ann}(L)=R$.

□

We note that if M and N are any two R-modules, then it is not hard to see that $\mathrm{Ann}(\mathrm{Hom}_R(M,N))$ and $\mathrm{Ann}(M\otimes_R N)$ contains both $\mathrm{Ann}(M)$ and $\mathrm{Ann}(N)$ and so, if $\mathrm{Ann}(M)+\mathrm{Ann}(N)=R$, then we necessarily have $\mathrm{Hom}_R(M,N)=0$ and $M\otimes_R N=0$. Of course in general neither $M\otimes_R N=0$ nor $\mathrm{Hom}_R(M,N)=0$ implies that $\mathrm{Ann}(M)+\mathrm{Ann}(N)=R$.

Proposition 5.2.2 *Let $\{M_i\}_{i\in\mathfrak{I}}$ be a family of R-modules such that for all pairs $i\neq j$ in $\mathfrak{I}$, $\mathrm{Ann}(M_i)+\mathrm{Ann}(M_j)=R$. Then*

(i) $\sum_{i\in\mathfrak{I}} M_i=\bigoplus_{i\in\mathfrak{I}} M_i$.

(ii) $\mathrm{Hom}_R(\bigoplus_{i\in\mathfrak{I}_1} M_i,\bigoplus_{j\in\mathfrak{I}_2} M_j)=0$, *for any two finite disjoint subsets $\mathfrak{I}_1$ and $\mathfrak{I}_2$ of $\mathfrak{I}$.*

(iii) $(\bigoplus_{i\in\mathfrak{I}_1} M_i)\otimes_R(\bigoplus_{j\in\mathfrak{I}_2} M_j)=0$, *for any two disjoint subsets $\mathfrak{I}_1$ and $\mathfrak{I}_2$ of $\mathfrak{I}$.*

(iv) $\mathrm{Ext}^k_R(\bigoplus_{i\in\mathfrak{I}_1} M_i,\bigoplus_{j\in\mathfrak{I}_2} M_j)=0$, *for any two finite disjoint subsets $\mathfrak{I}_1$ and $\mathfrak{I}_2$ of $\mathfrak{I}$ and all $k\geq 1$.*

(v) If moreover for each $i\in\mathfrak{I}$, M_i is simple, then $\mathrm{Tor}^R_1(\bigoplus_{j\in\mathfrak{I}_1} M_i,\bigoplus_{j\in\mathfrak{I}_2} M_j)=0$, *for any two disjoint subsets $\mathfrak{I}_1$ and $\mathfrak{I}_2$ of $\mathfrak{I}$.*

Proof. (i) Since every element of $\sum_{i\in\mathfrak{I}} M_i$ is contained in a submodule generated by a finite number of the M_i and since for any finite subset $\mathfrak{J}$ of $\mathfrak{I}$ not containing i, $\mathrm{Ann}(M_i)+\bigcap_{j\in\mathfrak{J}}\mathrm{Ann}(M_j)=R$, it follows that $M_i\cap\sum_{j\in\mathfrak{J}} M_j=0$, and so $\sum_{i\in\mathfrak{I}} M_i$ is a direct sum.

For the proofs of (ii) and (iii) use the fact that Hom is distributive over finite direct sum and the fact that $\mathrm{Hom}_R(M_i, M_j) = 0$, and Tensor product is distributive over arbitrary direct sum and the fact that $M_i \otimes_R M_j = 0$.

(iv) Since for each pair $i \neq j$ in $\mathfrak{I}$, $\mathrm{Ann}(M_i) + \mathrm{Ann}(M_j) \subseteq \mathrm{Ann}(\mathrm{Ext}_R^k(M_i, M_j))$, it follows that $\mathrm{Ext}_R^k(M_i, M_j) = 0$ for all $k \geq 1$. Hence $\mathrm{Ext}_R^k(\bigoplus_{i\in\mathfrak{I}_1} M_i, \bigoplus_{j\in\mathfrak{I}_2} M_j) = 0$ follows from the fact that Ext is distributive over finite direct sum.

