ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE

ENGINEERING AND TECHNOLOGY

SYMMETRY GROUP CLASSIFICATION OF SOME PROBLEMS IN MATHEMATICAL PHYSICS

Ph.D. THESIS

Özlem ORHAN

Department of Mathematical Engineering

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Özlem ORHAN, a Ph.D. student of ITU Graduate School of Science Engineering and Technology 509112080 successfully defended the thesis entitled " SYMMETRY GROUP CLASSIFICATION OF SOME PROBLEMS IN MATHEMATICAL PHYSICS ", which he/she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

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To my mother Ayten Orhan,

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SUMMARY

In this thesis, some problems in physics and engineering sciences are examined by symmetry methods. In the literature, there are a lot of methods to solve nonlinear differential equations and these methods play an important role. One of these methods is to use symmetry groups. We consider some symmetry group related methods to solve problems in mathematical physics.

Firstly, we deal with the Noether symmetry classification of the nonlinear fin equation, in which thermal conductivity and heat transfer coefficient are assumed to be functions of the temperature. This classification includes Noether symmetries, first integrals and some invariant solutions with respect to different choices of thermal conductivity and heat transfer coefficient functions. In this thesis, Noether symmetries of the fin equation are investigated using the partial Lagrangian approach.

Secondly, we consider Lienard II-type harmonic nonlinear oscillator equation as a nonlinear dynamical system. Firstly, we examine the first integrals in the form $A(t,x)\dot{x}+B(t,x)$ and corresponding exact solutions, the integrating factors. In addition, we analyze other types of the first integrals via λ-symmetry approach. It is shown that the equation can be linearized by means of nonlocal transformation, which is called Sundman transformation. Using the modified Prelle-Singer approach, time independent first integrals are derived for the Lienard II-type harmonic nonlinear oscillator equation.

The modified Prelle-Singer procedure is used for a class of second order nonlinear ordinary differential equations and several physically interesting nonlinear systems are solved. Prelle and Singer have proposed an algorithmic procedure to find the integrating factor for the system of first order ordinary differential equation. Once the integrating factor for the equation is determined then it leads to a time independent integral of motion for the first order ordinary differential equation. The Prelle-Singer method guarantees that if the first order ordinary differential equation has a first integral in terms of elementary functions then this first integral can be found. This method has been generalized to incorporate the integrals with non-elementary functions. Recently, this theory is generalized to obtain general solutions for second order and higher order ordinary differential equations without any integration.

Moreover, it is possible to consider some feasible algorithm to obtain first integral, integrating factor and invariant solution and one can apply this algorithm to nonlinear equation. The another method for application to nonlinear differential equation is the transformation method. Considering this transformation procedure, a nonlinear equation can be converted to a linear differential equation whose solutions are known. It is well-known that Lie proves the general algorithm that all second order nonlinear differential equations can be converted to second order linear differential equations

by the method of change of variables, which is called Lie linearization test. In fact, the mathematical procedure of linearizing transformation is quite diffucult work and this can be applied to only second order ordinary differential equations that have a eight-dimensional Lie algebra. Therefore, it is necessary to consider other type of transformation techniques of nonlinear differential equations for linearization of larger classes of equations. One of nonlocal transformations is of the form $X = F(t,x), \quad dT = G(t,x)dt$, which is called the generalized Sundman transformation. This transformation is also called S-transformation and the equations that can be linearized by means of S-transformations are called S-linearizable. In the second problem, λ-symmetries via Lie symmetries, integrating factors, first integrals and invariant solutions of Lienard II-type harmonic nonlinear oscillator equation are obtained.

In third problem, we examine first integrals, transformation pair and invariant solutions of fin equation by linearization methods. And we apply nonlocal transformation to fin equation. The important relations λ-symmetry with Lie point symmetry, Prelle Singer method with λ-symmetry and Lie symmetry are examined. The first integrals, integrating factors, Sundman transformation pair and invariant solutions of fin equation are found.

MATEMATİKSEL FİZİKTEKİ BAZI PROBLEMLERİN SİMETRİ GRUP SINIFLANDIRMALARI

ÖZET

Bir fonksiyonun türevleri arasındaki ya da fonksiyonun kendisi ve türevleri arasındaki ilişkiyi açık olarak belirten denkleme diferansiyel denklem denir. Diferansiyel denklemleri bağımsız değişkenlerin sayısına ve içerdikleri türevlerin türlerine göre sınıflandırabiliriz. Denklemin tek bir bağımsız değişkeni varsa denklem adi diferansiyel denklem, iki veya daha çok bağımsız değişken içeriyorsa kısmi diferansiyel denklem olarak adlandırılır. Diferansiyel denklemler fiziksel olayların modellemesinde kullanılmaktadır.

Doğa bilimleri ve mühendislikte önemli bir yere sahip olan ve fiziksel olayların bir modellemesi olarak elde edilen lineer olmayan diferensiyel denklemlerin integrallenebilirliği 1960'lardan beri uygulamalı matematiğin temel konularından biri olmuştur. Lineer olmayan diferansiyel denklemlerin çözümlerinin elde edilmesi her zaman mümkün olamamaktadır. Bu zorluktan dolayı öncelikli olarak bu tip denklemlerin integrallenebilirliği üzerinde çalışılmıştır. Bununla birlikte integrallenebilir lineer olmayan diferansiyel denklemlerin çözümlerini bulmak için bir çok yöntem geliştirilmiştir. Simetri grupları ve korunum kanunları, bu yöntemlerden bazılarıdır.

Bu tezde simetri grupları kullanılarak, fizik ve matematikteki bazı önemli problemler incelenmiştir. Lineer olmayan diferansiyel denklemlerin analitik çözümlerinin ve korunum kanunlarının bulunması problemi ele alınmıştır. Literatürde lineer olmayan diferansiyel denklemleri çözmek için bir çok yöntem geliştirilmiştir, simetri grupları bunlardan biridir. Öncelikle, analitik çözümlerin araştırılmasında en güçlü yöntemler arasında gösterilen Lie simetri grupları ele alınmıştır. Sophus Lie, adi diferansiyel denklemler bir dönüşüm altında değişmez kalırsa mertebelerinin bir derece düşürülebileceğini göstermiştir. Bu şekilde, lineer olmayan diferansiyel denklemlere Lie cebrini uygulayıp denklemi değişmez bırakarak mertebesini indirgeyip denklemin çözümünü elde edebiliriz. n. basamaktan bir diferansiyel denklemin Lie grubunu elde etmek için, bu Lie grubuna ait sonsuz küçük üreticin n. uzanımını diferansiyel denkleme uyguladığımız zaman sonuç sıfır çıkmalıdır. Bu uzanım diferansiyel denkleme uygulandığı zaman bulunan açılımdan çok belirli kısmi diferansiyel denklemler sistemi elde edilir ve bu denklemler belirleyici denklemler olarak adlandırılır. Lie grupları ile çalışmanın bir zorluğu, Lie grup teorisini uyguladıktan sonra elde edilen belirleyici denklemleri çözmektir, bu zorluğu aşmak için bazı matematiksel programlar kullanılabilir, bunun için bu tezdeki problemleri incelerken Mathematica programı kullanılmıştır. Bu tezde, Lie grup teorisi bazı fiziksel problemlere uygulanıp sonuçlar elde edilmiştir.

Fakat, bazı durumlarda Lie grup teorisi yetersiz kalır. Her diferansiyel denklem Lie simetrilerine sahip olmayabilir. Bu durumda, simetrileri elde etmek ve

sınıflandırmak için farklı yöntemler kullanılmaktadır. Bu yöntemlerden biri Noether Teoremi'dir. Noether Teoremi, Alman matematikçi Noether tarafından bulunmuştur. Bu teoremi uygulamak için öncelikle denklemin Lagrangian fonksiyonu elde edilir, Lagrangian fonksiyonu Euler-Lagrange denklemlerini sağlamalıdır. Daha sonra bu Lagrangian fonksiyonu yardımıyla denklemin ilk integralleri bulunur. Bu teorideki en önemli kısım Lagrangian fonksiyonunun belirlenmesidir. Standard Lagrangian'a sahip olmayan bir çok denklem vardır. Bu tür denklemler için kısmi Lagrangian yöntemi geliştirilmiştir. Standart Lagrangian'a sahip olmayan denklemler için kısmi Lagrangian kullanılarak Noether simetrileri ve ilk integralleri bulunabilir. Tezin bir bölümünde kısmi Lagrangian yöntemi ele alınmıştır. Bu yöntem yardımıyla standard Lagrangian fonksiyonuna sahip olmayan fiziksel bir denklem olan fin denklemi için kısmi Lagrangian fonksiyonu belirlenmiştir. Sonrasında Noether teoremi kullanılarak, denklemin Noether simetrileri ve ilk integralleri elde edilmiştir. Bu simetriler fin denkleminin ısı-sıcaklık katsayılarına göre sınıflandırılmıştır. Daha sonra bu ilk integraller kullanılarak denklemin değişmez çözümleri elde edilmiştir.

Lie simetrisine sahip olmayan denklemlerin simetrilerini elde etmek için diğer bir yöntem Muriel ve Romero tarafından 2001 yılında tanımlanmıştır. Yeni bir vektör alanı tanımlayarak, yeni bir uzanım formu elde etmişler ve elde ettikleri simetrileri λ-simetrileri olarak adlandırmışlardır. Muriel ve Romero, bu yeni teoride Lie simetrilerinden farklı olarak tanımladıkları yeni vektör alanını kullanarak elde edilen belirleyici denklemlerin çözümünü sonsuz küçük fonksiyonlar ve λ fonksiyonu cinsinden belirlemişlerdir. Bir diferansiyel denklemin λ-simetrileri, integrasyon çarpanları ve ilk integralleri arasında önemli bir ilişki vardır. Özellikle, λ-simetrileri, Lie simetrisi olmayan lineer ve lineer olmayan denklemler için integrasyon çarpanlarının ve ilk integrallerinin bulunmasında etkili bir yöntemdir. Bu tezde ele alınan bir diğer önemli ilişki Lie simetrilerinden λ-simetrilerinin elde edilmesidir. Daha sonrasında λ-simetrileri kullanılarak integrasyon çarpanı ve ilk integraller elde edilebilir.

Bu simetrileri bulmamızı sağlayan diğer bir yöntem Prelle-Singer yöntemidir. Bu yöntem, Prelle ve Singer tarafından 1993 yılında ele alınmıştır ve zaman içinde Duarte yöntemi geliştirmiştir. Prelle-Singer yönteminde R ve S fonksiyonları ile ifade edilen üç adet belirleyici denklem vardır, bu denklemler çözülerek simetriler elde edilmeye çalışılır. Muriel ve Romero 2009 yılında Prelle-Singer yöntemi ile λ-simetrileri arasında bir ilişki kurmuşlardır. Bu ilişkiye göre, λ-simetrisi ve μ integrasyon çarpanı olmak üzere $R = -\mu$ ve $S = -\lambda$ eşitlikleri elde edilir. Bu yöntem kullanılarak, bazı fiziksel denklemlerin λ-simetrileri, integrasyon çarpanları, ilk integralleri ve sırası ile çözümleri elde edilmiştir. Lie, λ ve Prelle Singer yöntemleri arasında önemli bir ilişki söz konusudur. λ-simetrileri kullanılarak Lie simetrileri, Prelle Singer yöntemi kullanılarak öncelikle λ-simetrileri ve λ-simetrileri kullanılarak sonrasında Lie simetrileri elde edilebilir.

Tezde kullanılan diğer bir yöntem ise lineerleştirmedir. Verilen denklem önce bazı fonksiyonlara gore sınıflandırılıp daha sonrasında ait olduğu sınıfa dair kullanılan algoritma ile farklı ilk integralleri elde edilmiştir. Lineeleştirme problemleri altında incelediğimiz diğer bir dönüşüm Sundman dönüşümüdür. Duarte tarafından ortaya atılan bu dönüşüm ile lineer olmayan bir denklem lineerleştirilebiliniyorsa, bu denklem Muriel ve Romero tarafından S-lineerleştirilebilir olarak adlandırılmıştır. Tezde S-dönüşümleri kullanılarak ilk integrallerin ve λ-simetrilerinin nasıl bulunduğu açıkca gösterilmiştir. Sonrasında bu λ-simetrilerinden denklemin integrasyon çarpanı ve farklı ilk integralleri elde edilmiştir. Devamında bu λ-simetri bilgisi kullanılarak Prelle-Singer yöntemine geçilmiş ve Hamiltonian ve Lagrangian fonksiyonları elde edilmiştir. Bu tezde, lineer olmayan bir denklem için bir çok farklı yöntemle ilk integraller, simetriler ve bunların yardımıyla denklemin çözümlerinin elde edilebildiği açıktır.

1. INTRODUCTION

Mathematical modeling of many problems in physics and engineering sciences involve nonlinear ordinary differential equations. Therefore, the methods to solve a nonlinear ordinary differential equations have been continuously developed in the literature; see for example, [1,12].

It is always not possible to obtain solution of nonlinear equations. We can obtain solutions of nonlinear equations by integrating these equations therefore integrability of nonlinear equations is important. Moreover, a lot of methods have been improved to obtain the solutions of nonlinear equations using integrability. Some of these methods are called as Lie point symmetry, Noether symmetry, λ-symmetry, nonlocal transformation, etc. In this thesis, we examine these methods and we obtain analytical solutions and conservation laws of nonlinear differential equations by using these methods.

1.1 Purpose of Thesis

The purpose of the thesis is to find analyze the analytic solutions and conservation laws for nonlinear differential equations. There are a lot of methods to obtain these solutions in literature. Some of these methods are Lie symmetry groups theory, Noether theory, linearization methods, nonlocal transformations. In this thesis, some conservation law methods are used and these methods are applied to the equations in mathematical physics. Moreover, we aim to obtain solutions of nonlinear differential equations using symmetry groups which are found by different conservation law methods.

1.2 Literature Review

In this chapter, firstly we discuss the basic properties of Lie symmetry groups necessary in later chapters for the study of differential equations. We use the study [2] as reference to explain Lie symmetry groups, it is an important tool to understand the other symmetry methods. Then we examine Noether theorem and some concept which is related to conservation laws and we consider the definition of partial Lagrangian by using the study [12]. Then we examine some important definitions and theorems corresponding to λ-symmetries. Moreover, we explain transformation methods with some definitions.

In the literature, symmetry classifications of differential equations with respect to Lie point symmetries and Noether symmetries have an important role for understanding possible solutions of differential equations [1,9]. Noether symmetries can also be used in finding the first integrals (conserved forms) of the nonlinear problems. The earliest studies on Noether symmetries based on the Noether theorem are due to German mathematician Emmy Noether [1]. Applications of the Noether theorem to differential equations can provide some important information about the problems in mechanics, physics, and engineering sciences [12,18]. In order to apply the Noether theorem, the differential equations should have a standard Lagrangian. On the other hand, one can apply the partial Lagrangian method to differential equations to investigate Noether symmetries and first integrals by using Euler-Lagrange equations [12].

1.2.1 Lie symmetry groups

Lie symmetry group was developed to deal with the solution of differential equations. Lie was influenced by lectures of Sylow on Galois theory, therefore Lie symmetry groups are the extension of Galois methods for the study of differential equations. The basic examination is that the simple constant that can by added to any indefinite integral of dy/dx is in fact an element of a continuous symmetry group that convert solutions of the differential equation into other solutions. This observation was used by Lie to develop an algorithm when a differential equation has an invariance. If such a group exists, then the order of a higher order ordinary differential equation can be reduced.

In this sense, Sophus Lie has introduced the concept of continuous groups in order to extend different solution methods for ordinary differential equations and these groups are called Lie groups. Lie proved that if an ordinary differential equation is invariant under a one parameter Lie group of point transformations, then the order of ordinary differential equation can be reduced by one.

A symmetry group of a system of differential equations is a group of transformations which maps any solution to another solution of the system. Lie symmetry groups include translations, rotations and scalings. First order ordinary differential equations define a one parameter Lie group of point transformations.

Lie's fundamental theorem demonstrates that groups can be described by their infinitesimal generators. Lie groups and their infinitesimal generators can be extended to follow up on the space of independent variables, dependent variables and derivatives of the dependent variables.

Special solutions of differential equations are called similarity solutions or invariant solutions, if a differential equation is invariant under Lie group of point transformations.

Now, we examine the basic definitions and theorems for Lie symmetry groups to explain the later concepts.

Definition 1.1 *Let ϕ be a law of composition. The set G is called a group, if the nonempty set G is satisfy the following axioms:*

(i) Closure property: For any elements a and b of G, $\phi(a,b)$ is an element of G.

(ii) Associative property: For any elements a, b and c of G,

$$\phi(a,\phi(b,c)) = \phi(\phi(a,b),c). \tag{1.1}$$

(iii) Identity element: There exists a unique identity element e of G such that for any element a of G,

$$\phi(a,e) = \phi(e,a) = a. \tag{1.2}$$

(iv) Inverse element: For any element a of G, there exists a unique inverse element a^{-1} in G such that

$$\phi(a,a^{-1}) = \phi(a^{-1},a) = e. \tag{1.3}$$

In addition to these conditions, if $\phi(a,b) = \phi(b,a)$ holds for all elements a and b in G, a group G is called *Abelian*.

Definition 1.2 *Let $x = (x_1, x_2, ..., x_n)$ belongs to region $D \subset R^n$. The set of transformations*

$$x^* = X(x;\varepsilon), \tag{1.4}$$

defined for each x in D, depending on parameter ε. $\phi(\varepsilon,\delta)$ is a law of composition of parameters ε and δ in S. A group of transformation on D satisfy the following properties:

(i) For each parameter ε in S the transformations are one-to-one onto D, in particular x^* belongs to D.

(ii) S with the law of composition ϕ forms a group G.

(iii) $x^* = x$ when $\varepsilon = e$, i.e.

$$X(x;\varepsilon) = x. \tag{1.5}$$

(iv) If $x^* = X(x;\varepsilon)$, $x^{**} = X(x^*;\delta)$, then

$$x^{**} = X(x;\phi(\varepsilon,\delta)). \tag{1.6}$$

Definition 1.3 *If a group of transformations satisfy the following conditions in addition (i)-(iv), this group is called one-parameter Lie group of transformations.*

(v) The identity element e for $\varepsilon = 0$.

(vi) X is differentiable with respect to x in D and an analytic function of ε in S.

(vii) $\phi(\varepsilon,\delta)$ is an analytic function of ε and δ, $\varepsilon \in S$ and $\delta \in S$.

Definition 1.4 *Let*

$$x^* = X(x;\varepsilon) \tag{1.7}$$

be an one-parameter Lie group of transformations with identity $\varepsilon = 0$ and law of composition ϕ. If we expanding (1.7) about $\varepsilon = 0$, we obtain

$$\begin{aligned} x^* &= x + \varepsilon\Big(\frac{\partial X}{\partial \varepsilon}(x;\varepsilon)|_{\varepsilon=0}\Big) + \frac{\varepsilon^2}{2}\Big(\frac{\partial^2 X}{\partial \varepsilon^2}(x;\varepsilon)|_{\varepsilon=0}\Big) + \ldots \\ &= x + \varepsilon\Big(\frac{\partial X}{\partial \varepsilon}(x;\varepsilon)|_{\varepsilon=0}\Big) + O(\varepsilon^2). \end{aligned} \tag{1.8}$$

And

$$\xi(x) = \frac{\partial X}{\partial \varepsilon}(x;\varepsilon)|_{\varepsilon=0}. \tag{1.9}$$

Thus $x + \varepsilon\xi(x)$ is called the infinitesimal transformation of the Lie group of transformations and the terms of $\xi(x)$ are called the infinitesimals of the equation (1.7).

