ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE
ENGINEERING AND TECHNOLOGY

THE ROLES OF MATRIX NORMS IN THE GAME THEORY

M.Sc. THESIS

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Mathematical Engineering Programme

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MATRİS NORMLARININ OYUN TEORİSİNDEKİ ROLLERİ

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To my father, my mother and my brother

FOREWORD

I would like to thank my supervisor for his excellent guidence, support and patience during this process. I also would like to thank my family for their supportive words and consideration.

Arpil 2018 Murat ÖZKAYA

TABLE OF CONTENTS

Page

FOREWORD ..... **ix**
TABLE OF CONTENTS ..... **xi**
ABBREVIATIONS ..... **xiii**
SYMBOLS ..... **xv**
LIST OF TABLES ..... **xvii**
LIST OF FIGURES ..... **xix**
SUMMARY ..... **xxi**
ÖZET ..... **xxiii**
1. INTRODUCTION **1**
1.1 History of Matrices 1
1.2 2-Dimensional Matrices 4
1.3 2-Dimensional Matrix Norms 6
2. 3-DIMENSIONAL MATRICES **9**
2.1 Basic Principals of 3-D Matrices 9
2.2 Applications of 3-D Matrices 13
2.3 3-Dimensional Matrix Norms in Literature 15
2.4 3-D Matrix Norm Inequalities 16
2.5 Applicatios of the 3-Dimensional Matrix Norm Inequalities 21
3. THE MATRIX NORMS IN THE GAME THEORY **23**
3.1 Basics of the Game Theory 23
3.2 A New Glance to the Game Value 25
3.3 Extrema for the Game Strategies 32
3.4 Applications 34
4. CONCLUSIONS **41**
REFERENCES **45**
CURRICULUM VITAE **47**

ABBREVIATIONS

3-D	**:**	3-dimensional
MT	**:**	Main Theorem
GMT	**:**	Generalized Main Theorem

SYMBOLS

$\mathbb{C}$	**:** Complex Numbers
v	**:** Game Value
$\mathbf{X}, \mathbf{Y}$	**:** Mixed Strategy Set of Players
p_i, q_i	**:** Elements of Mixed Strategy Set of Players

LIST OF TABLES

Page

Table 1.1 : Coefficients table of the 2-D norm inequalities.. 7
Table 2.1 : Coefficient Table of the 3-D Matrix Norm Inequalities. 21
Table 2.2 : 3-Dimensional Matrix Norm Obtained by Simulations 21
Table 2.3 : Matrix Norm Inequalities Coffiecient Table for $M \in \mathbb{C}^{1000\times 101\times 280}$ .. 22
Table 2.4 : Inequalities for the 3-D Matrix Norm Value obtained by Milstein and SRK Methods 22

LIST OF FIGURES

Page

Figure 3.1 : Flowchart for the solution and creation of the two person zero sum matrix games 40

THE ROLES OF MATRIX NORMS IN THE GAME THEORY

SUMMARY

In this thesis, we make some significant contributions and give a new perspective to the game theory and 3-dimensional matrix theory. We present our contributions and developments in three diffrent chapters as follows: In the first chaper, a brief history of the matrices is given. Some examples are given in order to demonstrate the usage in science for different purposes. In the second chapter, some definitions and properties for the 2-dimensional matrices are extended to the 3-dimensional matrices. The basic concepts of the 3-dimensional matrices are presented by extending the definitions for the 2-dimensional matrices. The 3-dimensional matrix product is defined as it is defined for the 2-dimensional matrices. Moreover, the matrix inversion of a 3-dimensional matrix, determinant vector and some other definitions are made.The condition number vectors for the 3-dimensional matrices is defined. In addition these definitions, the singular and nonsingular 3-dimensional matrices are defined based on the definition of the determinant vector. Furthermore, the definition of ill-conditioned and well-conditioned 3-dimensional matrices are presented by using the definition of the condition number vector. Beside these, Cauchy-Schwarz inequality is represented for the 3-dimensional matrices and proved by inducing the 3-dimensional matrix to 2-D matrix. Additionally, some other important inequalities related to the 3-dimensional matrix norms are demonstrated. Finally, in this chapter, the effects of the third dimension with the new definitions and inequalities by some examples are investigated.

In addition to these, the norm inequalities for 3-dimensional matrices are presented and comprehensively proved. The proofs are completed with the similiar methodology being used for the 2-dimensional matrix norm inequalities. That is, we first induce the 3-D matrix to the 2-dimensional matrix. Then, we use the 2-D matrix norm inequalities and necessary vector norm inequalities. Moreover, the relationships between these norms are showed and the coefficients of the 3-dimensional matrix norm inequalities are presented with a table in order to simplify the usage of these norm inequalities. Furthermore, the usefulness of these inequalities is illustrated for 3-dimensional matrices which are obtained from simulations and real data applications.