(v)If for each $i \in \mathfrak{I}$, M_i is simple, then $M_i \cong R/\mathrm{Ann}(M_i)$. But then from $\mathrm{Ann}(M_i) + \mathrm{Ann}(M_j) = R$ we have $\mathrm{Ann}(M_i)\mathrm{Ann}(M_j) = \mathrm{Ann}(M_i) \cap \mathrm{Ann}(M_j)$. Therefore $\mathrm{Tor}_1^R(M_i, M_j) \cong \mathrm{Ann}(M_i) \cap \mathrm{Ann}(M_j)/\mathrm{Ann}(M_i)\mathrm{Ann}(M_j) = 0$. Now, $\mathrm{Tor}_1^R(\bigoplus_{j\in\mathfrak{I}_1} M_i, \bigoplus_{j\in\mathfrak{I}_2} M_j) = 0$ is a consequence of the fact that Tor is distributive over arbitrary direct sum. □

It may be worth mentioning that if R is a Noetherian ring and M and N are two finitely generated R-modules with $\mathrm{Ann}(M) + \mathrm{Ann}(N) = R$, then the i^{th} local cohomology of M with respect to the ideal $\mathrm{Ann}N$ is zero. That is, $\mathrm{H}^i_{\mathrm{Ann}N}(M) = \varinjlim \mathrm{Ext}_R^i(R/(\mathrm{Ann}N)^n, M) = 0$, which easily follows from the proof of part (iv) of Proposition 5.2.2 above.

Proposition 5.2.3 *Let R be a ring and M and N be R-modules. Suppose that* $\mathrm{Ann}(M) \neq 0$ *and that* $\mathrm{Ann}(M)$ *is not contained in $Z(N)$, the set of zero divisors of N. Then* $\mathrm{Hom}_R(M, N) = 0$.

Proof. Suppose that $\mathrm{Hom}_R(M, N) \neq 0$, and let f be a non-zero element of $\mathrm{Hom}_R(M, N)$. Then there is a non-zero element m in M such that $f(m) \neq 0$ in N. Let now r be any non-zero element of $\mathrm{Ann}(M)$ which is not contained in $Z(N)$. Then $rf(m) = f(rm) = 0$. But this is a contradiction to the fact that r is not in $Z(N)$. Therefore $\mathrm{Hom}_R(M, N) = 0$. □

We note that if R is an integral domain and K is the field of fractions of R, then for any non-zero ideal I of R, $\mathrm{Hom}_R(R/I, K) = 0$. Now applying $\mathrm{Hom}(-, K)$ to the short exact sequence $0 \to I \to R \to R/I \to 0$ we obtain $\mathrm{Hom}_R(R,K) \cong \mathrm{Hom}_R(I,K)$, as K is an injective R-module. On the other hand applying $\mathrm{Hom}(R/I,-)$ to the same short exact sequence we obtain $\mathrm{Hom}_R(R/I,R/I) \cong \mathrm{H}_1^R(R/I,R/I)$, the first homology of the Hom functor applied to the given sequence. Also the conditions that $\mathrm{Ann}(M) \neq 0$ and $\mathrm{Ann}(M) \not\subseteq Z(N)$ in the statement of the above proposition has to be retained for its conclusion. For let $M = R$ and $N = K$, then clearly $\mathrm{Hom}_R(M,N) \neq 0$.

5.2.2 Divisible modules of finite length

Recall that an $R-$module M is divisible if for any nonzero divisor r in R, $M = rM$. In this section we examine conditions under which a divisible module is of finite length.

Proposition 5.2.4 *Let M be an R-module with $Z(M) \subseteq Z(R)$ and E be an injective R-module. Then* $\mathrm{Hom}_R(M,E)$ *is a divisible R-module.*

Proof. Let E be an injective R-module and M be any R-module with $Z(M) \subseteq Z(R)$, and let f be a non-zero element of $\mathrm{Hom}_R(M,E)$ and r be a non-zero divisor in R. We want to show that there exists a $g \in \mathrm{Hom}_R(M,E)$ such that $f = rg$. For this, we define $h : M \to M$ by $h(m) = rm$. Then it is clear that h is well-defined and one-to-one. Now using the injectivity of E, one obtains the following commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \xrightarrow{h} & M \\ & & {\scriptstyle f}\downarrow & \swarrow{\scriptstyle g} & \\ & & E & & \end{array}$$