Definition 1.5 *The gradient operator is*

$$\nabla = \Big(\frac{\partial}{\partial_{x_1}}, \frac{\partial}{\partial_{x_2}}, ..., \frac{\partial}{\partial_{x_n}}\Big). \tag{1.10}$$

The infinitesimal generator of the one-parameter Lie group of transformations (1.7) is defined as

$$X = X(x) = \xi(x).\nabla = \sum_{i=1}^{n} \xi_i(x)\frac{\partial}{\partial_{x_i}}. \tag{1.11}$$

For any differential function $F(x) = F(x_1, x_2, ..., x_n)$,

$$XF(x) = \xi(x).\nabla F(x) = \sum_{i=1}^{n} \xi_i(x)\frac{\partial F(x)}{\partial_{x_i}}. \tag{1.12}$$

Definition 1.6 *For any group transformation (1.7),*

$$F(x^*) \equiv F(x), \tag{1.13}$$

if and only if $F(x)$ is an invariant function of the Lie group of transformations (1.7).

Theorem 1.1 *$F(x)$ is invariant under (1.7) if and only if*

$$XF(x) \equiv 0. \tag{1.14}$$

Definition 1.7 *For a one parameter Lie group of transformations*

$$x^* = X(x,y;\varepsilon) = x + \varepsilon\xi(x,y) + O(\varepsilon^2),$$

$$y^* = Y(x,y;\varepsilon) = y + \varepsilon\eta(x,y) + O(\varepsilon^2), \tag{1.15}$$

the infinitesimal generator is

$$X = \xi(x,y)\frac{\partial}{\partial_x} + \eta(x,y)\frac{\partial}{\partial_y}. \tag{1.16}$$

Let $y_k = \frac{d^k y}{dx^k}$. We can extend (1.15) to $(x, y, y_1, ..., y_k)$ space for $k = 1, 2, ...,$ and k-prolongation is

$$y_k^* = Y_k(x,y,y_1,...,y_k;\varepsilon) = y_k + \varepsilon\eta^k(x,y,y_1,...,y_k) + O(\varepsilon^2), \tag{1.17}$$

and the infinitesimal generator for k-prolongation is

$$X_k = \xi(x,y)\frac{\partial}{\partial_x} + \eta(x,y)\frac{\partial}{\partial_y} + \eta^{(1)}(x,y,y_1)\frac{\partial}{\partial_{y_1}} + ... \tag{1.18}$$

$$+\eta^{(k)}(x,y,y_1,...,y_k)\frac{\partial}{\partial_{y_k}}, \tag{1.19}$$

where

$$\eta^{(k)}(x,y,y_1,...,y_k) = \frac{D\eta^{(k-1)}}{Dx} - y_k\frac{D\xi(x,y)}{Dx}, \quad \eta^{(0)} = \eta(x,y). \tag{1.20}$$

Now, we summarize these definitions and theorems, therefore we consider first order ordinary differential equation

$$\frac{dy}{dx} = g(x), \tag{1.21}$$

where x is the independent variable and y is the dependent variable. The solution of this equation is

$$y = G(x) = \int g(x)dx. \tag{1.22}$$

If we write the solutions of the form $y - G(x) = 0$, then $y + c - G(x) = 0$ is also a solution of the equation (1.21).

We study for a one-parameter group of transformations that leaves the surface equation invariant by changing variables in the (x,y) plane according to

$$x \to \bar{x}(\varepsilon) = x + \varepsilon\xi(x,y) \quad \bar{x}(\varepsilon = 0) = x \tag{1.23}$$

$$y \to \bar{y}(\varepsilon) = y + \varepsilon\eta(x,y) \quad \bar{y}(\varepsilon = 0) = y, \tag{1.24}$$

where one parameter group $x \to \bar{x}$ and $y \to \bar{y}$ in (1.21), so $\xi = 0$ and $\eta = 0$.

For $\xi(x,y)$ and $\eta(x,y)$, the first prolongation is

$$\frac{d\bar{y}}{d\bar{x}} = \frac{d\bar{y}/dx}{d\bar{x}/dx} = \frac{\varepsilon(\eta_x + \eta_y)}{1 + \varepsilon(\xi_x + \xi_y)} \to \varepsilon(\eta_x + (\eta_y - \xi_x) - \xi_y) \tag{1.25}$$

Thus,

$$\eta^{(1)}(x,y,y^{(1)}) = \eta_x + (\eta_y - \xi_x) - \xi_y). \tag{1.26}$$

The surface equation must be same under the one-parameter group of transformations, that is

$$F(x,y) = 0 \to F(\bar{x}(\varepsilon),\bar{y}(\varepsilon)) \to F(x+\varepsilon\xi, y+\varepsilon\eta) = F(x,y) + \varepsilon(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y})F(x,y). \tag{1.27}$$

And Lie point symmetry is

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}. \tag{1.28}$$

Thus, we obtain

$$F(x,y) = 0 \quad and \quad XF(x,y) = 0. \tag{1.29}$$

1.2.2 Noether theorem and first integrals

In this chapter, we use the study [12] as reference to explain some concept which is related conservation laws.

Noether showed that how the symmetries of action integral first placed to conservation laws for the corresponding Euler-Lagrange equations. Euler-Lagrange equations are invariant under variational symmetries. Now, we examine these Noether symmetries.

Suppose that x is the independent variable and $y = (y^1, \dots, y^m)$ is the dependent variable with coordinates y^α with respect to x are given as following form

$$y_x^\alpha = y_1^\alpha = D_x(y^\alpha), \quad y_s^\alpha = D_x^s(y^\alpha), \quad s \geq 2, \quad \alpha = 1, 2, \dots, m, \tag{1.30}$$

where D_x is the total derivative operator [2-7], with respect to x, which is defined as

$$D_x = \frac{\partial}{\partial x} + y_x^\alpha \frac{\partial}{\partial y^\alpha} + y_{xx}^\alpha \frac{\partial}{\partial y_x^\alpha}. \tag{1.31}$$

Here, the vector space of all differential functions of all finite orders is represented by $\mathscr{A}$ that is universal space. Also, operators apart from total derivative operator (1.31) are defined on space $\mathscr{A}$.

Definition 1.8 *The operator*

$$\frac{\delta}{\delta y^\alpha} = \frac{\partial}{\partial y^\alpha} + \sum_{s \geq 1} (-D_x)^s \frac{\partial}{\partial y_x^\alpha}, \quad \alpha = 1, 2, \dots, m, \tag{1.32}$$

is called the Euler operator or Euler-Lagrange operator.

Definition 1.9 *The generalized operator is given by*

$$X = \xi \frac{\partial}{\partial x} + \eta^\alpha \frac{\partial}{\partial y^\alpha} + \sum_{s \geq 1} \xi_s^\alpha \frac{\partial}{\partial y_s^\alpha}, \tag{1.33}$$

where

$$\xi_s^\alpha = D_x^s(W^\alpha) + \xi y_{s+1}^\alpha, \quad s \geq 2, \quad \alpha = 1, 2, \dots, m, \tag{1.34}$$

and W^α is the Lie characteristic function

$$W^\alpha = \eta^\alpha - \xi y_x^\alpha, \quad \alpha = 1, 2, \dots, m. \tag{1.35}$$

Here we can rewrite the generalized operator (1.33) in terms of characteristic function as below

$$X = \xi D_x + W^\alpha \frac{\partial}{\partial y^\alpha} + \sum_{s \geq 1} D_x^s(W^\alpha) \frac{\partial}{\partial y_s^\alpha} \tag{1.36}$$

and the Noether operator associated with a generalized operator X can be defined as

$$N = \xi + W^{\alpha}\frac{\partial}{\partial y^{\alpha}} + \sum_{s\geq 1} D_x^s(W^{\alpha})\frac{\partial}{\partial y_s^{\alpha}}. \tag{1.37}$$

Now let us consider a kth-order system of ordinary differential equation

$$E_{\alpha}(x,u,u_{(1)},u_{(2)},\ldots,u_{(k)}) = 0, \quad \alpha = 1,2,\ldots,m. \tag{1.38}$$

Definition 1.10 *The first integral of the system $I \in \mathscr{A}$ can be written in the following form*

$$D_x I = 0. \tag{1.39}$$

Then the expression (1.39) is called the local conservation law for system (1.38). Furthermore, $D_x I = Ç^{\alpha}E_{\alpha}$ is called the characteristic form of conservation law (1.39) where the functions $Ç^{\alpha} = (Ç^1,\ldots,Ç^m)$ are the associated characteristics of the conservation law (1.39).

Definition 1.11 *Let $L = L(x,(x,u,u_{(1)},u_{(2)},\ldots,u_{(\alpha)}) \in \mathscr{A}$, $\alpha \leq k$ and nonzero functions $f_{\alpha}^{\beta} \in \mathscr{A}$ be a partial Lagrangian and X be a Lie-Bäcklund operator of the form of (1.33). If there exists a vector $B \in \mathscr{A}$, $B \neq NL + C, C = constant$, we have the following relation*

$$X_{(\alpha)}L + LD_x(\xi) = W^{\alpha}\frac{\delta L}{\delta y^{\alpha}} + D_x(B), \tag{1.40}$$

where $W = (W^1,\ldots,W^m)$, $B(x,y)$ is the gauge function, and $W^{\alpha} \in \mathscr{A}$ then X is called a partial Noether operator coresponding to L and, $X_{(\alpha)}$ is the α^{th} prolongation of the generalized operator (1.36). If we apply Euler-Lagrange operator (1.32) to Lagrangian L, then we obtain following differential equations

$$\frac{\delta L}{\delta u^{\alpha}} = 0, \quad \alpha = 1,2,\ldots,m, \tag{1.41}$$

which are called Euler-Lagrange equations and the Lagrangian L is called a standard Lagrangian. However, if $\frac{\delta L}{\delta u^{\alpha}} \neq 0$, the Lagrangian L is called as a partial Lagrangian and the corresponding differential equations are called partial Euler-Lagrange equations.

Definition 1.12 *X is a Noether point symmetry corresponding to Lagrangian of the system of differential equations (1.38) if there exists a function $B(x,y)$. In addition, X* *is a Noether point symmetry corresponding to a Lagrangian of equation, then I is a first integral associated with X, which is given by the expression [10]*

$$I = \xi L + (\eta - y'\xi)L_{y'} - B. \tag{1.42}$$

1.2.3 Linearization methods

In this chapter, we use the study [20] as reference to explain some concept which is related linearization methods.

The linearization technics deal with obtaining the general solutions of the nonlinear equation by using the first integrals and λ-symmetries. Furthermore, it is a fact that using linearization methods, a nonlinear second order equation can be converted to a linear second order ordinary differential equation whose solutions are known. The first linearization problem for differential equations is solved by Lie [10]. He shows that a second order ordinary differential equation is linearizable by a change of variables if and only if the equation has the form

$$\ddot{x} + a_2(t,x)\dot{x}^2 + a_1(t,x)\dot{x} + a_0(t,x) = 0, \tag{1.43}$$

where t is the independent variable and x is the dependent variable of the equation and over dot denotes the derivative with respect to t [19].

Moreover, one of these methods is to obtain general solution by using the first integral of the equation. It is known that the some solutions remain invariant under symmetry group transformations; these solutions are called *invariant (or similarity)* solutions. It is assumed that the ordinary second order differential equation of the form (1.43) has the first integrals of the form

$$A(t,x)\dot{x} + B(t,x). \tag{1.44}$$

In order to find the first integrals of the form (1.44), one can use a standard procedure and apply it to the nonlinear differential equations. Then it is possible to present that the first integrals of the form $A(t,x)\dot{x} + B(t,x)$ can be obtained by using the linearization methods [20].

From the mathematical point of view, the process of linearization outlined above is a difficult task and it can only be applied to the second order ordinary differential equations. Therefore, it is necessary to consider other type of transformation techniques of the nonlinear differential equations for linearization. In the literature, it is shown that the equations of the form (1.43) can be transformed into the linear equations $X_{TT} = 0$ by means of nonlocal transformation of the form

$$X = F(t,x), \quad dT = G(t,x)dt, \tag{1.45}$$

which is known as the generalized Sundman transformation [21,24]. This transformation is also called S-transformation and the equations that can be linearized by means of S-transformation are called S-linearizable [21]. Duarte [22] proves that S-linearizable equations must be of the form (1.43). A detailed review for the available generalizations and recent contributions can be found in the references [25,26].

Another method to solve the nonlinear differential equations is to obtain λ-symmetries of the equations. Muriel and Romeo [20] prove that the equations of the form (1.43) have the first integrals of the form (1.44), λ-symmetries and the integrating factors $\mu = A(t,x)$. They also show that the equation of the form (1.43) admits $v = \partial_x$ for λ-symmetry of the form [21]

$$\lambda(t,x,\dot{x}) = \alpha(t,x)\dot{x} + \beta(t,x). \tag{1.46}$$

The other method which is called modified Prelle-Singer procedure [27,28] is used to apply it to a class of second order nonlinear ordinary differential equations and solved several physically interesting nonlinear systems and identified a number of important linearization procedures. Prelle and Singer have proposed an algorithmic procedure to find the integrating factor for the system of first order ordinary differential equations. Once the integrating factor for the equation is determined then it leads to a time independent integral of motion for the first order ordinary differential equation. The Prelle-Singer method guarantees that if the first order ordinary differential equation has a first integral in terms of elementary functions then this first integral can be found. This method has been generalized to incorporate the integrals with non-elementary functions. Recently, this theory is generalized to obtain general solutions for second order and higher order ordinary differential equations without any integration [28].

Now, we explain linearization methods in the following sections.

1.2.3.1 The first integrals of the form $A(t,x)\dot{x}+B(t,x)$

In this section we examine the equations of the form (1.43) that have first integrals of the form $A(t,x)\dot{x}+B(t,x)$ for $A \neq 0$. For this purpose we consider the following notations

$$S_1(t,x) = a_{1x} - 2a_{2t}, \tag{1.47}$$

$$S_2(t,x) = (a_0a_2 + a_{0x})_x + (a_{2t} - a_{1x})_t + (a_{2t} - a_{1x})a_1. \tag{1.48}$$

Then one can say that if $S_1 = 0$, then the equation (1.43) is *S-linearizable* if and only if $S_2 = 0$. By these definitions we have the following theorem to determine $A(t,x)$ and $B(t,x)$.

Theorem 1.2 *Let us assume that an equation (1.43) is S-linearizable, that is $S_1 = S_2 =$ 0. In addition, let $f(t)$ be the function defined by*

$$f(t) = a_0a_2 + a_{0x} - \frac{1}{2}a_{1t} - \frac{1}{4}{a_1}^2 \tag{1.49}$$

and $P = P(t,x)$ be a function such that

$$P_t = \frac{1}{2}a_1, \quad P_x = a_2. \tag{1.50}$$

Thus using an equation (1.50) one can determine the function $P = P(t,x)$ explicitly. Similarly, let $g = g(t)$ be a nonzero solution of the linear equation

$$g''(t) + f(t)g(t) = 0, \tag{1.51}$$

and $Q = Q(t,x)$ be a function such that

$$Q_t = a_0.g.e^P, \quad Q_x = \left(\frac{1}{2}a_1 - \frac{g'}{g}\right)g.e^P. \tag{1.52}$$

Then, one can obtain the function $Q = Q(t,x)$ from the equations (1.51) and (1.58). Finally, functions A and B are determined as follows

$$A = g.e^P, \quad B = Q. \tag{1.53}$$

Then one can say that if $S_1 \neq 0$, we have the following theorem to determine $A(t,x)$ and $B(t,x)$.

Theorem 1.3 *Let us assume that $S_1 \neq 0$. In this situation the functions $S_3 = S_4 = 0$. Let $f(t)$ be the function defined by*

$$f(t) = a_0a_2 + a_{0x} - \frac{1}{2}a_{1t} - \frac{1}{4}{a_1}^2 \tag{1.54}$$

and $P = P(t,x)$ be a function such that

$$P_t = a_1 + \frac{S_2}{S_1}, \quad P_x = a_2. \tag{1.55}$$

Thus one can determine the function $P = P(t,x)$ explicitly. Similarly, let $g = g(t)$ be a nonzero solution of the linear equation

$$g''(t) + f(t)g(t) = 0, \tag{1.56}$$

and $Q = Q(t,x)$ be a function such that

$$Q_t = a_0e^P, \quad Q_x = -\Big(\frac{S_2}{S_1}\Big)e^P. \tag{1.57}$$

Then there exist a function $Q = Q(t,x)$ due to the compatibility condition

$$[a_0e^P]_x, \quad [-\Big(\frac{S_2}{S_1}\Big)e^P]_t \tag{1.58}$$

Then, one can obtain the function $Q = Q(t,x)$ from the equations (1.51) and (1.58). Finally, functions A and B are determined as follows

$$A = e^P, \quad B = Q. \tag{1.59}$$

1.2.3.2 The λ-symmetries and the integrating factors

Let us consider a second-order ordinary differential equation

$$\ddot{x} = \Phi(t,x,\dot{x}). \tag{1.60}$$

Then one can say that the vector field $\upsilon = \partial_x$ is a λ-symmetry of (1.60) if and only if λ is a solution of the equation

$$\Phi_x + \lambda\Phi_{\dot{x}} = \lambda_t + \dot{x}\lambda_x + \Phi\lambda_{\dot{x}} + \lambda^2. \tag{1.61}$$

Using coefficients a_0, a_1, a_2 in (1.43), one can easily compute S_1, S_2 and then one can obtain λ-symmetry for (1.43).

Theorem 1.4 *If $S_1 = S_2 = 0$, then λ-symmetry for (1.43) is determined using following feasible algorithm.*

$$h'(t) + h^2(t) + f(t) = 0, \tag{1.62}$$

where $f(t)$ is defined by (1.49). Then the β function is found as

$$\beta(t,x) = h(t) - \frac{1}{2}a_1(t,x). \tag{1.63}$$

Thus we find the λ-symmetry is the following form

$$\lambda = -a_2(t,x)\dot{x} + h(t) - \frac{1}{2}a_1(t,x). \tag{1.64}$$

Theorem 1.5 *We know that if $S_1 \neq 0$, then the functions $S_3 = S_4 = 0$.*

In this situation, λ-symmetry for (1.43) is determined using following feasible algorithm.

Let $f(t)$ be the function defined by

$$f(t) = a_0a_2 + a_{0x} - \frac{1}{2}a_{1t} - \frac{1}{4}{a_1}^2 \tag{1.65}$$

$$\gamma_x + \frac{1}{2}S_1 = 0 \tag{1.66}$$

and

$$\gamma_t + \gamma^2 + f = 0 \tag{1.67}$$

From here,

$$S_1\gamma = -\frac{1}{2}S_{1t} + f_x. \tag{1.68}$$

Thus we find the λ-symmetry is the following form

$$\lambda = -a_2(t,x)\dot{x} + \left(\frac{S_2}{S_1}\right). \tag{1.69}$$

Theorem 1.6 *[1] If the equation (1.43) has the first integral of the form $I = A(t,x)\dot{x} + B(t,x)$, then the equation (1.43) has a integrating factor that has form $\mu = A(t,x)$.*

1.2.3.3 The nonlocal transformations

It is possible to show that the nonlinear second-order equations are linearizable by means of generalized Sundman transformation. These nonlinear equations are characterized in terms of the coefficients of the equations and constructive methods to derive the linearizing Sundman transformation can be presented. Thus the nonlinear ordinary differential equations can be solved by transforming them into the linear ordinary equations whose solutions are known. These equations are called S-linearizable. The second order S-linearizable equations have the first integrals of the form $A(t,x)\dot{x}+B(t,x)$. When a first integral of this form is known, we derive a method to construct the Sundman transformation that linearizes the equation. Conversely, if a linearizing Sundman transformation is known then a first integral of the form is obtained. Now, we present the following theorem that the characterizes S-linearizable equations by the coefficients of a given differential equations.

Theorem 1.7 *We assume that the equation (1.43) is S-linearizable. If $S_1 = S_2 = 0$ and $\varphi(t)$ is the solution of the equation*

$$\varphi_t + \varphi^2 + f = 0, \tag{1.70}$$

and the function $f(t)$ is defined by (1.49).