Thus for any $m \in M$, we have

$$f(m) = gh(m) = g(rm) = rg(m)$$

that is $f = rg$. Therefore $\mathrm{Hom}_R(M,E)$ is divisible. □

Corollary 5.2.5 *Let E be a torsion free injective module over an integral domain R. Then for any torsion free R-module M,* $\mathrm{Hom}_R(M,E)$ *is an injective R-module and in particular,* $\mathrm{End}_R(E)$ *is injective as an R-module.*

Proof. It is easy to see that $\mathrm{Hom}_R(M,E)$ is torsion free. Hence by Proposition 5.2.1, it is also divisible. Since over an integral domain a torsion free divisible module is injective, $\mathrm{Hom}_R(M,E)$ is an injective R-module. $\square$

Corollary 5.2.6 *Let M be an R-module and E be an injective R-module. Then* $\mathrm{Hom}_R(\mathrm{Hom}_R(M,R),E)$ *is a divisible R-module.*

Proof. Let $r \in Z(\mathrm{Hom}_R(M,R))$. Then there exists a nonzero element $f \in \mathrm{Hom}_R(M,R)$ such that $rf = 0$. Since f is nonzero, $0 \neq f(m) \in R$ for some $m \in M$. But then $rf(m) = 0$ and so $r \in Z(R)$. Therefore $Z(\mathrm{Hom}_R(M,R)) \subseteq Z(R)$ and the result follows from Proposition 5.2.4. $\square$

The following statement may be considered as the dual of Proposition 5.2.4 :

Proposition 5.2.7 *Let M be a divisible R-module. Then for any projective R-module P,* $\mathrm{Hom}_R(P,M)$ *is a divisible R-module.*

Proof. Let M be a divisible and P be a projective R-module, and let f be a non-zero element of $\mathrm{Hom}_R(P,M)$ and r be a non-zero divisor in R. We want to show that there exists a $g \in \mathrm{Hom}_R(P,M)$ such that $f = rg$. For this, we define $h : M \to M$ by $h(m) = rm$. Then from the divisibility of M, h is onto. Now using the projectivity of P, one obtains the following commutative diagram:

$$\begin{array}{ccccc} & & P & & \\ & {\scriptstyle g}\swarrow & \downarrow {\scriptstyle f} & & \\ M & \xrightarrow[h]{} & M & \longrightarrow & 0 \end{array}$$

Thus for any $p \in P$, we have

$$f(p) = h(g(p)) = rg(p)$$

that is $f = rg$. Therefore $\mathrm{Hom}_R(P,M)$ is divisible. $\square$

Proposition 5.2.8 *Let M be an Artinian R-module with $Z(M) \subseteq Z(R)$. Then M is divisible.*

Proof. Let $r \in R - Z(R) \subseteq R - Z(M)$. Then because M is Artinian, the chain

$$rM \supseteq r^2M \supseteq \cdots$$

must stabilize i.e. $r^nM = r^{n+1}M$ for some positive integer n. Let now $x \in M$ then $r^nx = r^{n+1}y$ for some $y \in M$. Hence $r^n(x - ry) = 0$ and since $r^n \in R - Z(M)$, $x - ry = 0$ implies $x = ry$. Therefore $M = rM$ for all $r \in R - Z(R)$ and so M is divisible. □

Proposition 5.2.9 *Over an integral domain R which is not a field the only finitely generated divisible module is the zero module.*

Proof. Let M be a finitely generated divisible module over the integral domain R. Then for any nonzero prime ideal P of R and any nonzero element r in P we have $rM = M$ and hence $\frac{r}{1}M_P = M_P$ as R_P-modules. But then by Nakayama's Lemma $M_P = 0$. Thus $M_P = 0$ for all prime ideals P of R and therefore $M = 0$. □

When R is not an integral domain there are cases where R possesses a nonzero finitely generated divisible module and we now establish these facts.

Proposition 5.2.10 *Let M be a nonzero finitely generated divisible R-module. Then any maximal ideal in the support of M consists of zero divisors.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of R such that $M_{\mathfrak{m}} \neq 0$ and r be a nonzero divisor in $\mathfrak{m}$. Then as M is divisible, $rM = M$ and hence $\frac{r}{1}M_{\mathfrak{m}} = M_{\mathfrak{m}}$ as $R_{\mathfrak{m}}$-modules. But then again by Nakayama's Lemma, $M_{\mathfrak{m}} = 0$. This contradiction shows that $\mathfrak{m}$ consists of zero divisors. □

Corollary 5.2.11 *Let R be a ring and M a finitely generated non-zero R-module. Suppose that the Jacobson radical, $J(R)$, of R is non-zero and that M is divisible. Then $J(R)$ consists of only zero divisors.*

Corollary 5.2.12 *Let R be a local ring with maximal ideal $\mathfrak{m}$ and M be a finitely generated non-zero R-module. Then M is divisible implies that $\mathfrak{m}$ consists of zero divisors.*

Proposition 5.2.13 *A reduced local ring with finitely many minimal prime ideals which possesses a non-zero finitely generated divisible module is of Krull dimension zero.*

Proof. Let R be a reduced local ring with finitely many minimal prime ideals which possesses a nonzero finitely generated divisible module and $\mathfrak{m}$ be a maximal ideal of R. Then it follows from the above corollary that $\mathfrak{m}$ consists of zero divisors. But then since R is reduced, we have

$$\mathfrak{m} = Z(R) = \cup_{i=1}^{n}\{P_i \mid P_i \; is \; a \; minimal \; prime \; ideal \; of \; R\}$$

which implies that $\mathfrak{m} = P_i$ for some i and so the height of $\mathfrak{m}$ is zero. Therefore R is of Krull dimension zero. □

It follows from Corollary 5.2.12 and Proposition 5.2.13 that if A is a Noetherian local ring which is not a field and possesses a nonzero finitely generated divisible module, then the maximal ideal of A consists of zero divisors and contains at least one nonzero nilpotent element. Therefore the reduced Noetherian local ring $A = k[[x,y]]/(xy)$ does not have a nonzero finitely generated divisible module.

On the other hand, let $R = k[[x,y]]/(x^2,xy)$. Then since every element of R is either a unit or a zero divisor, every R-module is divisible. Note also that R is of Krull dimension one and therefore non-Artinian. Thus there are Noetherian divisible modules that are not Artinian. One also knows that the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty) = \mathbb{Z}[1/p]/\mathbb{Z}$ is an Artinian divisible module which is not Noetherian.
With this in mind, we have the following result:

Theorem 5.2.14 *Over a reduced Noetherian ring R, a finitely generated divisible module M is Artinian and $Z(M) \subseteq Z(R)$.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of R containing $\operatorname{Ann} M$. Then by Proposition 5.2.10 and the fact that R is reduced, we have

$$\mathfrak{m} \subseteq Z(R) = \cup_{i=1}^{n}\{P_i \mid P_i \; is \; a \; minimal \; prime \; ideal \; of \; R\}$$

which implies that $\mathfrak{m} = P_i$ for some i and so height of $\mathfrak{M}$ is zero. Therefore $R/\operatorname{Ann} M$ is of Krull dimension zero and hence is Artinian. Since a finitely generated module over an Artinian ring is Artinian, M is Artinian as an $R/\operatorname{Ann} M$- module. But then since M as an R-module and as an $R/\operatorname{Ann} M$- module is one and the same it follows that M is an Artinian R-module. Let now $r \in R$ be a nonzero divisor in R and define $f : M \to M$ by $f(m) = rm$. It is clear from the divisibility of M that f is onto and also since M is Noetherian, f must be an isomorphism. Therefore $\operatorname{Ker} f = 0$ and so $rm = 0$ implies $m = 0$ which implies that $r \in R$ is a nonzero divisor of M. Thus $R - Z(R) \subseteq R - Z(M)$ and hence we have $Z(M) \subseteq Z(R)$. □

Proposition 5.2.15 *Over a Noetherian integral domain R of Krull dimension 1, a finitely generated module M with* $\operatorname{Ann} M \neq 0$ *is Artinian and so is of finite length.*

Proof. Since R is of Krull dimension 1, $rad(\operatorname{Ann} M)$, the radical of $\operatorname{Ann} M$, is a finite product of maximal ideals of R, and so $R/\operatorname{Ann} M$ is Artinian. Hence M is Artinian both as an R-module and an $R/\operatorname{Ann} M$-module. □

We also would like to mention that if R is any ring with $J(R) \neq 0$ and M is an Artinian R-module with $Z(M) \subseteq Z(R)$, then $J(R)$ is contained in the set of zero divisors of R. This easily follows from the proof of the following proposition.