Let $C(x,t)$ be a solution of the following equations

$$C_t = a_0 - C(\frac{a_1}{2} + \varphi), \tag{1.71}$$

$$C_x = (\frac{a_1}{2} - \varphi) - Ca_2. \tag{1.72}$$

If $F(x,t)$ is a solution of the equation

$$F_t = CF_x, \tag{1.73}$$

and

$$G = F_x \exp(-P - \int \varphi(t)dt), \tag{1.74}$$

and thus S-transformation pair F and G are defined.

Theorem 1.8 *We suppose that the equation (1.43) has a first integral as*

$$I = A(t,x)\dot{x} + B(t,x). \tag{1.75}$$

It can be written

$$A(t,x) = \frac{F_x}{G}, \quad B(t,x) = \frac{F_t}{G}. \tag{1.76}$$

*The equation (1.43) can be linearized by (**??**). D_t is total derivative operator and the equation*

$$A(\ddot{x} + a_2(t,x)\dot{x}^2 + a_1(t,x)\dot{x} + a_0(t,x)) = D_t(I(t,x,\dot{x})), \tag{1.77}$$

can be obtained. Consequently,

$$F(t,x) = \varphi(I(t,x)) \tag{1.78}$$

and

$$G(t,x) = \frac{F_x}{A} \quad or \quad G(t,x) = \frac{F_t}{B} \quad B \neq 0. \tag{1.79}$$

The transformation pair obtained by first integral.

1.2.3.4 Lagrangian and Hamiltonian description

Assuming the existence of a Hamiltonian

$$I(x,\dot{x}) = H(x,p) = p\dot{x} - L(x,\dot{x}) \tag{1.80}$$

where $L(x,\dot{x})$ is the Lagrangian and p is the canonically conjugate momentum, we have

$$\frac{\partial I}{\partial \dot{x}} = \frac{\partial H}{\partial \dot{x}} = \frac{\partial p}{\partial \dot{x}}\dot{x} + p - \frac{\partial L}{\partial \dot{x}} = \frac{\partial p}{\partial \dot{x}}\dot{x}. \tag{1.81}$$

From equation (4.1) we identify

$$p = \int \frac{I_{\dot{x}}}{\dot{x}} d\dot{x} + f(x), \tag{1.82}$$

where $f(x)$ is an arbitrary function of x. Equation (1.82) has also been derived recently by a different methodology. We take $f(x) = 0$ and substituting the known expression of I into equation (1.82) and integrating it, we can obtain the expression for the canonical momentum p.

2. ON SYMMETRY GROUP CLASSIFICATION OF FIN EQUATION [1]

In this chapter, we discuss the nonlinear fin equation and the corresponding determining equations. This section also includes different cases corresponding to different choices of thermal conductivity and heat transfer coefficient. Furthermore, Noether point symmetries and first integrals for each different case are presented. In addition, we present some invariant solutions.

2.1 Noether Symmetries of Fin Equation

In this section, we classify the Noether point symmetries of a fin equation. Noether symmetries can also be used in finding the first integrals of the nonlinear problems. We should apply Noether theorem to fin equation for obtaining Noether symmetries. In order to apply the Noether theorem, the differential equations should have a standard Lagrangian. On the other hand, one can apply the partial Lagrangian method to differential equations to investigate Noether symmetries and first integrals by using Euler-Lagrange equations. The fin equation has not standart Lagrangian, therefore, we determine the partial Lagrangian and Noether symmetries of the fin equation by applying partial Noether approach to a nonlinear fin equation.

We now consider Noether symmetry classification of the nonlinear fin equation [14-15]

$$y'' + \frac{K'(y)}{K(y)}(y')^2 - \frac{H(y)}{K(y)} = 0, \tag{2.1}$$

where K and H are thermal conductivity and heat transfer coefficient, respectively, which are considered as functions of temperature, and $y = y(x)$ is the temperature function and x is dimensional spatial variable. The Lie point symmetries equation (2.1) is investigated in the reference [16]. In this study, we consider the partial Noether approach to analyze Noether symmetries of equation (2.1).

[1] This chapter is based on the paper Orhan Ö., Gün G. and Özer T., On symmetry group classification of a fin equation. Journal of Inequalities and Applications, 2013, 2013:147, doi:10.1186/1029-242X-2013-147.

For the fin equation (2.1), we can write the *Euler-Lagrange* operator (1.32)

$$\frac{\delta}{\alpha y^{\alpha}} = \frac{\partial}{\partial y^{\alpha}} - D_x \frac{\partial}{\partial y_x} + D_x^2 \frac{\partial}{\partial y_{xx}}, \tag{2.2}$$

and the partial Lagrangian L for the fin equation (2.1) can be written as

$$L = \frac{1}{2}(y')^2 + \int \frac{H(y)}{K(y)} dy, \tag{2.3}$$

and if we apply Euler-Lagrange operator (2.2) to Lagrangian (2.3), then we obtain

$$\frac{\delta L}{\delta y} = \frac{H(y)}{K(y)} - y''. \tag{2.4}$$

In addition, if we rewrite the fin equation in the following form

$$-y'' + \frac{H(y)}{K(y)} = (y')^2 \frac{K'(y)}{K(y)}, \tag{2.5}$$

then, the equation (2.4) becomes

$$\frac{\delta L}{\delta y} = (y')^2 \frac{K'(y)}{K(y)}. \tag{2.6}$$

In relation (1.40), the partial Lagrangian (2.3) has at most first order derivatives and then we can take $\alpha = 1$ and write the following definition

$$W^1 \frac{\delta L}{\delta y} = (\eta - \xi y')\left(y'^2 \frac{K'(y)}{K(y)}\right) = \eta y'^2 \frac{K'(y)}{K(y)} - \xi y'^3 \frac{K'(y)}{K(y)}, \tag{2.7}$$

and $D_x(B)$ is defined in the form

$$D_x(B) = B_x + y' B_y. \tag{2.8}$$

By application of the first prolongation of the generalized operator (2.7) $X_{(1)}$ to Lagrangian (2.3), we get

$$X_{(1)} L = \eta \frac{H(y)}{K(y)} + \eta^1 y', \tag{2.9}$$

where η^1 is defined in the form [2-5]

$$\eta^1 = \eta_x + (\eta_y - \xi_x) y' - \xi_y (y')^2. \tag{2.10}$$

The expansion of form of (2.10) by using the definition of the first prolongation of the Noether operator and relations (2.6)-(2.10) is written below

$$\eta_x y' + (\eta_y - \xi_x) y'^2 - \xi_y y'^3 + \frac{1}{2}\xi_x y'^2 + \frac{1}{2}\xi_y y'^3 + \xi_x \int \frac{H(y)}{K(y)} dy +$$

$$+\xi_y y' \int \frac{H(y)}{K(y)} dy + \eta \frac{H(y)}{K(y)} + \xi \frac{K'(y)}{K(y)} y'^3 - \eta \frac{K'(y)}{K(y)} y'^2 - B_x - y' B_y = 0. \tag{2.11}$$

The usual separation by powers of derivatives of y (2.11) reduces to the following determining equations

$$-\frac{1}{2}\xi_y + \xi \frac{K'(y)}{K(y)} = 0, \tag{2.12}$$

$$\eta_y - \frac{1}{2}\xi_x - \eta \frac{K'(y)}{K(y)} = 0, \tag{2.13}$$

$$\eta_x + \xi_y \int \frac{H(y)}{K(y)} dy - B_y = 0, \tag{2.14}$$

$$\xi_x \int \frac{H(y)}{K(y)} dy + \eta \frac{H(y)}{K(y)} - B_x = 0. \tag{2.15}$$

To find the infinitesimals ξ and η the determining equations (2.12)-(2.15) should be solved together. First, from the solution of the equation (2.12) we have

$$\xi = K(y)^2 a(x), \tag{2.16}$$

where $a(x)$ is a function of x. The solution of equation (2.13) is

$$\eta = \frac{1}{2} a'(x) K(y) \int K(y) dy + K(y) b(x), \tag{2.17}$$

where $b(x)$ is a function of x. Thus, if we differentiate (2.14) with respect to x and (2.15) with respect to y then we can eliminate the function $B(x,y)$ from equations (2.14)-(2.15) and we obtain the following single equation

$$\begin{aligned} &\left(\frac{1}{2} a'(x) \int K(y) dy + b(x)\right) H'(y) + \frac{3}{2} K(y) a'(x) H(y) \\ &-\frac{1}{2} a'''(x) K(y) \left(\int K(y) dy\right) - K(y) b''(x) = 0 \end{aligned} \tag{2.18}$$

which is a differential equation including unknown functions $K(y)$, $H(y)$, $a(x)$ and $b(x)$. Using the equations (2.16)-(2.18) one can classify Noether symmetries and corresponding first integrals of the nonlinear fin equation (2.1) based on different forms of the thermal conductivity $K(y)$ and heat transfer coefficient $H(y)$ and differential relations for $a(x)$ and $b(x)$.

Case 1: $K(y) = k(constant)$

In equation (2.19) if we consider $K(y) = k(constant)$ then we obtain the following differential equation for $H(y)$ function

$$(2b(x)+a'(x)ky)H'(y)+3ka'(x)H(y)-k(2b''(x)-ka'''(x))=0. \tag{2.19}$$

Now we analyze differential equation (2.19) for different $H(y)$ functions corresponding to different solutions of (2.18) and we get differential relations between functions $a(x)$ and $b(x)$, which yield Noether symmetries and corresponding first integral for each case.

Case 1.1: $H(y) = h(constant)$

For this case the equation (2.19) becomes

$$3hka'(x)-2kb''(x)-k^2ya'''(x)=0. \tag{2.20}$$

In (2.20) it is clear that $a'''(x) = 0$, $3ha'(x)-2b''(x) = 0$. From the solutions of $a(x)$ and $b(x)$ we obtain the following infinitesimal functions

$$\xi = k^2(c_1+xc_2+x^2c_3),$$
$$\eta = \frac{1}{2}k^2y(c_2+2xc_3)+k\left(\frac{3}{4}hx^2c_2+\frac{1}{2}hx^3c_3+c_4+xc_5\right), \tag{2.21}$$

and the corresponding Noether symmetries

$$X_1 = k^2\frac{\partial}{\partial x}, \quad X_2 = k^2x\frac{\partial}{\partial x}+(\frac{1}{2}k^2y+\frac{3}{4}khx^2)\frac{\partial}{\partial y}, \quad X_3 = k^2x^2\frac{\partial}{\partial x}+(k^2xy+\frac{1}{2}hkx^3)\frac{\partial}{\partial y},$$
$$X_4 = k\frac{\partial}{\partial y}, \quad X_5 = kx\frac{\partial}{\partial y}\cdot \tag{2.22}$$

By using relations (2.14) and (2.15) the function $B(x,y)$ is found in the form below

$$B(x,y) = \frac{1}{4}h^2x^3c_2+\frac{3}{2}hkxyc_2+\frac{1}{8}h^2x^4c_3$$
$$+\frac{3}{2}hkx^2yc_3+\frac{1}{2}k^2y^2c_3+hxc_4+\frac{1}{2}hx^2c_5+kyc_5, \tag{2.23}$$

where $c_i, i = 1,\ldots,5$ are constants. Thus, the first integrals (conserved forms) for the nonlinear fin equation (2.1) can be calculated by using expression (2.12) and by considering each group parameter c_i.

$$I_1 = hky - \frac{1}{2}k^2(y')^2, \quad I_2 = \frac{1}{8}(-2h^2x^3 - 4hkxy + 2k(3hx^2 + 2ky)y' - 4k^2x(y')^2),$$

$$I_3 = \frac{1}{8}(-h^2x^4 - 4hkx^2y - 4k^2y^2 + 2k(2hx^3 + 4kxy)(y') - 4k^2x^2(y')^2), \tag{2.24}$$

$$I_4 = -hx + ky', \quad I_5 = -\frac{1}{2}hx^2 - ky + kxy'.$$

Case 1.2: $H(y) = y$

Based on the similar calculation in the first case if we take $H(y) = y$, we obtain the infinitesimals ξ and η by solving equations $4a'(x) - ka'''(x) = 0$, $b(x) - kb''(x) = 0$

$$\xi = k^2(\frac{1}{2}e^{\frac{-2x}{\sqrt{k}}}\sqrt{k}(e^{\frac{4x}{\sqrt{k}}}c_1 - c_2) + c_3),$$

$$\eta = \frac{1}{2}k^2y(2e^{\frac{2x}{\sqrt{k}}}c_1 - (e^{\frac{-2x}{\sqrt{k}}}(e^{\frac{4x}{\sqrt{k}}}c_1 - c_2)) + k(e^{\frac{x}{\sqrt{k}}}c_4 + e^{\frac{-x}{\sqrt{k}}}c_5), \tag{2.25}$$

where $c_i, i = 1, ..., 5$ are constants.

The corresponding generators are

$$X_1 = (\frac{1}{2}e^{\frac{2x}{\sqrt{k}}}k^{\frac{5}{2}})\frac{\partial}{\partial x} + (\frac{1}{2}e^{\frac{2x}{\sqrt{k}}}k^2y)\frac{\partial}{\partial y},$$

$$X_2 = (-\frac{1}{2}e^{\frac{-2x}{\sqrt{k}}}k^{\frac{5}{2}})\frac{\partial}{\partial x} + (\frac{1}{2}e^{\frac{-2x}{\sqrt{k}}}k^2y)\frac{\partial}{\partial y},$$

$$X_3 = k^2\frac{\partial}{\partial x}, \quad X_4 = (e^{\frac{x}{\sqrt{k}}}k)\frac{\partial}{\partial y}, \quad X_5 = (e^{\frac{-x}{\sqrt{k}}}k)\frac{\partial}{\partial y}, \tag{2.26}$$

and the gauge function is

$$B(x,y) = e^{\frac{-2x}{\sqrt{k}}}\sqrt{k}\left(\frac{1}{2}e^{\frac{4x}{\sqrt{k}}}ky^2c_1 - \frac{1}{2}ky^2c_2 + e^{\frac{3x}{\sqrt{k}}}yc_4 - e^{\frac{x}{\sqrt{k}}}c_5y\right) + c_6, \tag{2.27}$$

where c_6 is an arbitrary constant and the first integrals are found by using the expression (2.12)

$$I_1 = \frac{1}{4}e^{\frac{2x}{\sqrt{k}}}\left(-k^{\frac{3}{2}}y^2 + 2k^2yy' - k^{\frac{5}{2}}(y')^2\right), \tag{2.28}$$

$$I_2 = \frac{1}{4}e^{\frac{-2x}{\sqrt{k}}}\left(k^{\frac{3}{2}}y^2 + 2k^2yy' + k^{\frac{5}{2}}(y')^2\right), \tag{2.29}$$

$$I_3 = \frac{1}{2}\left(ky^2 - k^2(y')^2\right), \quad I_4 = e^{\frac{x}{\sqrt{k}}}(ky' - \sqrt{k}y), \tag{2.30}$$

$$I_5 = e^{\frac{-x}{\sqrt{k}}}(\sqrt{k}y + ky'). \tag{2.31}$$

Case 1.3: $H(y) = y^n, n > 1$

In equation (2.19), if we take $H(y) = y^n$ then we obtain

$$2ny^{(-1+n)}b(x) + 3ky^na'(x) + kny^na'(x) - 2kb''(x) - k^2ya'''(x) = 0, \tag{2.32}$$

which gives $a'(x) = 0$ and $b(x) = 0$ gives ξ and η

$$\xi = k^2 c_1, \quad \eta = 0, \quad B(x,y) = 0, \tag{2.33}$$

where c_1 is a constant and then the infinitesimal generator corresponding to (3.30) is

$$X = k^2 \frac{\partial}{\partial x}, \tag{2.34}$$

and the first integral is written similar to the previous case

$$I = -\frac{-2ky^{1+n} + k^2(1+n)(y')^2}{2(1+n)}. \tag{2.35}$$

Case 1.4: $H(y) = Exp(y)$

For this case it is clear that the infinitesimal functions are

$$\xi = k^2 c_1, \quad \eta = 0, \quad B(x,y) = 0, \tag{2.36}$$

where c_1 is a constant and the generator is

$$X = k^2 \frac{\partial}{\partial x}, \tag{2.37}$$

and the first integral is

$$I = e^y k - \frac{1}{2} k^2 (y')^2. \tag{2.38}$$

Case 1.5: $H(y) = \frac{1}{my+n}$

For this case the infinitesimals are found as below

$$\xi = k^2 c_1, \quad \eta = 0, \quad B(x,y) = 0, \tag{2.39}$$

where c_1 is an arbitrary constant and the generator is

$$X = k^2 \frac{\partial}{\partial x}, \tag{2.40}$$

and the first integral is

$$I = \frac{k \log(k(n+my))}{m} - \frac{1}{2} k^2 (y')^2. \tag{2.41}$$

Case 1.6: *Arbitrary function* $H(y)$

We find that

$$\xi = k^2 c_1, \quad \eta = 0, \quad B(x,y) = 0, \tag{2.42}$$

where c_1 is a constant and the generator is

$$X = k^2 \frac{\partial}{\partial x}, \tag{2.43}$$

and the first integral is

$$I = k \int H(y) dy - \frac{1}{2} k^2 (y')^2. \tag{2.44}$$

Case 2: $K(y) = kExp(\alpha y)$, k and β are constants.

In the equation (2.18) if we take $K(y) = kExp(\beta y)$, we obtain the following differential equation in terms of $H(y)$ function

$$(2\alpha b(x) + a'(x) e^{y\alpha} k) H'(y) + 3e^{y\alpha} k\alpha a'(x) H(y) - e^{y\alpha} k (2\alpha b''(x) + e^{y\alpha} k a'''(x)) = 0, \tag{2.45}$$

and consider following cases as the solutions of (2.42) and get the mathematical relations between functions $a(x)$ and $b(x)$.