Proposition 5.2.16 *Let R be a ring with nonzero Jacobson radical $J(R)$ and M be an Artinian R-module. Then $J(R) \subseteq Z(M)$.*

Proof. Suppose $J(R) \nsubseteq Z(M)$ and let $r \in J(R) - Z(M)$. Then for any nonzero $x \in M$, the Nakayama's Lemma would give a non-stationary descending chain of submodules of M

$$Rx \supsetneq rRx \supsetneq r^2Rx \supsetneq \cdots$$

But then this yields a contradiction. Therefore $J(R) \subseteq Z(M)$. □

5.2.3 Local cohomology modules of finite length

Let $(R, \mathfrak{m})$ be a Noetherian local ring and I be an ideal of R. Then for any finitely generated R-module M with dimension d, one knows that for all i, $\mathrm{H}^i_{\mathfrak{m}}(M)$ and $\mathrm{H}^d_I(M)$ are Artinian modules. Here in this section, we use the information of Section 5.2.2 to give necessary and sufficient conditions for $\mathrm{H}^d_I(M)$ to be of finite length. The following is yet another proof (that uses Proposition 5.2.9) of the so called Grothendieck's non-vanishing theorem, see for example Section 6.1.4 of [19] and [36].

Theorem 5.2.17 *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and M be a finitely generated R- module of dimension d. Then $\mathrm{H}^d_{\mathfrak{m}}(M)$ is finitely generated if and only if $d = 0$.*

Proof. By the Independence of Base[[37], Proposition 2.14], we may place R by $R/\operatorname{Ann} M$. Therefore we may assume that $\operatorname{Ann} M = 0$ and so $d = \dim_R M = \dim_R R$. On the other hand, as is well-known that $\mathrm{H}^i_{\hat{\mathfrak{m}}}(\hat{M}) \cong \mathrm{H}^i_{\mathfrak{m}}(M)$, we may also assume that R is complete, here $\hat{M}$ denotes the $\mathfrak{m}$-adic completion of M. Then by Cohen's structure theorem, R is the homomorphic image of a complete regular Noetherian local ring T with dimension $n \geq d$ and thus there is a surjective homomorphism $\phi : T \to R$ and clearly $I = Ker\phi$ is an ideal of T with height $n-d$. Since every regular local ring is Cohen-Macaulay, I contains a regular sequence $(x_1, x_2, \cdots, x_{n-d})$ and so $T/(x_1, x_2, \cdots, x_{n-d})$ is a regular local ring. Let $S = T/(x_1, x_2, \cdots, x_{n-d})$. Then clearly $\dim S = d$. Let now $\mathfrak{m}_S$ be the maximal ideal of S and $E_S(S/\mathfrak{m}_S)$ be the injective hull of the residue field $S/\mathfrak{m}_S$ of S and so again by the Independence of Base, $\mathrm{H}^d_{\mathfrak{m}}(M) \cong \mathrm{H}^d_{\mathfrak{m}_S}(M)$. But then by the local duality theorem [[37], Theorem 4.4], we have

$$\mathrm{H}^d_{\mathfrak{m}}(M) \cong \mathrm{H}^d_{\mathfrak{m}_S}(M) \cong \operatorname{Hom}_S(\operatorname{Hom}_S(M,S), E_S(S/\mathfrak{m}_S))$$

Since every regular local ring is an integral domain, $0 \in \operatorname{Ass}(\operatorname{Hom}_S(M,S))$ which implies that $\operatorname{Hom}_S(M,S)$ is nonzero and then again by local duality $\mathrm{H}^d_{\mathfrak{m}}(M)$ is nonzero. On the other hand, by Corollary 5.2.6

$$\operatorname{Hom}_S(\operatorname{Hom}_S(M,S), E_S(S/\mathfrak{m}_S))$$

is a divisible S-module. Then by Proposition 5.2.9, $\mathrm{H}^d_{\mathfrak{m}}(M)$ is finitely generated only if S is Artinian and so $d = \dim S = 0$.
Conversely, suppose $\dim M = 0$. Then the result follows from the fact that $\mathrm{H}^0_{\mathfrak{m}}(M) \subseteq M$.
□