Case 2.1: $H(y) = h(constant)$.
For this case the differential equation (2.42) yields

$$e^{y\alpha} k \left(3h\alpha a'(x) + 2\alpha b''(x) + e^{y\alpha} k a'''(x)\right) = 0. \tag{2.46}$$

In (3.43) $k \neq 0$ then the term in the parenthesis must be zero, which gives $a'''(x) = 0$ and $3ha'(x) + 2b''(x) = 0$, then the infinitesimal functions are found as below

$$\xi = e^{2y\alpha} k^2 (c_1 + x c_2 + x^2 c_3),$$
$$\eta = \frac{e^{2y\alpha} k^2 (c_2 + 2x c_3)}{2\alpha} + e^{y\alpha} k (\frac{3}{4} h x^2 c_2 + \frac{1}{2} h x^3 c_3 + c_4 + x c_5), \tag{2.47}$$

where $c_i, i = 1, \ldots, 5$ are constants and we have following five infinitesimal generators

$$X_1 = (e^{2y\alpha} k^2) \frac{\partial}{\partial x}, \quad X_2 = (e^{2y\alpha} k^2 x) \frac{\partial}{\partial x} + (\frac{3}{4} e^{y\alpha} h k x^2 + \frac{e^{2y\alpha} k^2}{2\alpha}) \frac{\partial}{\partial y},$$
$$X_3 = (e^{2y\alpha} k^2 x^2) \frac{\partial}{\partial x} + (\frac{1}{2} e^{y\alpha} h k x^3 + \frac{e^{2y\alpha} k^2 x}{\alpha}), \quad X_4 = e^{y\alpha} k \frac{\partial}{\partial y}, \quad X_5 = e^{y\alpha} k x \frac{\partial}{\partial y}, \tag{2.48}$$

and we have the gauge function

$$B(x,y) = \frac{1}{8\alpha^2} 4\left(e^{2y\alpha} k^2 c_3 - 4 e^{y\alpha} k \alpha (h(4c_1 + x(c_2 + x c_3)) - 2c_5)\right)$$

$$+hx\alpha^2\left(hx^2(2c_2+xc_3)+8c_4+4xc_5\right)\Big), \tag{2.49}$$

and the corresponding first integrals

$$I_1=\frac{e^{\alpha y}hk}{\alpha}-\frac{1}{2}e^{2\alpha y}k^2\alpha^2(y')^2,$$

$$I_2=-\frac{2e^{y}hkx\alpha+h^2x^3\alpha^2-e^{\alpha y}k\alpha(2e^{\alpha y}k+3hx^2\alpha)y'+2e^{2\alpha y}k^2x\alpha^2(y')^2}{4\alpha^2},$$

$$I_3=-\frac{4e^{2\alpha y}k^2+4e^{\alpha y}hkx^2\alpha+h^2x^4\alpha^2-2e^{\alpha y}k\alpha(4e^{\alpha y}kx+2hx^3\alpha)(y')+4e^{2\alpha y}k^2x^2\alpha^2(y')^2}{8\alpha^2},$$

$$I_4=e^{\alpha y}k\alpha^2y'-hx,\quad I_5=e^{\alpha y}kxy'-\frac{1}{2}hx^2-\frac{e^{\alpha y}k}{\alpha}\cdot \tag{2.50}$$

Case 2.2: *Arbitrary* $H(y)$

For an arbitrary $H(y)$ function we obtain infinitesimal functions in the form

$$\xi=e^{2y\alpha}k^2c_1,\quad \eta=0, \tag{2.51}$$

where c_1 is a constant and the infinitesimal generator is

$$X=e^{2\alpha y}k^2\frac{\partial}{\partial x}, \tag{2.52}$$

and the gauge function is

$$B(x,y)=2k\alpha c_1\int e^{2\alpha y}\left(\int e^{-\alpha y}H(y)dy\right)dy, \tag{2.53}$$

and the first integral is calculated as follow

$$I=-\frac{1}{2}k\left(-2e^{2\alpha y}\int e^{-\alpha y}H(y))dy+4\alpha\int e^{2\alpha y}(\int e^{-\alpha y}H(y)dy)dy+e^{2\alpha y}k(y')^2\right)\cdot \tag{2.54}$$

Case 2.3: $H(y)=\frac{h}{(\beta y+\gamma)^2}$, β and γ are arbitrary constants
For this case the infinitesimals ξ and η are

$$\xi=e^{2y\alpha}k^2c_1,\quad \eta=0, \tag{2.55}$$

where c_1 a constant and the infinitesimal generator is

$$X=e^{2\alpha y}k^2\frac{\partial}{\partial x}, \tag{2.56}$$

and the gauge function is

$$B(x,y) = -\frac{e^{-\frac{\alpha\gamma}{\beta}}hk\alpha c_1}{\beta^2}\big(e^{\frac{2\alpha(y\beta+\gamma)}{\beta}}ExpIntegralEi(-\frac{\alpha(y\beta+\gamma)}{\beta})$$
$$+ExpIntegralEi(\frac{\alpha(y\beta+\gamma)}{\beta})\big), \tag{2.57}$$

where ExpIntegralEi is a special function on the complex plane, for real nonzero values of x, the exponential integral $Ei(x)$ is defined as

$$Ei(x) = \int_{-\infty}^{x}(\frac{e^t}{t})dt, \tag{2.58}$$

and the first integral is

$$I = \frac{1}{2\beta^2(\gamma+\beta y)}e^{-\frac{\alpha\gamma}{\beta}}\big(-2(e^{\frac{\alpha\gamma}{\beta}+\alpha y}hk\beta - hk\alpha ExpIntegralEi(\alpha\frac{(\gamma+y\beta}{\beta}))(\gamma+\beta y))$$
$$-e^{-\frac{\alpha\gamma}{\beta}+2\alpha y}k^2\beta^2(\gamma+\beta y)(y')^2\big). \tag{2.59}$$

Case 3: $K(y) = ky^{\beta}$, $\beta \neq -1$

If we take $H(y) = h$ is constant, then we obtain the following equation

$$-3h(1+\beta)a'(x) + 2(1+\beta)b''(x) + ky^{1+\beta}a'''(x) = 0, \tag{2.60}$$

and we find the infinitesimals functions from solutions of $a'''(x) = 0$ and $-3h(1+\beta)a'(x)+$
$2(1+\beta)b''(x) = 0$

$$\xi = k^2y^{2\beta}(c_1 + xc_2 + x^2c_3),$$
$$\eta = \frac{k^2y^{1+2\beta}(c_2+2xc_3)}{2(1+\beta)} + ky^{\beta}(\frac{3}{4}hx^2c_2 + \frac{1}{2}x^3c_3 + c_4 + xc_5), \tag{2.61}$$

where $c_i, i = 1,...,5$ are constants. In this case we have following five infinitesimal generators

$$X_1 = k^2y^2\beta\frac{\partial}{\partial x}, \quad X_2 = k^2xy^{2\beta}\frac{\partial}{\partial x} + (\frac{3}{4}hkx^2y^{\beta} + \frac{k^2y^{1+2\beta}}{2(1+2\beta)})\frac{\partial}{\partial y}, \tag{2.62}$$

$$X_3 = k^2x^2y^{2\beta}\frac{\partial}{\partial x} + (\frac{1}{2}kx^3y^{\beta} + \frac{k^2xy^{1+2\beta}}{(1+2\beta)})\frac{\partial}{\partial y},$$
$$X_4 = ky^{\beta}\frac{\partial}{\partial y}, \quad X_5 = xky^{\beta}\frac{\partial}{\partial y} \tag{2.63}$$

and the gauge function is

$$B(x,y) = \frac{1}{8}\big(hx(hx^2(2c_2+xc_3)+8c_4+4xc_5) + \frac{4ky^{1+\beta}}{(\beta-1)(\beta+1)^2}\big(-h(1+\beta)(3x(c_2+xc_3)$$

$$+\beta(4c_1+x(c_2+xc_3)))+(\beta-1)(ky^{\beta}c_3+2(\beta+1)c_5))\big). \tag{2.64}$$

Using this equation (2.59) we obtain three first integrals

$$I_1=-\frac{ky^{\beta}(-2hy+k(1+\beta)y^{\beta}y'^2)}{2(1+\beta)},$$

$$I_2=-\frac{1}{4(1+\beta)}\left((hx-ky^{\beta}y')(hx^2(1+\beta)+2ky^{1+\beta}-2kx(1+\beta)y^{\beta}y')\right),$$

$$I_3=-\frac{1}{8(1+\beta)^2}(hx^2(1+\beta)+2ky^{1+\beta}-2kx(1+\beta)y^{\beta}y')^2, \tag{2.65}$$

$$I_4=-hx+ky^{\beta}y', \quad I_5=\frac{hx^2}{2}-\frac{ky^{(1+\beta)}}{1+\beta}+kxy^{\beta}y'.$$

Case 2.1: $K(y)=ky^{\beta}$, $\beta=-1$

For this case the equation (2.18) is equal to

$$3ha'(x)-2b''(x)-ka'''(x)lny=0, \tag{2.66}$$

and by using the (2.61) the infinitesimals functions become that

$$\xi=\frac{k^2}{y^2}(c_1+xc_2+x^2c_3),$$

$$\eta=\frac{1}{y}\left(k(\frac{3}{4}hx^2c_2+\frac{1}{2}hx^3c_3+c_4+xc_5)+k^2(c_2+2xc_3)lny\right), \tag{2.67}$$

where $c_i, i=1,...,5$ are constants and the infinitesimal generators are

$$X_1=\frac{k^2}{y^2}\frac{\partial}{\partial x}, \quad X_2=\frac{k^2x}{y^2}\frac{\partial}{\partial x}+\frac{k(3hx^2+2klny)}{4y}\frac{\partial}{\partial y},$$

$$X_3=\frac{k^2x^2}{y^2}\frac{\partial}{\partial x}+\frac{k(2hx^3+4hkxlny)}{4y}\frac{\partial}{\partial y}, \tag{2.68}$$

$$X_4=\frac{k}{y}\frac{\partial}{\partial y}, \quad X_5=\frac{kx}{y}\frac{\partial}{\partial y},$$

and the gauge function is

$$B(x,y)=\frac{1}{8}(hx(4k(c_2+xc_3)+hx^2(2c_2+xc_3)+8c_4+4xc_5)$$

$$+4k(h(-2c_1+x(c_2+xc_3)+2c_5)lny+4k^2c_3lny^2). \tag{2.69}$$

And we have four first integrals

$$I_1=\frac{k}{2}(h+2hlny-\frac{ky'^2}{y^2}),$$

$$
\begin{aligned}
I_2 &= \frac{ky' - hxy}{4y^2}\Big((hx^2 + 2klny)y - 2kxy'\Big), \\
I_3 &= -\frac{1}{8y^2}\Big((hx^2 + 2klny)y - 2kxy'\Big)^2, \\
I_4 &= \frac{ky'}{y} - hx, \quad I_5 = kx\frac{y'}{y} - klny - \frac{hx^2}{2}.
\end{aligned} \tag{2.70}
$$

2.2 Invariant Solutions

Some group invariant solutions of nonlinear fin equation (2.1) can be constructed from the Noether symmetries and the first integrals. In this section we consider some different special cases to present invariant solutions of (2.1).

Case 1. For the case $K(y) = k(constant)$ and $H(y) = h(constant)$ the first conservation law is

$$
I = hky - \frac{1}{2}k^2(y')^2, \tag{2.71}
$$

then the expression $D_x I = 0$ gives the following invariant solution of the fin equation (2.1)

$$
y(x) = \frac{4c + 2h^2x^2 - 2\sqrt{2}h^2kxc_1 + h^2k^2c_1^2}{4hk}, \tag{2.72}
$$

where c, c_1 are constants.

Case 2. As another case if we consider $K(y) = k(constant)$ and $H(y) = y$, then the first integral becomes

$$
I = \frac{1}{4}e^{\frac{-2x}{\sqrt{k}}}\Big(k^{\frac{3}{2}}y^2 + 2k^2yy' + k^{\frac{5}{2}}(y')^2\Big), \tag{2.73}
$$

and $D_x = 0$ yields the following solution

$$
y(x) = -\sqrt{c}\frac{e^{\frac{x}{\sqrt{k}}}}{k^{\frac{3}{4}}} + e^{-\frac{x}{\sqrt{k}}}c_2, \tag{2.74}
$$

where c, c_2 are constants. This solution (2.4) is the group invariant solution that satisfies the original the fin equation (2.1).

Case 3. As the third case we consider $K(y) = k(constant)$ and $H(y) = y$ and find the conserved form as below

$$
I = \frac{1}{2}e^{\frac{-2x}{\sqrt{k}}}\Big(e^{\frac{2x}{\sqrt{k}}}ky^2 - e^{\frac{2x}{\sqrt{k}}}k^2(y')^2\Big), \tag{2.75}
$$

and (2.5) gives the following invariant solution

$$y(x) = \frac{e^{-\frac{x+kc_3}{\sqrt{k}}}(e^{2\sqrt{k}c_3} + 8ce^{\frac{2x}{\sqrt{k}}})}{4k}. \tag{2.76}$$

Case 4. The choice of $K(y) = kExp(\alpha y)$ and $H(y) = h(constant)$ yields the conservation law

$$I = \frac{e^{\alpha y}hk}{\alpha} + \frac{1}{2}e^{2\alpha y}k^2\alpha^2(y')^2, \tag{2.77}$$

and by integration of (2.8) we find the group invariant solution in the following form

$$y(x) = \frac{1}{\alpha}ln(\frac{c\alpha}{hk} + \frac{1}{2k}hx^2\alpha + \frac{hx\alpha^{\frac{3}{2}}}{\sqrt{2}}c_4 + \frac{1}{4}hk\alpha^2c_4^2), \tag{2.78}$$

where c_4 is a constant and which satisfies the fin equation (2.1).

Table 2.1 : The table of Noether symmetries of fin equation

Thermal	Heat coefficient	Infinitesimal and first integrals
$k(constant)$	$H(y)$	$\xi = k^2 c_1, \quad \eta = 0, \quad I = k\int H(y)dy - \frac{1}{2}k^2 y'^2$
$k(constant)$	h	$\xi = k^2(c_1 + xc_2 + x^2 c_3)$ $\eta = \frac{1}{2}k^2 y(c_2 + 2xc_3) + k\left(\frac{3}{4}hx^2 c_2 + \frac{1}{2}hx^3 c_3 + c_4 + xc_5\right)$ $I_1 = hky - \frac{1}{2}k^2 y'^2$ $I_2 = \frac{1}{8}(-2h^2 x^3 - 4hkxy + 2k(3hx^2 + 2ky)y' - 4k^2 xy'^2$ $I_3 = \frac{1}{8}(-h^2 x^4 - 4hkx^2 y - 4k^2 y^2 + 2k(2hx^3 + 4kxy)y'$ $I_4 = -hx + ky', \quad I_5 = -\frac{1}{2}hx^2 - ky + kxy'$
$k(constant)$	$\frac{1}{my+n}$	$\xi = k^2 c_1, \quad \eta = 0, \quad I = \frac{k\log(k(n+my))}{m} - \frac{1}{2}k^2 y'^2$
$k(constant)$	e^y	$\xi = k^2 c_1, \quad \eta = 0, \quad I = e^y k - \frac{1}{2}k^2 y'^2$
$k(constant)$	y	$\xi = k^2(\frac{1}{2}e^{\frac{-2x}{\sqrt{k}}}\sqrt{k}(e^{\frac{4x}{\sqrt{k}}}c_1 - c_2) + c_3)$ $\eta = \frac{1}{2}k^2 y(2e^{\frac{2x}{\sqrt{k}}}c_1 - (e^{\frac{-2x}{\sqrt{k}}}(e^{\frac{4x}{\sqrt{k}}}c_1 - c_2)) + k(e^{\frac{x}{\sqrt{k}}}c_4 + e^{\frac{-x}{\sqrt{k}+c_5}})$ $I_1 = \frac{1}{4}e^{\frac{2x}{\sqrt{k}}}\left(-k^{\frac{3}{2}}y^2 + 2k^2 yy' - k^{\frac{5}{2}}(y')^2\right)$ $I_2 = \frac{1}{4}e^{\frac{-2x}{\sqrt{k}}}\left(k^{\frac{3}{2}}y^2 + 2k^2 yy' + k^{\frac{5}{2}}(y')^2\right)$ $I_3 = \frac{1}{2}\left(ky^2 - k^2(y')^2\right), \quad I_4 = e^{\frac{x}{\sqrt{k}}}(ky' - \sqrt{k}y)$ $I_5 = e^{\frac{-x}{\sqrt{k}}}(\sqrt{k}y + ky')$
$k(constant)$	y^n	$\xi = e^{2y\alpha}k^2 c_1, \quad \eta = 0, \quad I = \frac{2ky^{1+n} - k^2(1+n)(y')^2}{2(1+n)}$
$kExp(\alpha y)$	$H(y)$	$\xi = e^{2y\alpha}k^2 c_1, \quad \eta = 0$ $I = k(2e^{2\alpha y}\int e^{-\alpha y}H(y))dy$ $-2\alpha\int e^{2\alpha y}(\int e^{-\alpha y}H(y)dy)dy - \frac{1}{2}e^{2\alpha y}ky'^2$
$kExp(\alpha y)$	$h(sabit)$	$\xi = e^{2y\alpha}k^2(c_1 + xc_2 + x^2 c_3)$ $\eta = \frac{e^{2y\alpha}k^2(c_2 + 2xc_3)}{2\alpha} + e^{y\alpha}k(\frac{3}{4}hx^2 c_2 + \frac{1}{2}hx^3 c_3 + c_4 + xc_5)$ $I_1 = \frac{e^{\alpha y}hk}{\alpha} - \frac{1}{2}e^{2\alpha y}k^2\alpha^2(y')^2$ $I_2 = -\frac{2e^y hkx\alpha + h^2 x^3\alpha^2 - e^{\alpha y}k\alpha(2e^{\alpha y}k + 3hx^2\alpha)y' + 2e^{2\alpha y}k^2 x\alpha^2(y')^2}{4\alpha^2}$ $I_3 = -\frac{1}{8\alpha^2}(4e^{2\alpha y}k^2 + 4e^{\alpha y}hkx^2\alpha + h^2 x^4\alpha^2$ $-2e^{\alpha y}k\alpha(4e^{\alpha y}kx + 2hx^3\alpha)(y') + 4e^{2\alpha y}k^2 x^2\alpha^2(y')^2$ $I_4 = e^{\alpha y}k\alpha^2 y' - hx, \quad I_5 = e^{\alpha y}kxy' - \frac{1}{2}hx^2 - \frac{e^{\alpha y}k}{\alpha}$
$kExp(\alpha y)$	$\frac{h}{(\beta y+\gamma)^2}$	$\xi = e^{2y\alpha}k^2 c_1, \quad \eta = 0$ $I = \frac{1}{2\beta^2(\gamma+\beta y)}e^{-\frac{\alpha\gamma}{\beta}}\left(-2(e^{\frac{\alpha\gamma}{\beta}+\alpha y}hk\beta\right.$ $\left.-hk\alpha ExpIntegralEi(\alpha\frac{(\gamma+y\beta}{\beta}))(\gamma+\beta y))\right)$

3. ANALYSIS OF LIENARD II-TYPE OSCILLATOR EQUATION BY SYMMETRY TRANSFORMATION METHODS [1]

In this chapter, we discuss the nonlinear Lienard II-type harmonic nonlinear oscillator equation and the corresponding linearization methods. Furthermore, the first integral, the λ-symmetry, the integrating factor and transformation pair are presented. We apply modified Prelle-Singer method to the Lienard II-type harmonic nonlinear oscillator equation to obtain Lie symmetries, the first integrals, λ-symmetries, the integrating factors and the Lagrangian-Hamiltonian functions.

3.1 The First Integral, λ-symmetry and the Integrating Factor of Lienard II-type Harmonic Nonlinear Oscillator Equation

We consider the following nonlinear Lienard II-type harmonic nonlinear oscillator equation, which possesses exact periodic solution, exhibiting the characteristic amplitude-dependent frequency of nonlinear oscillator in spite of the sinusoidal nature of the solution of equation [29]

$$\ddot{x}(t) - \frac{2\dot{x}^2(t)}{3x(t)} + \frac{\omega^2 x(t)}{3} = 0, \tag{3.1}$$

where x is the position coordinate, which is a function of time t and ω is the strength of the forcing, in which these parameters indicate nonlinearity. The Lienard II-type harmonic nonlinear oscillator equation has a natural generalization in three dimensions and these systems can be also quantized exhibiting many interesting features and can be interpreted as an oscillator constrained to move on a three-sphere. In this section, we investigate the first integral of the form $A(t,x)\dot{x} + B(t,x)$ of equation (3.1).

Proposition 3.1 *The equation of the form*

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0, \tag{3.2}$$

[1]This chapter is based on the paper Orhan Ö. and Özer T., Analysis of Lienard II-type oscillator equation by symmetry transformation methods. Advance in Difference Equations, 2016, 2016:259, doi:10.1186/s13662-016-0966-4.

where $f(x)$ and $g(x)$ are arbitrary functions of x and over dots denote differentiation with respect to t, is called quadratic Lienard-type equation. Quadratic Lienard type equations are linearizable if and only if these equations must satisfy the following condition

$$g'(x)+f(x)g(x)=\gamma, \tag{3.3}$$

where γ is an arbitrary constant. This condition is called the isochronous condition. A dynamical system is called isochronous if it features in its phase space an open, fully-dimensional region where all its solutions are periodic in all its degrees of freedom with the same, fixed, period. In order to the Lienard II-type harmonic nonlinear oscillator equation belongs to this class, it must satisfy isochronous condition to be linearized. Thus if we apply (3.3) to the equation (3.1), we see that Lienard II-type harmonic nonlinear oscillator equation satisfy isochronous condition.

Proof: Firstly, we compute S_1 function for the nonlinear Lienard II-type harmonic nonlinear oscillator equation to classify equation. The Lienard II-type harmonic nonlinear oscillator equation is form of (1.43) and we know the coefficients of (3.1) by like that

$$a_2(t,x)=-\frac{2\dot{x}^2(t)}{3x(t)},\quad a_1(t,x)=0,\quad a_0(t,x)=\frac{\omega^2x(t)}{3}. \tag{3.4}$$

Using these coefficients, we obtain $S_1=0$, which is given by (1.47). Thus, we know from Theorem 1 that S_2 must be zero if $S_1=0$. The function S_2 is compute for the Lienard II-type harmonic nonlinear oscillator equation and the function S_2 is found zero.

3.1.1 The first integral of the form $A(t,x)\dot{x}+B(t,x)$ and the invariant solutions

It can be shown that the Lienard II-type harmonic nonlinear oscillator equation (3.1) has the first integral of the form $A(t,x)\dot{x}+B(t,x)$ by determining functions A and B using a procedure given above. Then, the equation can be integrated by using this first integral and the exact solution of the equation can be obtained.

For this purpose, let $P=P(t,x)$ be a function such that

$$P_t=0,\quad P_x=\frac{-2}{3x}. \tag{3.5}$$

Using (1.50) we obtain function $P=P(x)$ like this

$$P(x)=\frac{-2logx}{3}. \tag{3.6}$$

If we compute $f(t)$ using the formula (1.49) we obtain

$$f(t) = \frac{\omega^2}{9}. \tag{3.7}$$

Let us $g = g(t)$ is a nonzero solution of the equation (1.51) then if we substitute (3.7) in equation (1.51), we obtain the following equation

$$g''(t) + \frac{\omega^2}{9} g(t) = 0. \tag{3.8}$$

The solution of this differential equation yields

$$g(t) = c_1 cos(\frac{\omega t}{3}) + c_2 sin(\frac{\omega t}{3}). \tag{3.9}$$

If we substitute the functions $P(x)$ and $g(t)$ into (1.58) we obtain the following equations

$$Q_t = \frac{1}{3} x^{\frac{1}{3}} \omega^2 (c_1 cos(\frac{\omega t}{3}) + c_2 sin(\frac{\omega t}{3})), \tag{3.10}$$

$$Q_x = \frac{-\frac{1}{3}\omega(c_1 cos(\frac{\omega t}{3}) + c_2 sin(\frac{\omega t}{3}))}{x^{\frac{2}{3}}}. \tag{3.11}$$

From solutions of (3.10) and (3.11) we have

$$Q(t,x) = c_3 - \omega x^{\frac{1}{3}} (c_2 cos(\frac{\omega t}{3}) - c_1 sin(\frac{\omega t}{3})). \tag{3.12}$$

Then, we substitute these solutions into (1.59) we obtain the functions $A(t,x)$ and $B(t,x)$ as follows

$$A(t,x) = \frac{(c_1 cos(\frac{\omega t}{3}) + c_2 sin(\frac{\omega t}{3}))}{x^{\frac{2}{3}}}, \tag{3.13}$$

$$B(t,x) = c_3 - \omega x^{\frac{1}{3}} (c_2 cos(\frac{\omega t}{3}) - c_1 sin(\frac{\omega t}{3})). \tag{3.14}$$

Thus, the first integral of the Lieanard II-type nonlinear harmonic oscillator equation (3.1) is written as

$$I = c_3 - \omega x^{\frac{1}{3}} \left(c_2 cos(\frac{\omega t}{3}) - c_1 sin(\frac{\omega t}{3})\right) + \frac{\left(c_1 cos(\frac{\omega t}{3}) + c_2 sin(\frac{\omega t}{3})\right)\dot{x}}{x^{\frac{2}{3}}}, \tag{3.15}$$

and from Theorem 3, the integrating factor is

$$\mu = \frac{(c_1 cos(\frac{\omega t}{3}) + c_2 sin(\frac{\omega t}{3}))}{x^{\frac{2}{3}}}. \tag{3.16}$$

Group invariant solutions of this nonlinear equation can be constructed from the first integral, that is, from (4.2) the invariant solution of the equation (3.1) is determined

$$x(t) = \frac{\omega c_1^2 c_5 cos(\frac{\omega t}{3}) + (c_4 - c_3 + \omega c_1 c_2 c_5 sin(\frac{\omega t}{3}))^3}{\omega^3 c_1^3}, \tag{3.17}$$

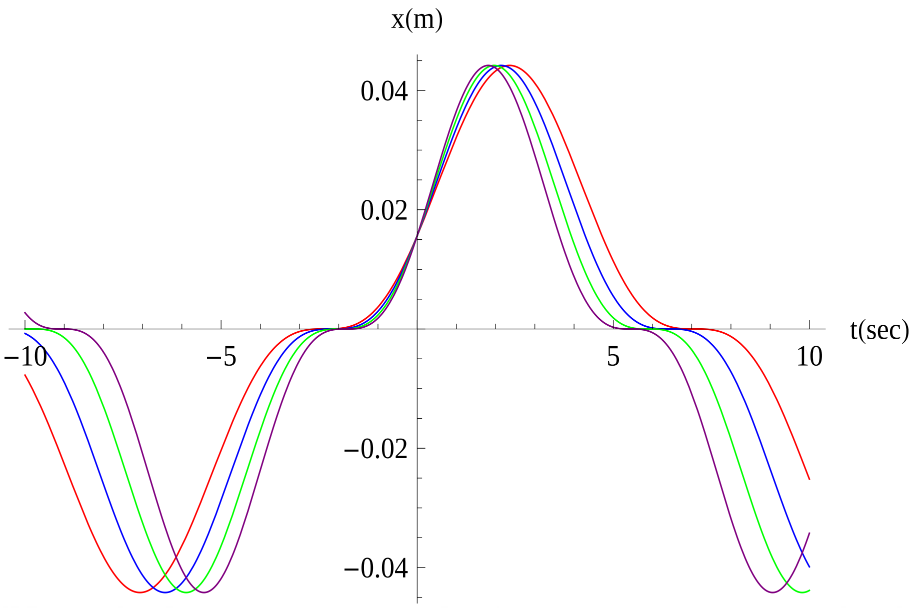

Figure 3.1 : Phase portrait of the equation (3.17) for different values of ω.

where c_1, c_2, c_3, c_4, c_5 are constants.

The phase plane method refers to graphically determining the existence of limit cycles in the solutions of the oscillator equations. The solutions to the nonlinear differential equation are a family of functions. Graphically, this can be plotted in the phase plane like a two-dimensional vector field. Vectors representing the derivatives of the points with respect to a parameter time t at representative points are drawn. With enough of these arrows in place the system behavior over the regions of plane in analysis can be visualized and limit cycles can be identified. Then a phase portrait is a geometric representation of the trajectories of a dynamical system in the phase plane. Each set of initial conditions is represented by a different curve, or point.

The phase portraits are an invaluable tool in studying dynamical systems. They consist of a plot of typical trajectories in the state space. This reveals information such as whether an attractor, a repeller or limit cycle is present for the chosen parameter value. The concept of topological equivalence is important in classifying the behavior of systems by specifying when two different phase portraits represent the same qualitative dynamic behavior. A phase portrait graph of a dynamical system depicts the system's trajectories.

Remark 1: We see in the Figure 3.1, the solution of the oscillator equation constitute *limit cycle* in the phase plane. The limit cycle begins as circle and, with varying ω, become increasingly sharp.

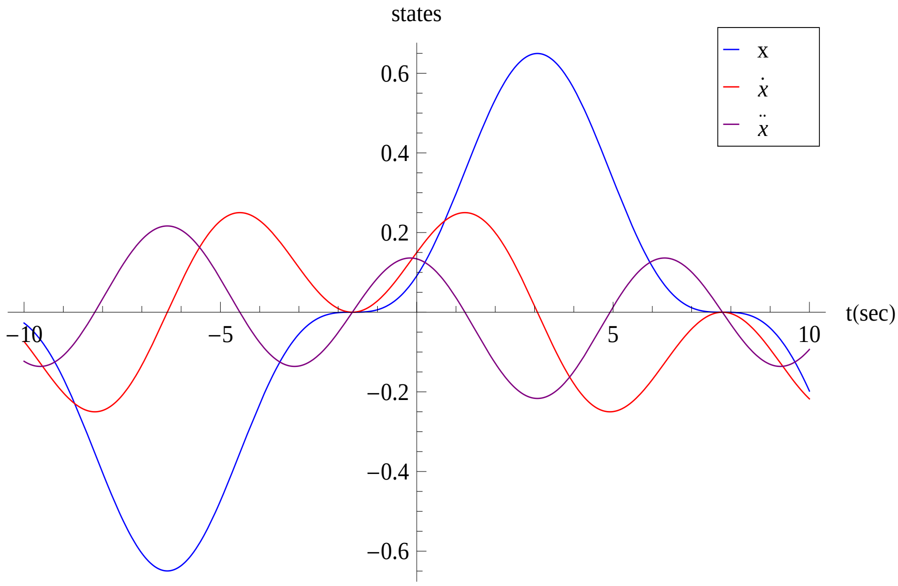

Figure 3.2 : Phase portrait of the equation of (3.17) for values $x(m)$, $\dot{x}(m/sec)$ and $\ddot{x}(m/sec^2)$.

Remark 2: The blue line shows the position x over time, and the red line shows the rate of change x, in other words the velocity $\dot{x}$, over time and the purple line shows the rate of change of the velocity, that is acceleration, over time in Figure 3.2. These are the three states of the system, simulated over time. The way to interpret this simulation is, if we start the system at $x = 0$, $\dot{x} = 0$ and $\ddot{x} = 0$, and simulate for 20 seconds, this is how the system would behave.

Furthermore, one can determine the corresponding Hamiltonian form related with the first integral (4.52). First, the canonical conjugate momentum is

$$p = \frac{\log(\dot{x})(c_1\cos(\frac{t\omega}{3}) + c_2\sin(\frac{t\omega}{3}))}{x^{\frac{2}{3}}} \tag{3.18}$$

The Hamiltonian function that corresponding to canonical conjugate momentum

$$H = c_3 - \omega(c_2\cos(\frac{t\omega}{3}) - c_1\sin(\frac{t\omega}{3}))x^{\frac{1}{3}} + \frac{c_1\cos(\frac{t\omega}{3}) + c_2\sin(\frac{t\omega}{3}))\dot{x}}{x^{\frac{2}{3}}} \tag{3.19}$$

Then the corresponding Lagrangian is

$$L = \frac{\omega(c_2\cos(\frac{t\omega}{3}) - c_1\sin(\frac{t\omega}{3}))x + (\log(\dot{x}) - 1)(c_1\cos(\frac{t\omega}{3}) + c_2\sin(\frac{t\omega}{3}))\dot{x} - c_3x^{\frac{2}{3}}}{x^{\frac{2}{3}}}. \tag{3.20}$$

Now, one can see the graph of the solution. The graph of a dynamical system depicts stable steady states and unstable steady states in a state space. The axes are of state variables. In this case we deal with the rate of change (velocity) $\dot{x}$ and the rate of change of the velocity (i.e the acceleration) denoted $\ddot{x}$ as states of the equation.

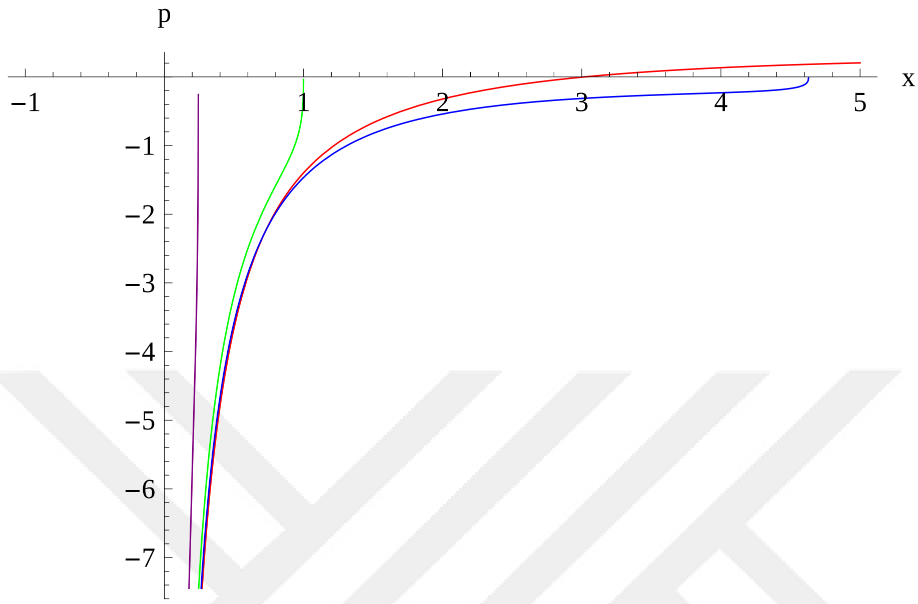

Figure 3.3 : The graph of the conjugate momentum p is given (3.18) depending on position x.

The graph of the (3.17) corresponding to the conjugate momentum (3.18) for four different values $\omega = 0.1$, $\omega = 0.3$, $\omega = 0.5$ and $\omega = 0.8$ with ω is shown with four different colors in Figure 3.3

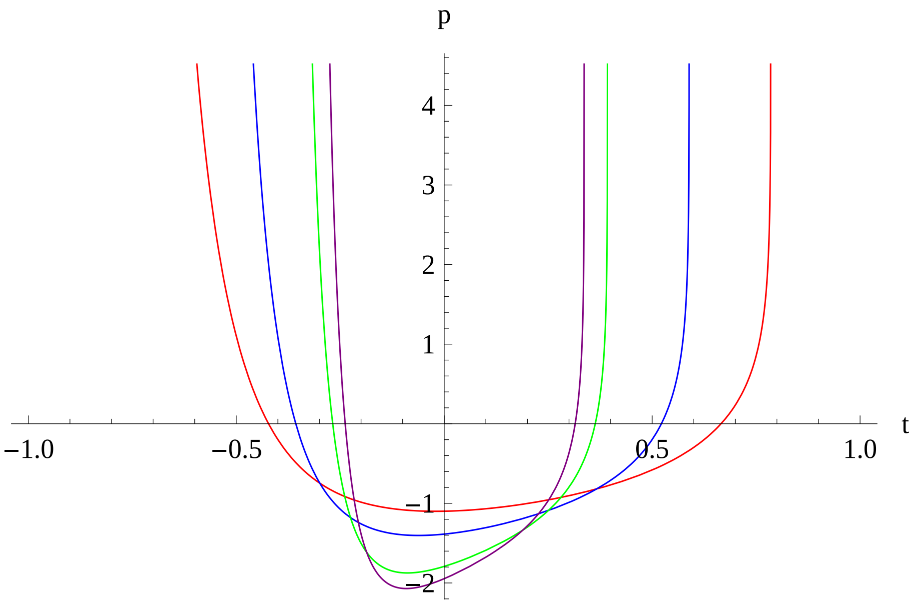

Figure 3.4 : The graph of the conjugate momentum p over t.

Remark 3: The trajectories are open curves representing unbounded motions in Figure 3.4.

Now, we can obtain the following contour plot graph for conjugate momentum. Firstly, we find the argument t in terms of x and $\dot{x}$. Using this relation we can rewrite canonical conjugate momentum p in terms of x and $\dot{x}$ and we obtain this diagram for different values of ω.

A contour plot is a graphical technique for representing a 3-dimensional surface by plotting constant z slices, called contours, on a 2-dimensional format. That is, given a value for z, lines are drawn for connecting the (x, y) coordinates where that z value occurs. The contour plot is an alternative to a 3-D surface plot. The independent variables are usually restricted to a regular grid. An additional variable may be required to specify the z values for drawing the iso-lines. If the function do not form a regular grid, you typically need to perform a 2-D interpolation to form a regular grid. The contour plot is used to answer the question "How does z change as a function of x and y?"

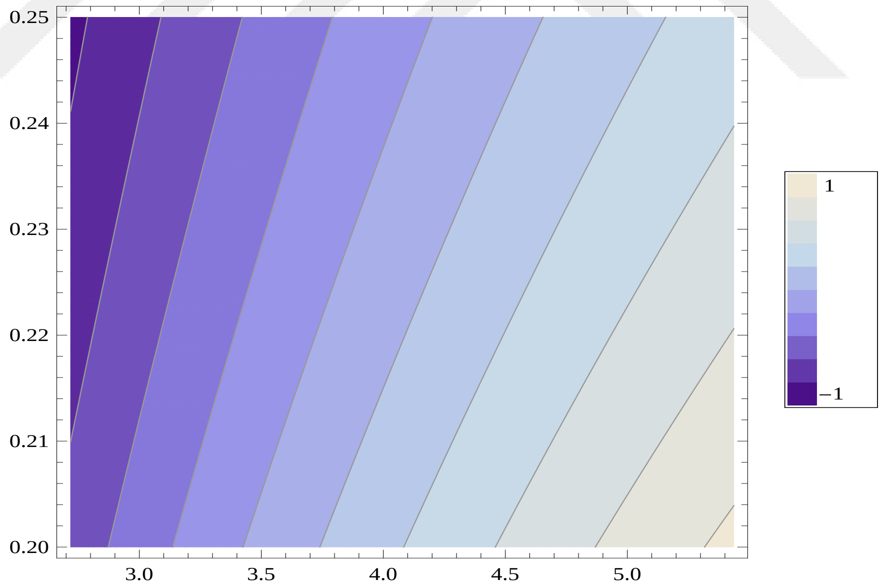

Figure 3.5 : The contour plot for conjugate momentum p.

3.2 The λ-symmetry and the Nonlocal Transformation Pair of the Equation

We can characterize the second-order ordinary differential equation that can be linearized by means of nonlocal transformations. This characterization is given in terms of the coefficients of the equation and determines the second-order ordinary differential equations that admit λ-symmetries. There is a systematic method to find λ-symmetries. These λ-symmetries can be used to reduce order of equation. Second order ordinary differential equation can be integrated by a unified procedure based on λ-symmetries. The equation of the form (1.43) admit $v = \partial_x$ as λ-symmetries for some function λ of the form

$$\lambda(t,\dot{x},\ddot{x}) = \alpha(t,x)\dot{x} + \beta(t,x). \tag{3.21}$$

Proposition 3.2 *We consider an equation (3.1) and S_1, S_2 are the functions defined by (1.47), (1.48). The condition $S_1 = S_2 = 0$ is satisfied if and only if ∂_x is a λ-symmetry of (3.1) for $\lambda = \frac{1}{3}\omega\tan(\frac{1}{3}(-\omega t + 9\omega c_1)) - \frac{2\dot{x}}{3x}$.*

Proof: To obtain the λ-symmetry of the equation, firstly we substitute the function $f(t)$ (3.7) into (1.62) and the following differential equation is found

$$h'(t) + h^2(t) + \frac{\omega^2}{9} = 0. \tag{3.22}$$

From the solution of this differential equation we have

$$h(t) = \frac{1}{3}\omega\tan(\frac{1}{3}(-\omega t + 9\omega c_1)). \tag{3.23}$$

We substitute the function $h(t)$ in (1.63), the function β is found as

$$\beta(t,x) = \frac{1}{3}\omega\tan(\frac{1}{3}(-\omega t + 9\omega c_1)). \tag{3.24}$$

And thus we find the λ-symmetry of the form (1.64)

$$\lambda = \frac{1}{3}\omega\tan(\frac{1}{3}(-\omega t + 9\omega c_1)) - \frac{2\dot{x}}{3x}. \tag{3.25}$$

Proposition 3.3 *The equation (3.1) has a transformation pair F and G then the equation can be linearized. Then the first integral is obtained from this transformation pair.*

Proof: For given equation (3.1), we know $S_1 = 0$ and thus $S_2 = 0$. In this situation, we first obtain the transformation pair F and G. For this purpose, we consider an algorithm to determine nonlocal transformation pair of oscillator equation which is linearizable under the nonlocal transformation. If $\varphi(t)$ is the solution of equation

$$\varphi_t+\varphi^2+\frac{\omega^2}{9}=0, \tag{3.26}$$

And then $f(t)$ is computed in (3.7). Solving (4.9) we obtain $\varphi(t)$ function like this

$$\varphi=\frac{1}{3}\omega\tan(\frac{1}{3}(-\omega t+9\omega)). \tag{3.27}$$

Let $C(t,x)$ be a solution of the following equations

$$C_t=\frac{\omega^2 x}{3}-C\frac{1}{3}\omega\tan(\frac{1}{3}(9\omega-\omega t)), \tag{3.28}$$

$$C_x=-\frac{1}{3}\omega\tan(\frac{1}{3}(9\omega-\omega t))+\frac{2C}{3x}, \tag{3.29}$$

Solving these equations we can obtain

$$C(t,x)=x\omega\tan(\frac{1}{3}(\omega t-9\omega)). \tag{3.30}$$

If $F(t,x)$ is a solution of the equation

$$F_t=CF_x, \tag{3.31}$$

And the following partial differential equation is obtained if we substitute the function $C(t,x)$ in (4.15)

$$F_t-F_x x\omega\tan(\frac{1}{3}(\omega t-9\omega))=0, \tag{3.32}$$

If we solve this partial differential equation, the function $F(t,x)$ is found

$$F(t,x)=\psi(x\sec(\frac{1}{3}(\omega t-9\omega))^3). \tag{3.33}$$

And if we substitute these functions, G is given

$$G=x^{\frac{2}{3}}\sec\frac{1}{3}(\omega t-9\omega))^4\psi'(x\sec(\frac{1}{3}(\omega t-9\omega))^3), \tag{3.34}$$

And thus S-transformation pair F and G are found

$$F=\psi(x\sec(\frac{1}{3}(\omega t-9\omega))^3),\quad G=x^{\frac{2}{3}}\sec\frac{1}{3}(\omega t-9\omega))^4\psi'(x\sec(\frac{1}{3}(\omega t-9\omega))^3). \tag{3.35}$$

And one can integrate the equation using this nonlocal transformation.