Theorem 5.2.18 *Let R be a reduced Noetherian local ring and M be a finitely generated R-module of dimension d with the property that $Z(M) \subseteq Z(R)$. Then for any ideal I of R, $\mathrm{H}^d_I(M)$ is a nonzero finitely generated R-module if only if R is a field.*

Proof. Suppose $\mathrm{H}_I^d(M)$ is a nonzero finitely generated R-module and $x \in R - Z(R) \subseteq R - Z(M)$. Then the short exact sequence

$$0 \longrightarrow M \xrightarrow{.x} M \longrightarrow M/xM \longrightarrow 0$$

yields the following long exact sequence

$$\begin{array}{llllll} 0 \longrightarrow & \mathrm{H}_I^0(M) & \xrightarrow{.x} \mathrm{H}_I^0(M) & \longrightarrow \mathrm{H}_I^0(M/xM) & \longrightarrow & \cdots \\ \longrightarrow & \mathrm{H}_I^{d-1}(M/xM) & \longrightarrow \mathrm{H}_I^d(M) & \xrightarrow{.x} \mathrm{H}_I^d(M) & \longrightarrow & \mathrm{H}_I^d(M/xM) \\ \longrightarrow & \cdots & & & & \end{array}$$

Since $\dim M/xM < d$, $\mathrm{H}_I^d(M/xM)$ is zero and so the map $\mathrm{H}_I^d(M) \xrightarrow{.x} \mathrm{H}_I^d(M)$ is surjective. Therefore for any nonzero divisor x of R, we have $\mathrm{H}_I^d(M) = x\mathrm{H}_I^d(M)$ which implies that $\mathrm{H}_I^d(M)$ is divisible. Then by Proposition 5.2.13, R is Artinian. The result now follows from the fact that a reduced Artinian ring is nothing but a field.
The converse is obvious.

□

6. CONCLUSION

In this thesis, we dealt with local cohomology modules and their relations with radically perfect ideals. However, since local cohomology modules are not finitely generated in many instances, it is not very easy to obtain considerable results on these modules by using classical tools. To overcome this problem, one needs to establish a relation between a local cohomology module and a finitely generated module. In this regard, Lyubeznik applied the theory of $\mathscr{D}$-modules in the rings of characteristic zero and also he developed the theory of F-modules for the case when the underlying ring is of characteristic $p > 0$; [38], [23]. Since all local cohomology modules $H_I^i(R)$ have natural F- finite F- module (resp. holonomic $\mathscr{D}-$module) structures, the class of F- finite F- modules (resp. holonomic $\mathscr{D}-$modules) has significant applications to local cohomology modules in characteristic $p > 0$ (resp. characteristic zero). By using these applications, we obtained the result which established Conjecture 2 in equicharacteristic $p > 0$. In our future work, we would like to apply the $\mathscr{D}-$ module theory to local cohomology modules to establish Conjecture 2 in equicharacteristic zero case.

Furthermore, in this thesis, we mentioned about several definitions of local cohomology modules all of which are equivalent when the underlying ring is Noetherian. However, to obtain more specific results on radically perfect ideals, we need an alternative definition which is both compatible with all these definitions and valid over the rings that need not to be Noetherian. Fortunately, J.P. Greenlees and J.P.C. May give such a definition of local cohomology modules in their very popular paper by using the notions from algebraic topology, [39]. In the future, we also would like to examine this definition and try to relate it with radically perfect ideals.

REFERENCES

[1] **Forster, O.** (1964). Uber die Ahzahl der Erzeugenden eines Ideals in einem Noetherschen Ring, *Math. Z.*, *84*, 80–87.

[2] **Lyubeznik, G.** (1992). The number of defining equations of affine algebraic sets, *Amer. J. Math.*, *114*, 413–463.

[3] **Hellus, M.**, (2007). Local Cohomology and Matlis Duality, habilitation dissertation, University of Leipzig, Germany.

[4] **Hellus, M.** (2005). On the associated primes of Matlis Duals of top local cohomology modules, *Comm. Alg.*, *33*, 3997–4009.

[5] **Hellus, M.** (2007). Matlis Duals of top local cohomology modules and the arithmetic rank of an ideal, *Comm. Alg.*, *35*, 1421–1432.

[6] **Hellus, M. and Schenzel, P.** (2014). Notes on local cohomology and duality, *J. Algebra*, *401*, 48–61.

[7] **Hellus, M. and Stückrad, J.** (2008). Matlis Duals of top local cohomology modules, *Proc. Amer. Math. Soc.*, *136*, 489–498.

[8] **Lyubeznik, G. and Yıldırım, T.** (2018). On the Matlis dual of local cohomology modules, *Proc. Amer. Math. Soc.*, *146*(9), 3715–3720.

[9] **Goto, S. and Ogawa, T.** (1983). A note on rings with finite local cohomology, *Tokyo J. Math.*, *6*, 403–411.

[10] **Erdoğdu, V. and McAdam, S.** (1994). Coprimely packed Noetherian polynomial rings, *Comm. Algebra*, *22*, 6459–6470.

[11] **Erdoğdu, V. and Harman, S.** (2013). Commutative rings whose prime ideals are radically perfect, *J. Commut. Algebra*, *5*(4), 527–544.

[12] **Erdoğdu, V.** (2004). Coprime packedness and set theoretic complete intersection of ideals in polynomial rings, *Proc. Amer. Math. Soc.*, *132*, 3467–3471.

[13] **Erdoğdu, V.** (2010). Efficient generation of prime ideals in polynomial rings up to radical, *Comm. Algebra*, *38*(5), 1802–1807.

[14] **Erdoğdu, V.** (2009). Radically perfect prime ideals in polynomial rings, *Arch. Math.*, *93*, 213–217.

[15] **Mimouni, A.** (2013). Radically Perfect Prime Ideals in Polynomial Rings Over Prüfer and Pullback Rings, *Comm. Algebra*, *41*(4), 1377–1385.

[16] **Matsumura, H.**, (1986). Commutative ring theory, publisher = Cambridge Studies in Advanced Mathematics. Cambridge University Press, volume=8, address = Cambridge,

[17] **Weibel, C.**, (1994). An introduction to homological algebra, publisher = Cambridge Studies in Advanced Mathematics, Cambridge University Press, volume=38, address = Cambridge,

[18] **Grothendieck, A. (notes by Hartshorne, R.**, (1967). Local Cohomology, Lecture Notes, volume 41, Springer Verlag.

[19] **Brodmann, M.P. and Sharp, R.Y.**, (2013). Local Cohomology: An algebraic introduction with geometric applications., volume136, Cambridge University Press, Cambridge, 2nd ed. edition.

[20] **Schenzel, P.** (2003). Proregular sequences, local cohomology, and completion, *Math. Scand.*, *92*, 161–180.

[21] **Matlis, E.** (1958). Injective modules over Noetherian rings., *Pacific J. Math.*, *8*(4), 511–528.

[22] **Iyengar S, e.a.**, (2007). Twenty-four hours of local cohomology, publisher = Graduate Studies in Mathematics. Amer. Math. Soc., volume=87, address = Providence,

[23] **Lyubeznik, G.** (1997). F-modules: applications to local cohomology and D-modules in characteristic p>0, *J. reine angew. Math.*, *491*, 65–130.

[24] **Peskine, C. and Szpiro, L.** (1973). Dimension Projective Finie et Cohomologie Locale, *I.H.E.S. Publ. Math.*, *42*, 47–119.

[25] **Kunz, E.** (1969). Characterizations of regular local rings of characteristic p, *Amer. J. Math.*, *91*, 772–784.

[26] **Erdoğdu, V. and Yıldırım, T.** (2018). Cohomological Dimension and Top Local Cohomology Modules, *Submitted*.

[27] **Eisenbud, D. and Evans, E.G.** (1973). Every algebraic set in n-space is the intersection of n hypersurfaces, *Invent. Math.*, *19*, 107–112.

[28] **Lyubeznik, G.** (2002). A partial survey of local cohomology, local cohomology and its applications, *Lect. Notes Pure Appl. Math.*, *226*, 121–154.

[29] **Huneke, C.**, (1992). Problems in local cohomology, Free Resolutions in Commutative Algebra and Algebraic Geometry Sundance 90, volume 2, Jones and Barlett Publishers, Boston.