Now, we can derive the first integral from transformation pair. Firstly we find the functions $A(t,x)$ and $B(t,x)$ using these equations

$$A(t,x)=\frac{F_x}{G},\quad B(t,x)=\frac{F_t}{G}. \tag{3.36}$$

And $A(t,x)$ and $B(t,x)$ are obtained like this

$$A(t,x)=\frac{\cos(3\omega-\frac{t\omega}{3})}{x^{\frac{2}{3}}},\quad B(t,x)=x^{\frac{1}{3}}\omega\cos(3\omega-\frac{t\omega}{3})\tan(\frac{1}{3}(\omega t-9\omega)). \tag{3.37}$$

And the first integral is found using transformation pair,

$$I=x^{\frac{1}{3}}\omega\cos(3\omega-\frac{t\omega}{3})\tan(\frac{1}{3}(\omega t-9\omega))+\frac{\cos(3\omega-\frac{t\omega}{3})}{x^{\frac{2}{3}}}\dot{x}. \tag{3.38}$$

The solution of equation corresponding to this first integral is found

$$x(t)=\frac{1}{\omega^3}(c_1^3\omega^3\cos(\frac{1}{3}(t-9)\omega)^3+3c_1^2c_2\omega^2\cos(\frac{1}{3}(t-9)\omega)^2\sin(\frac{1}{3}(t-9)\omega))+$$
$$+3c_1c_2^2\omega\cos(\frac{1}{3}(t-9)\omega)\sin(\frac{1}{3}(t-9)\omega)^2+c_2^3\sin(\frac{1}{3}(t-9)\omega)^3). \tag{3.39}$$

And the conjugate momentum is corresponding to this solution is given

$$p=\frac{\cos(\frac{1}{3}(t-9)\omega)\log(\dot{x})}{x^{\frac{2}{3}}}. \tag{3.40}$$

The Lagrangian is obtained

$$L=\frac{-\omega\sin(\frac{1}{3}(t-9)\omega)x+\cos(\frac{1}{3}(t-9)\omega)(\log(\dot{x})-1)\dot{x}}{x^{\frac{2}{3}}}. \tag{3.41}$$

Finally the Hamiltonian function corresponding to conjugate momentum p is

$$H=\frac{\omega\sin(\frac{1}{3}(t-9)\omega)x+\cos(\frac{1}{3}(t-9)\omega)\dot{x}}{x^{\frac{2}{3}}}. \tag{3.42}$$

Hence, one can obtain the graphs of these solutions by Figure 3.6 and Figure 3.7.

Remark 4: The solution (3.39) of the oscillator equation constitute *limit cycle*. The limit cycle begins as circle and, is changing for different choices of ω.

3.3 The Extended Prelle-Singer Method and λ-symmetry Relation

In this section, we consider other types of the first integrals and the exact solutions by using the Prelle-Singer method procedure and its relation with λ-symmetry. This method provides not only the first integrals but also integrating factors. Moreover, one can define the Hamiltonian and Lagrangian forms of the differential equations by using the extended Prelle-Singer method. In this section, we consider the first integrals and exact solutions of the Lienard II-type harmonic nonlinear oscillator equation by the approach related with the Prelle-Singer, λ-symmetry and Lie point symmetry as a different concept from the mathematical point of view.

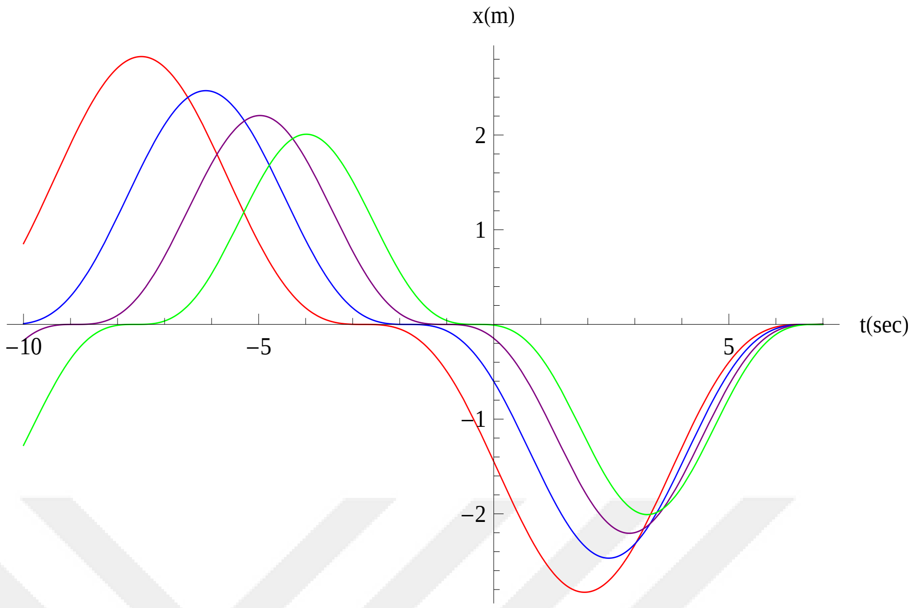

Figure 3.6 : The graph of the position is given in (3.39) over time t for different values of ω.

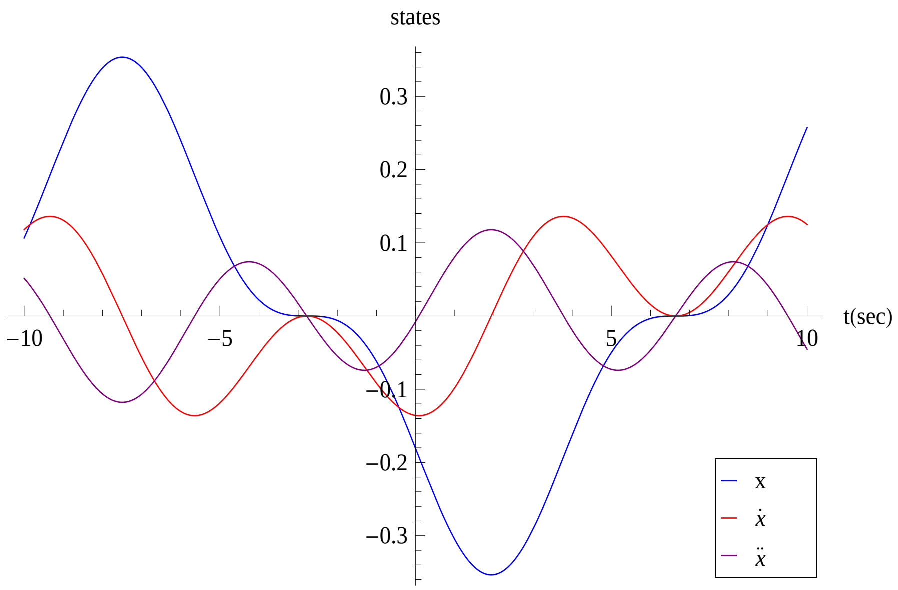

Figure 3.7 : The graphs of the position x is given (3.39), the velocity $\dot{x}$ and the acceleration $\ddot{x}$ over time t for different values of ω.

3.3.1 The time-independent first integrals

For the Lienard II-type oscillator equation (3.1) one can write

$$\phi = \frac{2\dot{x}^2(t)}{3x(t)} - \frac{\omega^2 x(t)}{3} \tag{3.43}$$

If this equation has a first integral $I(t,x,\dot{x}) = C$, with a constant C, then the total differential for the first integral can be written

$$dI = I_t dt + I_x dx + I_{\dot{x}} d\dot{x} = 0. \tag{3.44}$$

Substituting equation (3.44) in the formula $\phi dt - d\dot{x} = 0$ and adding a null term $S(t,x,\dot{x})\dot{x}dt - S(t,x,\dot{x})dx$, we obtain the following relation

$$(\phi + S\dot{x})dt - Sdx - d\dot{x} = 0. \tag{3.45}$$

Multiplying (3.44) by the factor $R(t,x,\dot{x})$ is called the integrating factor, hence we obtain

$$dI = R(\phi + S\dot{x})dt - RSdx - Rd\dot{x} = 0. \tag{3.46}$$

It is clear that equations (3.44) and (3.46) yield the following relations

$$I_t = R(\phi + S\dot{x}), \quad I_x = -RS, \quad I_{\dot{x}} = -R. \tag{3.47}$$

Then using the compatibility conditions, namely $I_{tx} = I_{xt}$, $I_{t\dot{x}} = I_{\dot{x}t}$, $I_{x\dot{x}} = I_{\dot{x}x}$, (3.47) provide us the following system of coupled nonlinear differential equations in terms of S, R and ϕ

$$S_t + \dot{x}S_x + \phi S_{\dot{x}} = -\phi_x + \phi_{\dot{x}}S + S^2, \tag{3.48}$$

$$R_t + \dot{x}R_x + \phi R_{\dot{x}} = -(\phi_{\dot{x}} + S)R, \tag{3.49}$$

$$R_x - SR_{\dot{x}} - RS_{\dot{x}} = 0, \tag{3.50}$$

where the last equation (3.50) is called *compatibility equation*. In addition one can determine the first integral I by using R and S functions with the following relations

$$I = r_1 - r_2 - \int [R + \frac{d}{d\dot{x}}(r_1 - r_2)]d\dot{x}, \tag{3.51}$$

where

$$r_1 = \int R(\phi + \dot{x}s)dt, \quad r_2 = \int (RS + \frac{d}{dx}r_1)dx. \tag{3.52}$$

First of all, we consider the time-independent first integral case, that is $I_t = 0$. One can easily find S from the first equation in (3.47) like this

$$S = \frac{-\phi}{\dot{x}} = \frac{x\omega^2}{3\dot{x}} - \frac{2\dot{x}}{3x}, \tag{3.53}$$

for ϕ (3.43). Substituting this form of S into equation (3.49) we get

$$R(\frac{2\dot{x}^2 + x^2\omega^2}{3x\dot{x}}) + R_{\dot{x}}(\frac{2\dot{x}^2}{3x} - \frac{x\omega^2}{3}) + R_x\dot{x} + R_t = 0. \tag{3.54}$$

The equation (3.54) is a first order linear partial differential equation. To solve this equation we assume R of the form

$$R = \frac{\dot{x}}{(A(x) + B(x)\dot{x} + C(x)\dot{x}^2)^r}, \tag{3.55}$$

where $A(x)$, $B(x)$ and $C(x)$ are functions of x and r is a constant. If we substitute (3.55) into the equation (3.54), then we obtain a set of equations in terms of $\dot{x}$ and its powers. From the solutions of these equations we have

$$A(x) = c_1 x^{\frac{2}{3}+\frac{4}{3r}}\omega^2 + c_3 x^{\frac{4}{3r}}, \tag{3.56}$$

$$B(x) = c_2 x^{-\frac{2(r-2)}{3r}}, \tag{3.57}$$

$$C(x) = c_1 x^{-\frac{4(r-1)}{3r}}, \tag{3.58}$$

where c_1, c_2 and c_3 are arbitrary constants. If we substitute these functions into the equation (3.55) then we find

$$R = \dot{x}\left(c_3 x^{\frac{4}{3r}} + c_1 x^{-\frac{4(r-1)}{3r}}\dot{x}^2 + c_1 x^{\frac{2}{3}+\frac{4}{3r}}\omega^2\right)^{-r}, \tag{3.59}$$

and if we substitute the functions R (3.59) and S (3.53) into the equations (3.48)-(3.50), it is possible to check that these equations are satisfied. Thus, one can determine the first integral of the Lienard II-type equation from the relation (3.51)

$$I = \frac{x^{\frac{4(-1+r)}{3r}}\left(x^{\frac{4}{3}(-1+\frac{1}{r})}(c_3 x^{\frac{4}{3}} + c_1(\dot{x}^2 + x^2\omega^2))\right)^{1-r}}{2c_1(r-1)}, \tag{3.60}$$

and for example $r=-2$, invariant solution of the Lienard II-type equation is

$$x(t) = e^{-i(t+c_4\sqrt{c_1})\omega}\left(e^{\frac{2}{3}i(t+c_4\sqrt{c_1})\omega} - c_3 c_1 \omega^2\right)^3. \tag{3.61}$$

Furthermore, one can determine the corresponding conjugate momentum related with the first integral (3.60),

$$p = -\frac{\frac{c_1^2\dot{x}^5}{5} + \frac{2}{3}c_1 x^{\frac{4}{3}}\dot{x}^3(c_3 + c_1 x^{\frac{2}{3}}\omega^2) + x^{\frac{8}{3}}\dot{x}(c_3 + c_1 x^{\frac{2}{3}}\omega^2)^2}{x^4} \tag{3.62}$$

Then the corresponding Lagrangian is,

$$L = -\frac{\dot{x}\left(\frac{c_1^2\dot{x}^5}{5} + \frac{2}{3}c_1 x^{\frac{4}{3}}\dot{x}^3(c_3 + c_1 x^{\frac{2}{3}}\omega^2) + x^{\frac{8}{3}}\dot{x}(c_3 + c_1 x^{\frac{2}{3}}\omega^2)^2\right)}{x^4} + \frac{(c_3 x^{\frac{4}{3}} + c_1(\dot{x}^2 + x^2\omega^2))^3}{6c_1 x^4} \tag{3.63}$$

And the corresponding Hamiltonian form related with the first integral (3.60),

$$H = -\frac{(c_3 x^{\frac{4}{3}} + c_1(\dot{x}^2 + x^2\omega^2))^3}{6c_1 x^4}. \tag{3.64}$$

Thus, we can examine the relation between Hamiltonian function and the position x with the following contour plot graph.

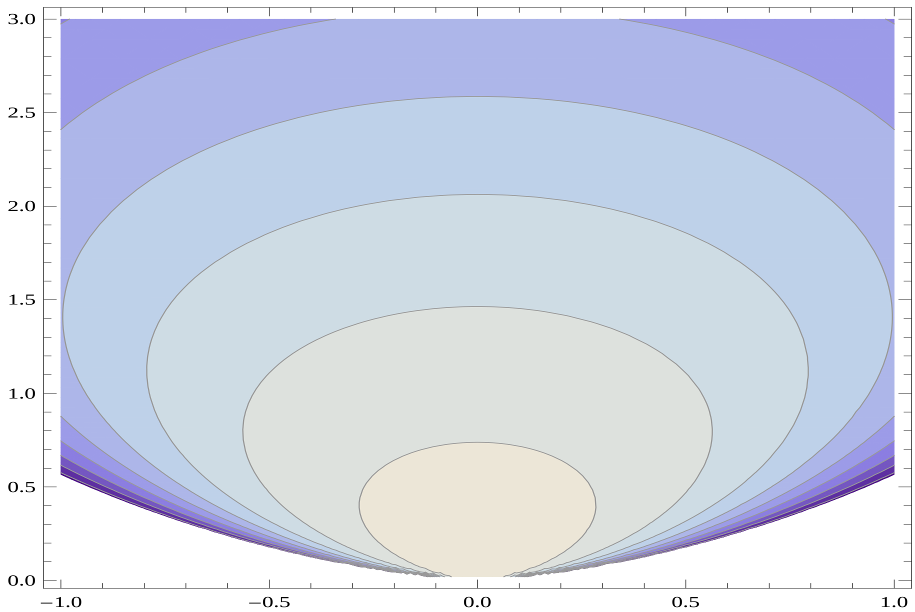

Figure 3.8 : The contour plot of Hamiltonian function in terms of x and $\dot{x}$.

3.3.2 The solution of the nonlinear oscillator harmonic equation using the λ-symmetries based on linearization method

In this section, we examine an another method to investigate symmetries of the nonlinear equations. We construct the first integral directly from λ-symmetry. The procedure essentially involves the following four steps.

1. Find a first integral $w(t,x,\dot{x})$ of $\upsilon^{[\lambda,(1)]}$, that is particular solution of the equation

$$w_x + \lambda w_{\dot{x}} = 0. \tag{3.65}$$

where subscripts denote partial derivative with respect to that variable and $\upsilon^{[\lambda,(1)]}$ is the first order λ-prolongation of the vector field υ.

2. Evaluate $A(w)$ and express $A(w)$ in terms of (t,w) as $A(w) = F(t,w)$ and the operator A is defined the following form

$$A = \partial_t + \dot{x}\partial_x + \phi(t,x,\dot{x})\partial_{\dot{x}}. \tag{3.66}$$

3. Find a first integral G of $\partial_t + F(t,w)\partial_w$.

4. Evaluate $I(t,x,\dot{x}) = G(t,w(t,x,\dot{x}))$. Then $I(t,x,\dot{x})$ is a first integral and $\mu(t,x,\dot{x}) = I_{\dot{x}}$ is an integrating factor of the given second order equation.