[30] **Aghapournahr, M. and Melkersson, L.** (2014). Artinianness of local cohomology modules, *Ark. Mat*, *52*, 1–10.

[31] **Dibaei, M.T. and Vahidi, A.** (2011). Artinian and Non-Artinian Local Cohomology Modules, *Canad. Math. Bull.*, *54*, 619–629.

[32] **Hellus, M. and Stückrad, J.** (2009). Artinianness of local cohomology, *J. Commut. Algebra*, *1*(2), 269–274.

[33] **Huneke, C. Katz, D. and Marley, T.** (2009). On the support of local cohomology, *J. Algebra*, *322*, 3194–3211.

[34] **Harman, S. and Yıldırım, T.** (2018). Modules of Finite Length, *Math. Reports*, *Accepted.*

[35] **Hochster, M.** Local Cohomology, unpublished notes, http://www.math.lsa.umich.edu/~hochster/615W11/loc.pdf.

[36] **Puthenpurakal, T.** (2009). An elementary proof of Grothendieck's nonvanishing theorem, *Comm. Alg*, *37*, 2994–2996.

[37] **Huneke, C.** (2007). Lectures on Local Cohomology, *Contemp. Math.*, *436*, 51–99.

[38] **Lyubeznik, G.** (1993). Finiteness properties of local cohomology modules (an application of D-modules to commutative algebra), *Invent. Math.*, *113*, 41–55.

[39] **Greenlees, J.P.C. and May, J.P.** (1992). Derived functors of I-adic completion and local homology, *J. Algebra*, *149*(2), 438–453.

CURRICULUM VITAE

Name Surname: Tuğba Yıldırım

Place and Date of Birth: Rize/ 02.07.1989

E-Mail: tugbayildirim@itu.edu.tr

EDUCATION:

- **B.Sc.:** 2011, Istanbul University, Science Faculty, Department of Mathematics

PROFESSIONAL EXPERIENCE:

- 2011-Present, İstanbul Technical University. Faculty of Science and Letters, Department of Mathematics, Research Assistant.
- 09.05.2016-29.05.2017, University of Minnesota, School of Mathematics, Visiting Research Scholar.
(Under TUBITAK 2214/A Scholarship Programme)

JOURNAL PUBLICATIONS ON THE THESIS:

- G. Lyubeznik and **T. Yıldırım**, "On The Matlis Duals of Local Cohomology Modules", Proc. Amer. Math. Soc., **146** (2018),no. 9, 3715–3720. DOI: https://doi.org/10.1090/proc/14038 , (SCI).
- S. Harman and **T. Yıldırım**, "Modules of Finite Length", Math. Reports (Accepted)(SCI-E).
- V. Erdoğdu and **T. Yıldırım**, "Cohomological Dimension and Top Local Cohomology Modules", (2018, Submitted.)

PRESENTATIONS ON THE THESIS:

- **T. Yıldırım**, "On the Artinianness of Top Local Cohomology Modules", ICRAPAM 2018, Karadeniz Technical University Prof. Dr. Osman Turan Convention Center, Trabzon (July 23-27, 2018), http://2018.icrapam.org
- **T. Yıldırım** and G. Lyubeznik, "On the Matlis duals of local cohomology modules in characteristic $p > 0$", Antalya Algebra Days XX, Nesin Mathematics Village, İzmir (May 16-20, 2018), http://85.111.17.208/nvetkinlik/events/2018-aad/acg_2018.pdf.
- **T. Yıldırım**, "On the Matlis duals of F-finite F-modules", Istanbul University, Department of Mathematics Seminars, (December 7, 2017).

- **T. Yıldırım** and V. Erdoğdu, "On the top local cohomology modules", Midwest Commutative Algebra Conference, Purdue University, Indiana, USA (August 3-7, 2015) (Poster Presentation), http://www.math.purdue.edu/~ulrich/Purdue_2015/index.html.

- **T. Yıldırım** and V. Erdoğdu, "On the cohomological dimension of local cohomology modules", Antalya Algebra Days XVII, Nesin Mathematics Village, İzmir (May 20-24, 2015), http://85.111.17.208/nvetkinlik/etkinlikler/2015_acg/acg.pdf.