Now we introduce a first integral and an exact solution of the nonlinear oscillator harmonic equation by using λ-symmetry (4.8) which is found by linearization method. We first consider the λ-symmetry (4.8) of the nonlinear oscillator harmonic equation (3.1). The null S function can be written

$$S = -\lambda = \frac{1}{3}\omega\tan(\frac{1}{3}(-\omega t + 9\omega c_1)) - \frac{2\dot{x}}{3x}. \tag{3.67}$$

From (3.65) we have

$$w = \frac{\dot{x} - x\omega\tan(3\omega - \frac{t\omega}{3})}{x^{\frac{2}{3}}}. \tag{3.68}$$

Hence one can evaluate $A(\omega)$ as the application of the operator A (3.66) to w (3.68)

$$A(w) = A = \partial_t + \dot{x}\partial_x + (\frac{\sigma x}{1+\sigma x^2}\dot{x}^2 - \frac{\omega^2 x}{1+\sigma x^2})\partial_{\dot{x}}, \tag{3.69}$$

and derive $A(\omega)$ in terms of (t,ω) as $A(\omega) = F(t,\omega)$, that is,

$$F(t,w) = -\frac{1}{6}w\omega\sec(\frac{1}{3}(t-9)\omega)\sin(6\omega - \frac{2t\omega}{3}). \tag{3.70}$$

In the last step one can find a first integral G of $\partial_t + F(t,\omega)\partial_\omega$ from the first order partial differential equation of the form

$$G_t + \left(-\frac{1}{6}w\omega\sec(\frac{1}{3}(t-9)\omega)\sin(6\omega-\frac{2t\omega}{3})\right)G_w = 0. \qquad (3.71)$$

which the solution is

$$G(t,w) = c_1\left(w\cos(\frac{1}{3}(t-9)\omega)\right), \qquad (3.72)$$

where c_1 is an arbitrary constant. Finally, one can express $G(t,w)$ in terms of $(t,x,\dot{x})$ using (3.68) to find the first integral

$$I = \frac{\cos(\frac{1}{3}(t-9)\dot{x} - x\omega\tan(3\omega-\frac{t\omega}{3}))}{x^{\frac{2}{3}}}. \qquad (3.73)$$

The integrating factor can be deduced from the first integral by differentiating it with respect to $\dot{x}$. Thus we find the integrating factor of the form

$$\mu = \frac{\left(\cos(\frac{1}{3}(t-9)\omega)\right)}{x^{\frac{2}{3}}}. \qquad (3.74)$$

And the function R can be written like this

$$R = -\mu = -\frac{\left(\cos(\frac{1}{3}(t-9)\omega)\right)}{x^{\frac{2}{3}}}. \qquad (3.75)$$

It is easy to check again that the functions S and R satisfy equations (3.48)-(3.50). Thus, the different exact solution of the Lienard II- type nonlinear harmonic oscillator equation is

$$x(t) = \frac{(c_1\omega\cos(\frac{1}{3}(t-9)\omega) + c\sin(\frac{1}{3}(t-9)\omega))^3}{\omega^3}, \qquad (3.76)$$

where c is an arbitrary constant. Now, we see the the graph of the corresponding to the equation (3.76) by Figure 3.9. Then, Figure 3.10 shows the rate of $x(m)$ is given by equation (3.76), $\dot{x}(m/sec)$ and $\ddot{x}(m/sec^2)$ depend on time t.

Furthermore, it is possible to show that one can find other forms of the first integrals and the integrating factors rather then the forms given by (3.73) and (3.74) for the same null S function. With this aim, we consider again (3.65) and substitute this form of S into the equation (3.49) to find

$$(R_t + \dot{x}R_x)3x + R(2\dot{x} + 2\omega\tan(\frac{1}{3}(-\omega t + 9\omega c_1))) + R_{\dot{x}}(2\dot{x}^2 - x^2\omega^2) = 0. \qquad (3.77)$$

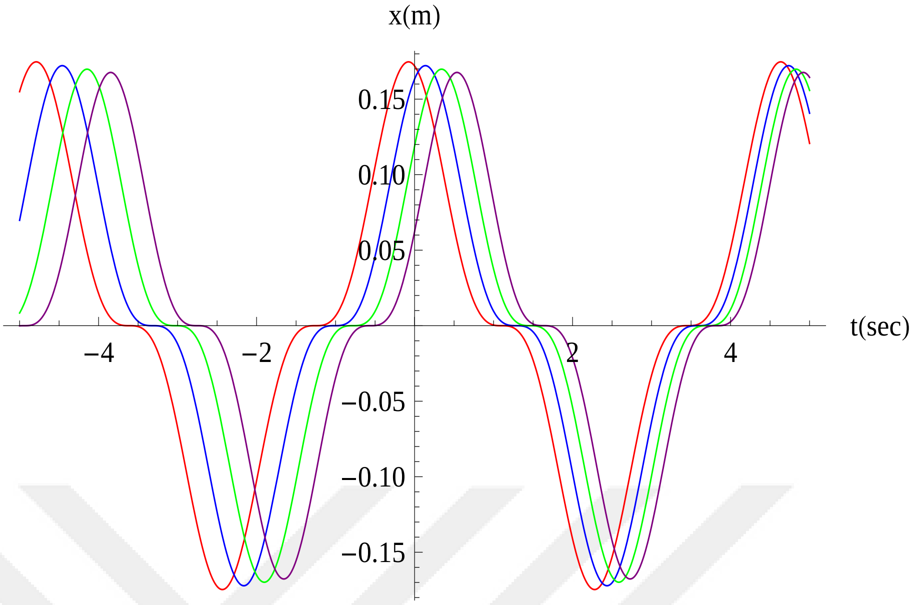

Figure 3.9 : The graph of the position is given in (3.76) versus time t.

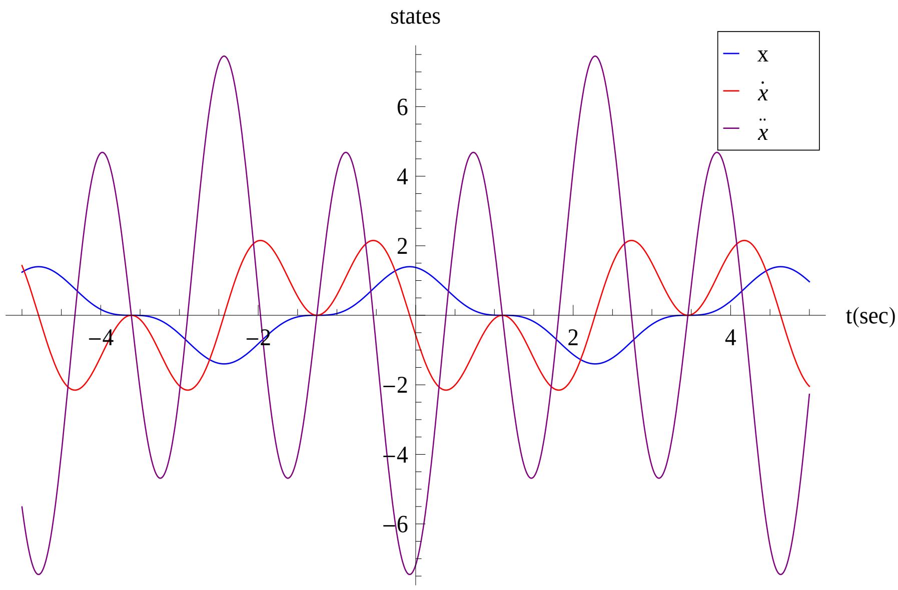

Figure 3.10 : The graphs of the position is given by equation (3.76), the velocity and the acceleration versus time t.

The equation (3.77) is a first order linear partial differential equation in terms of $R = R(t,x,\dot{x})$ and it is known that any particular solution is sufficient to construct an integral motion. For this purpose, to seek a particular solution for R one can make a suitable ansatz instead of looking for the general solution by assuming R to be of the form

$$R = \frac{3x}{(A(t,x)+B(t,x)\dot{x})^r}, \tag{3.78}$$

where A and B are functions of their arguments and r is a constant, which are all to be determined. The denominator of the function S should be numerator of the function R. Since the denominator of S is $3x$, we fix a numerator of R as $3x$. Then substituting (3.78) into (3.77) yields

$$-3(A(t,x)(-5\dot{x}+x\omega\tan(3\omega-\frac{t\omega}{3}))+B(t,x)((-5+2r)\dot{x}^2-rx^2\omega^2+x\dot{x}\omega\tan(3\omega-\frac{t\omega}{3}))+ \\ 3rx(A_t(t,x)+\dot{x}(B_t(t,x)+A_x(t,x)+\dot{x}B_x(t,x)))) = 0. \tag{3.79}$$

From the solutions of $A(x,t)$ and $B(x,t)$ the integrating factor R using (3.51), for example, r=-1, is written

$$R = -\mu = 3\cos(\frac{1}{3}(t-9)\omega)(c_1x^{\frac{2}{3}}+c_2\dot{x}\cos(\frac{1}{3}(t-9)\omega)-c_2x\omega\sin(3\omega-\frac{t\omega}{3})), \tag{3.80}$$

and the corresponding time-dependent first integral is

$$I = -\frac{3}{4x^{\frac{4}{3}}}(c_2\dot{x}^2+c_2x^2\omega^2+c_2(\dot{x}^2-x^2)\cos(\frac{2}{3}(t-9)\omega)-4c_1x^{\frac{5}{3}}\omega\sin(3\omega-\frac{t\omega}{3})+ \\ 4\cos(\frac{1}{3}(t-9)\omega)(c_1x^{\frac{2}{3}}\dot{x}-c_2x\dot{x}\omega\sin(3\omega-\frac{t\omega}{3})), \tag{3.81}$$

where c_1 and c_2 are arbitrary constants. But it is clear that it is not easy to find an explicit solution for (3.81). Then, one can obtain the Hamiltonian function corresponding to the first integral (3.81), the conjugate momentum is given

$$p = -\frac{3\cos(\frac{1}{3}(t-9)\omega)(x^{\frac{2}{3}}\log(\dot{x}))(c_1-c_2x^{\frac{1}{3}}\omega\sin(3\omega-\frac{t\omega}{3})+c_2\cos(\frac{1}{3}(t-9)\omega)\dot{x}}{x^{\frac{4}{3}}}. \tag{3.82}$$

The corresponding Lagrangian is

$$L = \frac{1}{2x^{\frac{4}{3}}}\Big(3c_2x^2\omega^2\sin(\frac{1}{3}(t-9)\omega)^2-6c_1x^{\frac{5}{3}}\omega\sin(3\omega-\frac{t\omega}{3})+3x^{\frac{2}{3}}(\log(\dot{x})-1)$$

$$(-2c_1\cos(\frac{1}{3}(t-9)\omega)+c_2x^{\frac{1}{3}}\omega\sin(6\omega-\frac{2t\omega}{3})\dot{x}-3c_2\cos(\frac{1}{3}(t-9)\omega)^2\dot{x}^2\Big), \quad (3.83)$$

and the Hamiltonian is

$$H=-\frac{3}{2x^{\frac{4}{3}}}\Big(c_2x^2\omega^2\sin(\frac{1}{3}(t-9)\omega)^2-2c_1x^{\frac{5}{3}}\omega\sin(3\omega-\frac{t\omega}{3})$$

$$+\dot{x}(2c_1x^{\frac{2}{3}}\cos(\frac{1}{3}(t-9)\omega)-c_2x\omega\sin(6\omega-\frac{2t\omega}{3})+c_2\cos(\frac{1}{3}(t-9)\omega)^2\dot{x}^2)\Big). \quad (3.84)$$

4. LINEARIZATION PROPERTIES, FIRST INTEGRALS, NONLOCAL TRANSFORMATION FOR HEAT TRANSFER EQUATION [1]

In this section, we examine fin equation belongs to this class of the equation

$$\ddot{x}+a_2(t,x)\dot{x}^2+a_1(t,x)\dot{x}+a_0(t,x)=0. \tag{4.1}$$

Fin is used in a large number of applications to increase the heat transfer from surfaces. Interest has been instilled by frequent encounters of fin problems in many engineering applications to enhance heat transfer. Typically, the fin material has a high thermal conductivity. The fin is exposed to a flowing fluid, which cools or heats it, with the high thermal conductivity allowing increased heat being conducted from the wall through the fin. The design of temperature reduction fin is encountered in many situations and we thus examine heat transfer in a fin as a way of defining some criteria for design.

To obtain first integral, integrating factor and invariant solution, it is possible to consider some feasible algorithm and one can apply this algorithm to nonlinear fin equation that is the form (4.1). The another method for application to nonlinear differential equation is transformation method. Considering this transformation procedure, a nonlinear equation can be converted to a linear second order ordinary differential equation whose solutions are known. It is well-known that Lie [10] proves the general algorithm that all second order nonlinear differential equation can be converted to linear differential equations by the method of change of variables, which is called Lie linearization test. In fact, the mathematical procedure of linearizing transformation is quite diffucult work and this can be applied to only second order ordinary differential equations that have a eight-dimensional Lie algebra. Therefore, it is necessary to consider other type of transformation techniques of nonlinear differential equations for linearization of larger classes of equations. In recent years,

[1] This chapter is based on the paper Orhan Ö. and Özer T., Linerization properties, first integrals, nonlocal transformation for heat transfer equation. International Journal of Modern Physics B, 2016, 30, 1640024, doi:10.1142/S0217979216400245

some studies on the linearization through transformation involving nonlocal terms has been carried out [30,31]. Then we apply Sundman transformation to fin equation.

4.1 The First Integral of the Form $A(t,x)\dot{x}+B(t,x)$ and Integrating Factor of Fin Equation

We now consider the nonlinear fin equation,

$$\ddot{x}+\frac{K'(x)}{K(x)}\dot{x}^2-\frac{H(x)}{K(x)}=0, \tag{4.2}$$

where $K(x)$ and $H(x)$ are thermal conductivity and heat transfer coefficient, respectively, which are considered as functions of temperature, and $x=x(t)$ is the temperature function and t is dimensional spatial variable. The Noether symmetries of Eq. (4.2) is investigated and obtained first integrals corresponding to Noether symmetries in [2].

Proposition 4.1 *If $S_1=S_2=0$, then there is the following relation between $K(x)$ and $H(x)$*

$$K(x)=-\frac{H'(x)}{\sigma}, \tag{4.3}$$

where σ is a constant.

Proof: From the Eq. (4.2), we have

$$a_2(t,x)=\frac{K'(x)}{K(x)},\quad a_1(t,x)=0,\quad a_0(t,x)=-\frac{H(x)}{K(x)}. \tag{4.4}$$

Using these coefficients, we obtain $S_1=0$, which is given by (1.47). Thus, we know from Theorem 1 that S_2 must be zero if $S_1=0$. Now we obtain the relation the functions $K(x)$ and $H(x)$ using this knowledge. The function S_2 is

$$S_2=\left(-\frac{H(x)}{K(x)}\frac{K'(x)}{K(x)}-\left(\frac{H'(x)K(x)-H(x)K'(x)}{K(x)^2}\right)\right)_x. \tag{4.5}$$

By simplifying (4.5) one can have

$$S_2=\left(\frac{-H'(x)}{K(x)}\right)_x, \tag{4.6}$$

Since S_2 must be zero for $S_1=0$, then we write

$$S_2=\left(\frac{-H'(x)}{K(x)}\right)_x=0. \tag{4.7}$$

The integration of (4.7) gives

$$S_2 = \left(\frac{-H'(x)}{K(x)}\right) = \sigma, \tag{4.8}$$

thus we have the following relation

$$K(x) = -\frac{H'(x)}{\sigma}, \tag{4.9}$$

where σ is a constant.

4.1.1 The first integrals of the form $A(t,x)\dot{x} + B(t,x)$ and the invariant solutions

The fin equation has the first integrals of the form $A(t,x)\dot{x} + B(t,x)$, we can calculate the functions A and B using a following procedure for the equation. Then, the equation can be integrated by these first integrals and solutions of the equation can be obtained using these first integrals.

Proposition 4.2 *The Eq. (4.2) has the first integral of the following form*

$$I = \frac{(c_2\cos(\sqrt{\sigma}t) - c_1\sin(\sqrt{\sigma}t))H(x) + c_3\sqrt{\sigma}}{\sqrt{\sigma}} - \frac{(c_1\cos(\sqrt{\sigma}t) + c_2\sin(\sqrt{\sigma}t)H'(x)\dot{x}}{\sigma}, \tag{4.10}$$

where c_1, c_2, c_3, σ are arbitrary constants.

Proof: Let $P = P(t,x)$ be a function such that

$$P_t = 0, \quad P_x = \frac{K'(x)}{K(x)}. \tag{4.11}$$

Using (4.11), we obtain function $P = P(t,x)$ like this

$$P(x) = logK(x). \tag{4.12}$$

We know $K(x) = \frac{-H'(x)}{\sigma}$ from (4.9) and substituting this relation into (4.5), we obtain function $P = P(t,x)$

$$P(x) = log\left(\frac{-H'(x)}{\sigma}\right). \tag{4.13}$$

Using the formula (1.49), we obtain

$$f(t) = -\frac{H'(x)}{K(x)}, \tag{4.14}$$

and by using a relation $K(x) = \frac{-H'(x)}{\sigma}$ in (4.14) we have

$$f(t) = \sigma, \tag{4.15}$$

where σ is a constant. If we substitute (4.15) in Eq. (1.51), we obtain the following equation

$$g''(t) + \sigma g(t) = 0. \tag{4.16}$$

The solution of the equation (4.16) is

$$g(t) = c_1 \cos(\sqrt{\sigma} t) + c_2 \sin(\sqrt{\sigma} t). \tag{4.17}$$

If we substitute $P(x)$ and $g(t)$ functions into (1.58), then one can write the system of equations

$$Q_t = H(x)(c_1 \cos(\sqrt{\sigma} t) + c_2 \sin(\sqrt{\sigma} t)), \tag{4.18}$$

$$Q_x = -\frac{(-\sqrt{\sigma} c_2 \cos(\sqrt{\sigma} t) + \sqrt{\sigma} c_1 \sin(\sqrt{\sigma} t))H'(x)}{\sigma}, \tag{4.19}$$

which gives the solution

$$Q(t,x) = \frac{(c_2 \cos(\sqrt{\sigma} t) - c_1 \sin(\sqrt{\sigma} t))H(x) + c_3 \sigma}{\sqrt{\sigma}}. \tag{4.20}$$

As a result the functions $A(t,x)$ and $B(t,x)$

$$A(t,x) = -\frac{(c_1 \cos(\sqrt{\sigma} t) + c_2 \sin(\sqrt{\sigma} t))H'(x)}{\sigma}, \tag{4.21}$$

$$B(t,x) = \frac{(c_2 \cos(\sqrt{\sigma} t) - c_1 \sin(\sqrt{\sigma} t))H(x) + c_3 \sigma}{\sigma}. \tag{4.22}$$

are found.

Proposition 4.3 *$I = A\dot{x} + B$ is a first integral of (4.2). In this case, the function A is an integrating factor, thus an integrating factor of (4.2) is obtained*

$$\mu = -\frac{(c_1 \cos(\sqrt{\sigma} t) + c_2 \sin(\sqrt{\sigma} t))H'(x)}{\sigma}. \tag{4.23}$$

Now, using the equations (4.10) and (4.23) one can classify first integrals and integrating factor of the nonlinear fin equation (4.2) based on different forms of heat transfer coefficient $H(x)$.

Case 1: Firstly we take $H(x) = x$, we find the first integral of (4.2) for the function $H(x)$,

$$I = \frac{(c_2 \cos(\sqrt{\sigma} t) - c_1 \sin(\sqrt{\sigma} t))x + c_3 \sqrt{\sigma}}{\sqrt{\sigma}} - \frac{(c_1 \cos(\sqrt{\sigma} t) + c_2 \sin(\sqrt{\sigma} t))\dot{x}}{\sigma}. \tag{4.24}$$

Some group invariant solutions of a nonlinear fin equation can be constructed from the first integrals. Now, we consider the first integral of special cases $H(x) = x$ to present the invariant solutions of (4.2).

For the case $H(x) = x$, the first conservation law is found (4.24) then the expression $D_t I = 0$ gives the following invariant solution of fin equation (4.2),

$$x(t) = \frac{\sqrt{\sigma}(-c_5 + c_3 \sin(\sqrt{\sigma}t))}{c_1} + c_4(c_1 \cos(\sqrt{\sigma}t) + c_2 \sin(\sqrt{\sigma}t)), \tag{4.25}$$

where c_1, c_2, c_3, c_4, c_5 and σ are constants.

If we take $H(x) = x$, we find the integrating factor of (4.2)

$$\mu = -\frac{c_1 \cos(\sqrt{\sigma}t) + c_2 \sin(\sqrt{\sigma}t)}{\sigma}. \tag{4.26}$$

Case 2: We now take $H(x) = e^x$ and we find the first integral of (4.2) for $H(x)$ functions,

$$I = \frac{e^x(c_2 \cos(\sqrt{\sigma}t) - c_1 \sin(\sqrt{\sigma}t)) + c_3\sqrt{\sigma}}{\sqrt{\sigma}} - \frac{e^x(c_1 \cos(\sqrt{\sigma}t) + c_2 \sin(\sqrt{\sigma}t))\dot{x}}{\sigma}. \tag{4.27}$$

Then the invariant solution of fin equation corresponding to first integral (4.27) is

$$x(t) = Log(\frac{c_1^2 c_4 \cos(\sqrt{\sigma}t) + (\sqrt{\sigma}(-c_5 + c_3) + c_1 c_2 c_4) \sin(\sqrt{\sigma}t)}{c_1}) \tag{4.28}$$

and the integrating factor of (4.2) for $H(x) = e^x$ is,

$$\mu = -\frac{(c_1 \cos(\sqrt{\sigma}t) + c_2 \sin(\sqrt{\sigma}t))e^x}{\sigma}. \tag{4.29}$$

Case 3: $H(x) = \frac{1}{mx+n}$; m, n are constants. For this case, the first integral is

$$I = \frac{(c_2 \cos(\sqrt{\sigma}t) - c_1 \sin(\sqrt{\sigma}t)) + c_3\sqrt{\sigma}(n+mx)}{\sqrt{\sigma}(n+mx)} + \frac{m(c_1 \cos(\sqrt{\sigma}t) + c_2 \sin(\sqrt{\sigma}t)\dot{x}}{\sigma(n+mx)^2} \tag{4.30}$$

and the invariant solution that corresponding to (4.30) is

$$x(t) = \frac{c_1^2 c_4 \cos(\sqrt{\sigma}t) + (\sqrt{\sigma}(-c_5 + c_3) + c_1 c_2 c_4) \sin(\sqrt{\sigma}t) + \cos(\sqrt{\sigma}t) - c_1 \sin(\sqrt{\sigma}t))}{c_1} \tag{4.31}$$

where c_1, c_2, c_3, c_4, c_5 and σ are constants and the integrating factor is

$$\mu = -\frac{m(c_1 \cos(\sqrt{\sigma}t) + c_2 \sin(\sqrt{\sigma}t))}{\sigma(mx+n)^2}. \tag{4.32}$$

Case 4: $H(x) = \frac{h}{(\beta x+\gamma)^2}$; h, β and γ are arbitrary constants.

$$I = \frac{h(c_2\cos(\sqrt{\sigma}t) - c_1\sin(\sqrt{\sigma}t)) + c_3\sqrt{\sigma}(x\beta+\gamma)^2}{\sqrt{\sigma}(x\beta+\gamma)^2} + \frac{2h\beta(c_1\cos(\sqrt{\sigma}t) + c_2\sin(\sqrt{\sigma}t)\dot{x}}{\sigma(x\beta+\gamma)^3} \tag{4.33}$$

and the invariant solution is

$$x(t) = -\frac{\sigma c_5\beta\gamma - \sigma\beta\gamma c_3}{\sigma c_5\beta^2 - \sigma\beta^2 c_3} - \frac{1}{c_1\cos(\sqrt{\sigma}t) + c_2\sin(\sqrt{\sigma}t)c_4 - \frac{\sigma\beta^2(c_5-c_3)\sin(\sqrt{\sigma}t)}{hc_1\cos(\sqrt{\sigma}t)+c_2\sin(\sqrt{\sigma}t)}} \tag{4.34}$$

and the integrating factor is

$$\mu = -\frac{2h\beta(c_1\cos(\sqrt{\sigma}t) + c_2\sin(\sqrt{\sigma}t))}{\sigma(\beta x+\gamma)^3}. \tag{4.35}$$

Case 5: $H(x)$ is a general power law. In this case, we have $H(x) = hx^{\beta}$, $\beta \neq -1$. The choice of $H(x) = hx^{\beta}$ yields

$$I = \frac{hx^{\beta}(c_2\cos(\sqrt{\sigma}t) - c_1\sin(\sqrt{\sigma}t)) + c_3\sqrt{\sigma}}{\sqrt{\sigma}} - \frac{hx^{\beta-1}\beta(c_1\cos(\sqrt{\sigma}t) + c_2\sin(\sqrt{\sigma}t)\dot{x}}{\sigma} \tag{4.36}$$

and the integrating factor of (4.2) corresponding to this choose is given by

$$\mu = -\frac{hx^{-1+\beta}\beta(c_1\cos(\sqrt{\sigma}t) + c_2\sin(\sqrt{\sigma}t))}{\sigma}. \tag{4.37}$$

Case 6: $H(x)$ is a general power law. In this case, we have $H(x) = hx^{\beta}$, $\beta = -1$. We obtain the first integral

$$I = \frac{hx^{-\beta}(c_2\cos(\sqrt{\sigma}t) - c_1\sin(\sqrt{\sigma}t)) + c_3\sqrt{\sigma}}{\sqrt{\sigma}} + \frac{hx^{-1-\beta}\beta(c_1\cos(\sqrt{\sigma}t) + c_2\sin(\sqrt{\sigma}t)\dot{x}}{\sigma} \tag{4.38}$$

and by integration of (4.38) we find the group invariant solution in the following form

$$x(t) = \left(\frac{hc_1^2c_4\cos(\sqrt{\sigma}t) + (\sqrt{\sigma}(-c_5+c_3) + hc_1c_2c_4)\sin(\sqrt{\sigma}t)}{hc_1}\right)^{\frac{-1}{\beta}}. \tag{4.39}$$

The integrating factor is

$$\mu = -\frac{hx^{-1-\beta}\beta(c_1\cos(\sqrt{\sigma}t) + c_2\sin(\sqrt{\sigma}t))}{\sigma}. \tag{4.40}$$

where c_1, c_2, c_3, c_4, σ are arbitrary constants.

Now, one can see the graph of the solution using phase plane method in Figure 4.1 and Figure 4.2.

The phase plane method refers to graphically determining the existence of limit cycles in the solutions of the equations. The solutions to the nonlinear differential equation are a family of functions. Graphically, this can be plotted in the phase plane like a two-dimensional vector field. Vectors representing the derivatives of the points with respect to a parameter time t at representative points are drawn. With enough of these arrows in place the system behavior over the regions of plane in analysis can be visualized and limit cycles can be identified. A phase portrait graph of a system depicts the system's trajectories and stable steady states and unstable steady states in a state space. The axes are of state variables. In this case we deal with the rate of heat transfer $\dot{x}$ and the rate of change of heat transfer denoted $\ddot{x}$ as states of the equation.

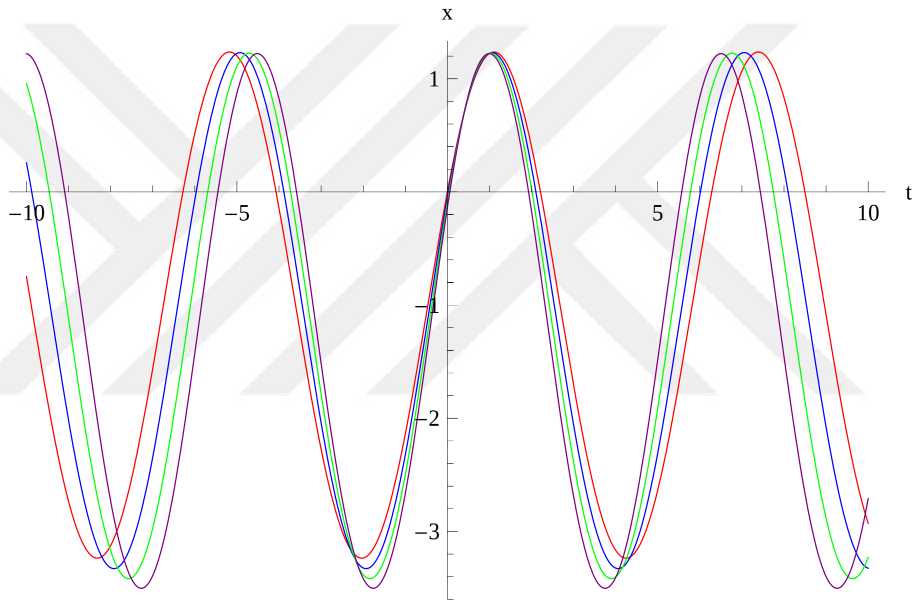

Figure 4.1 : The graph of the equation (4.25) for case 1 by different values of σ.

Remark 1: Red curve is found for σ=1, blue curve is found for σ=1.1, green curve is sketched for σ=1.2 and purple curve is denoted for σ=1.3. In the Figure 4.1 can be seen that the solution of the equation constitutes *limit cycle* in the phase plane. The limit cycle begins as circle and, with varying σ, becomes increasingly sharp.

Remark 2: The blue line shows the solution x over time, the red line shows the rate of heat transfer that is $\dot{x}$ and the purple line shows the rate of change of heat transfer over time in Figure 4.2. These are the three states of the system, simulated over time.

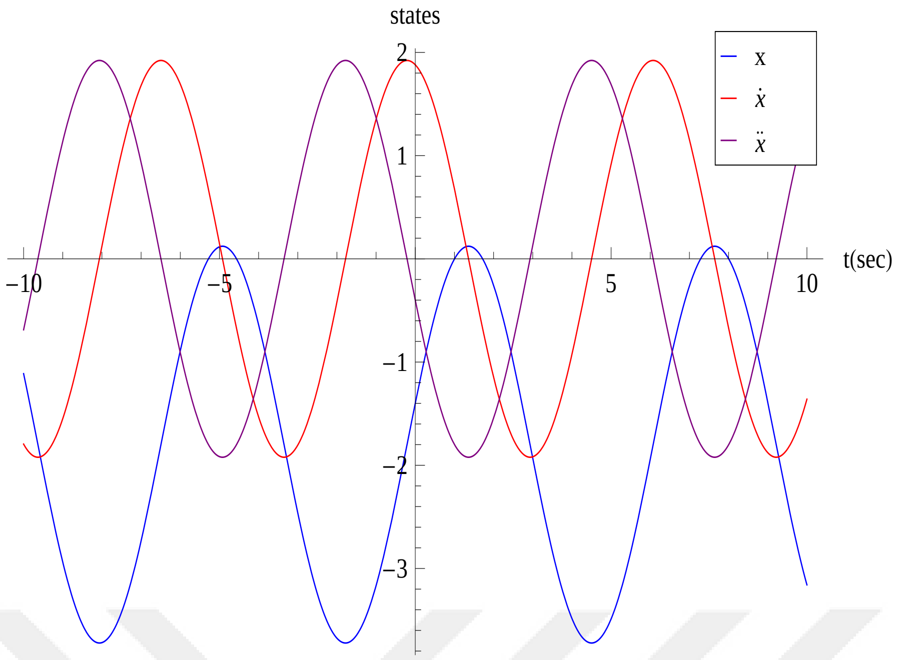

Figure 4.2 : The graph of the equation (4.25) of for values x, $\dot{x}$ and $\ddot{x}$ of case 1.

4.2 Nonlocal Transformation Pair of Fin Equation

The second order nonlinear differential equation can be linearized by the nonlocal transformation.

Proposition 4.4 *The fin equation has the following transformation pair F and G and then this transformation pair can linearize the fin equation.*

Proof: For given equation (4.2), it is known that $S_1 = 0$ and thus $S_2 = 0$. In this situation, we use the following procedure to obtain the transformation pair F and G. If one substitutes $f(t) = \sigma$ is into (1.70), then the following differential equation is obtained,

$$\omega_t + \omega^2 + \sigma = 0, \tag{4.41}$$

And from solution of (4.41), we find

$$\omega(t) = \sqrt{\sigma}\cot(t\sqrt{\sigma}). \tag{4.42}$$

And the function $C(t,x)$ must satisfy the following systems

$$C_t(t,x) = -\sqrt{\sigma}\cot(t\sqrt{\sigma})C(t,x) + \frac{\sigma H(x)}{H'(x)}, \tag{4.43}$$

$$C_x(t,x) = -\sqrt{\sigma}\cot(t\sqrt{\sigma}) - \frac{C(t,x)H''(x)}{H'(x)}. \tag{4.44}$$

From (4.43) we have

$$C(t,x) = \csc(t\sqrt{\sigma}) - \frac{\sqrt{\sigma}\cot(t\sqrt{\sigma})H(x)}{H'(x)}. \tag{4.45}$$

Since the Eq. (4.45) should satisfy the Eq. (4.44), the function $H(x)$ must be the following form

$$H(x) = c_1 + c_2 x. \tag{4.46}$$

By (1.73), $F(t,x)$ is found

$$F(t,x) = \varphi\Big(\frac{1}{c_2\sqrt{\sigma}}(-c_2\cot(t\sqrt{\sigma}) + c_1\sqrt{\sigma}\csc(t\sqrt{\sigma})) + c_2 x\sqrt{\sigma}\csc(t\sqrt{\sigma})\Big). \tag{4.47}$$

And $G(t,x)$ would be determined by

$$G(t,x) = \frac{1}{c_2}\csc(t\sqrt{\sigma})^2\varphi'\Big(-\frac{\cot(t\sqrt{\sigma})}{\sqrt{\sigma}} + \frac{(c_1+c_2x)\csc(t\sqrt{\sigma})}{c_2}\Big). \tag{4.48}$$

The pair F and G linearizes Eq. (4.2) by means of the Sundman transformation.

Now we obtain the first integrals of the form $A\dot{x}+B$ of Eq. (4.2) by using this transformation pair.

The functions A and B are obtained by

$$A = c_2\sin(t\sqrt{\sigma}), \tag{4.49}$$

and

$$B = c_2 - (c_1+c_2x)\sqrt{\sigma}\cos(t\sqrt{\sigma}), \tag{4.50}$$

and the first integral can be obtained in the following form

$$I = c_2\sin(t\sqrt{\sigma})\dot{x} + c_2 - (c_1+c_2x)\sqrt{\sigma}\cos(t\sqrt{\sigma}). \tag{4.51}$$

The corresponding invariant solution is

$$x(t) = c_3\sin(t\sqrt{\sigma}) + \frac{1}{c_2}\Big(\frac{c_2\cot(t\sqrt{\sigma})}{\sqrt{\sigma}} - c_1\csc(t\sqrt{\sigma})\Big)\sin(t\sqrt{\sigma}), \tag{4.52}$$

where c_1, c_2, c_3, σ are constants.

Now, one can see the graph of the solution.

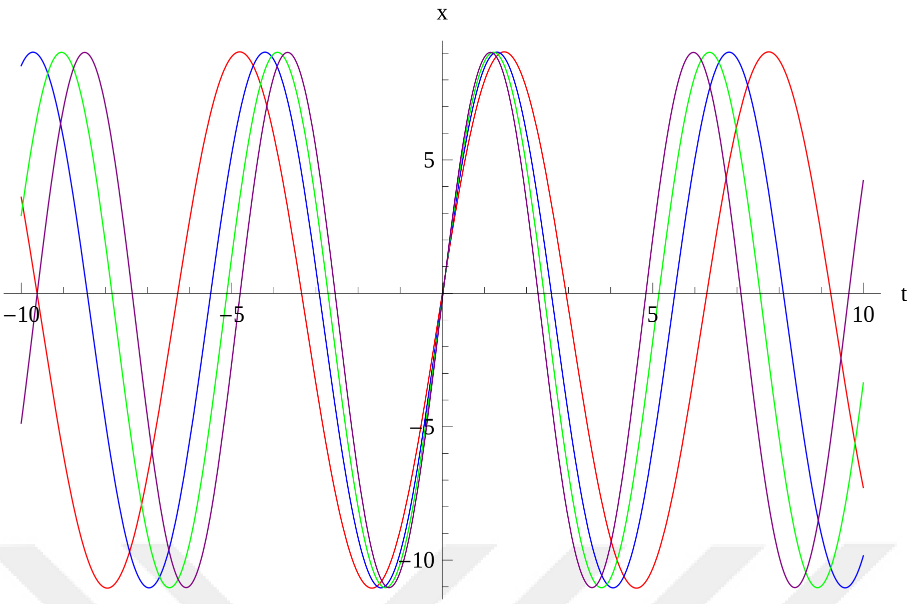

Figure 4.3 : The position-time graph of (4.52) for different values of σ.

5. CONCLUSIONS AND RECOMMENDATIONS

In this thesis, the first problem is to analyze Noether symmetry group classification of nonlinear fin equation, which is second order nonlinear ordinary differential equation. Here, we consider thermal conductivity and heat transfer coefficient as variable functions of temperature, and the nonlinear fin equation is considered in a one-dimensional model describing heat transfer in rectangular fins. From the mathematical point of view, it can be said that this problem is highly nonlinear. Here, we consider to apply partial Lagrangian approach for the classification in this problem. For different heat transfer coefficient and thermal conductivity functions we obtain Noether point symmetry algebras. Finally, we find the corresponding new first integrals for each case, the results are presented in a table and for each case some invariant solutions are obtained from the first integrals (conserved forms). This study can be considered as one of the first studies on Noether symmetry classification of differential equations in the literature. In addition, it is important to mention that λ-symmetry method is another new approach to find first integrals for differential equations.

In this first problem, one dimensional heat transfer of nonlinear fin with temperature dependent both thermal conductivity and heat transfer coefficient investigated with some methods. In this study we analyze first integrals, integrating factor and nonlocal transformation pair of fin equation, which is second order nonlinear ordinary differential equation. Here, we consider thermal conductivity and heat transfer coefficient as variable functions of temperature and the nonlinear fin equation is considered in a one-dimensional model describing heat transfer in rectangular fins. For different heat transfer coefficient and thermal conductivity functions we obtain first integrals, integrating factor and nonlocal transformation pair. Finally, we find the corresponding first integrals for each case.

The second problem is that the Lienard II-type nonlinear harmonic oscillator equation has a natural generalization in three dimensions and these systems can be also quantized exhibiting many interesting features and can be interpreted as an oscillator constrained to move on a three-sphere. As such the considered problem is highly nonlinear. In this problem, we analyze the first integral of the form $A(t,x)\dot{x}+B(t,x)$, the λ-symmetries and the integrating factors of the Lienard II-type nonlinear harmonic oscillator equation, which is second order nonlinear ordinary differential equation.

We have characterized the second order nonlinear ordinary differential equations and this characterization is given by the coefficients of the equation and also determines the first integral, the λ-symmetry and the integrating factor. Thus, the Lienard II-type nonlinear harmonic oscillator equation is classified by using functions S_1 and S_2 and the first integral of the form $A(t,x)\dot{x}+B(t,x)$ is obtained by an algorithm. Moreover, it is presented some properties and characterization of the equation that admits a vector field as λ-symmetry. Linearization, the symmetries and the transformation of equations play a crucial role. Furthermore, the nonlinear second order ordinary differential equations can be linearized by Sundman transformation. Finally, we apply Sundman transformation to Lienard II-type nonlinear harmonic oscillator equation.

Then, we have identified the time independent first integrals for the Lienard II-type nonlinear harmonic oscillator equation using the modified Prelle-Singer approach. Moreover, we have constructed the appropriate functions Lagrangian and Hamiltonian from the time independent first integrals and transformed the corresponding Hamiltonian forms to standart Hamiltonian forms. The important point of the Prelle-Singer procedure lies in finding the explicit solutions satisfying all three determining equations (3.48)-(3.50). In our study, we have taken specific ansatz forms to determine the null forms S, and the integrating factor R. Finally, from our detailed analysis we have shown these results with the phase portraits depending on the choice of parameters and using these phase portraits we interpret geometric meanings of the solutions. And using the Hamiltonian and the conjugate momentum function we demonstrate relation between solution and Hamiltonian and conjugate momentum by contour plot portrait.

The third problem is linearization methods for fin equation which is one dimensional heat transfer of nonlinear fin with temperature dependent both thermal conductivity and heat transfer coefficients. In this problem, we analyze first integrals, integrating factor and nonlocal transformation pair of fin equation, which is second order nonlinear ordinary differential equation. Here, we consider thermal conductivity and heat transfer coefficients as variable functions of temperature and the nonlinear fin equation is considered in a one-dimensional model describing heat transfer in rectangular fins. From the mathematical point of view, it can be said that this problem highly nonlinear. For different heat transfer coefficient and thermal conductivity functions we obtain first integrals, integrating factor and nonlocal transformation pair using linearization metods. Finally, we find the corresponding first integrals for each case for this problem.

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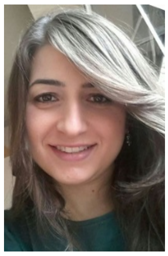

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