In the third chapter, a novel approach to solve and create a two person zero sum matrix game by using matrix norms is presented. Especially, we show how to obtain approximated game value, v_{app}, for any zero sum matrix game without solving any equations using our approaches. Firstly, some lemmas are given and the results of these lemmas for the game value depend on the matrix norms of the payoff matrix and some constants k containing the game value v. Then, the row-wise and column-wise induced matrix for the payoff matrix are introduced. Moreover, the proposed approaches are improved and the game value in the constant k is vanished off. Then, some new improved theorems for the game value are presented in order

to obtain some inequalities which depend on only the $1-norm$ and $\infty-norm$ of the payoff matrix. Furthermore, the min-max theorem for p_{max} and p_{min} is stated and clearly proved, where p_{max} and p_{min} are the maximum and minimum elements of the mixed strategy set, respectively. The min-max theorem shows the relationship between p_{max} and p_{min}. Additionally, this theorem provides an opportunity to obtain more optimal interval for the game value. We also illustrate and show the consistency of our approaches with some test examples. To the best of our knowledge, this is the first study in the literature that the game theory meets the matrix norms.

MATRİS NORMLARININ OYUN TEORİSİNDEKİ ROLLERİ

ÖZET

Bu tezde, genel olarak 3-boyutlu matrisler için bazı temel tanımlar ve özellikler sunuldu. Buna ek olarak, 3-boyutlu matris norm eşitsizlikleri ispatlandı ve bu normlar arasındaki ilişkileri gösteren katsayılar eşitsizliklerin daha kolay kullanılabilmesi için bir tablo halinde verildi. Son olarak, matris normları, literatürde ilk olmak üzere, oyun teorisi ile bir araya getirildi. Bütün bunlar tez boyunca üç farklı başlık altında detaylı bir şekilde incelendi. Bu bölümlerdeki içerikler şu şekildedir: Giriş bölümünde, matrislerin tarihi ve kullanım alanlarıyla ilgili bazı örnekler verildi. Bu verilen örneklerle matrislerin farklı bilim dallarında farklı amaçlarla kullanıldığının vurgulanması hedeflendi. Daha sonra, 2-boyutlu matrislerle ilgili literatür taraması mahiyetinde geçmişten bugüne kadar kullanılan bazı temel tanımlar ve özellikler verildi. Birinci bölümde, 2-boyutlu matrisler için bazı temel tanım ve özellikler, 2-boyutlu matrislerdeki tanımlar üzerinden, 3-boyutlu matrisler için genelleştirildi. Bunun sonucunda, 3-boyutlu matrisler için 2-boyutlu matrislere dayanarak temel konsept sunuldu. 3-boyutlu matrisleri temsil etmek üzere bir notasyon belirlendi. Ayrıca, 3-boyutlu matrislerde çarpma işlemi 2-boyutlu matrislerdeki çarpma işlemine benzer şekilde açıkça tanımlandı. Bir 3-boyutlu matrisin tersinin nasıl alınması gerektiği açıklandı. 3-boyutlu bir matrisin determinant vektörü tanımlandı ve bu tanıma bağlı olarak bir 3-boyutlu matrisin tekil olma, tekil olmama ve hemen hemen tekil olma durumları açıklandı. 2-boyutlu matrislerdeki kondisyon sayısı 3-boyutlu matrisler için genelleştirildiğinde bir vektör elde edildi ve bu vektör 3-boyutlu bir matrisin kondisyon sayısı vektörü olarak tanımlandı. 2-boyutlu matrislerdeki kondisyon sayısı hesaplanırken matrisin tersi kullanıldığı gibi 3-boyutlu bir matrisin kondisyon sayısı vektörünü hesaplarken daha önceden açıklanmış olan 3-boyutlu matrislerdeki ters alma işlemi kullanıldı. 2-boyutlu matrisler için tanımlanmış olan kondisyon sayısının tanımına bağlı kalınarak kötü ve iyi koşullu matris tanımları verildi. Bunların yanı sıra, 2-boyutlu matris normları için önemli bir eşitsizlik olan Cauchy-Schwarz eşitsizliği, 3-boyutlu matrislerin 2-boyutlu matrislere indirgenmesi yöntemiyle, 3-boyutlu matrisler için kanıtlandı. Buna ek olarak bazı 3-boyutlu matris normları için önemli eşitsizlikler sunulup, bu eşitsizlikler açık bir şekilde ispat edildi. Örneğin, bir matrisin spektral yarı çapının 3-boyutlu bir matrisin herhangi bir normundan küçük ya da eşit olduğu gösterildi. 3-boyutlu bir A matrisinin hermisyen bir matris olması durumunda bu matrisin spektral yarıçapının, 3-boyutlu matrisler için verilen $2-norm$una eşit olduğu ispatlandı. Bunların yanında 3-boyutlu matrisler için tanımlanmış $Frobenius-norm$un üniter matrislerle çarpım durumda değişmez olduğu sunulup, kanıtlandı. Benzer şekilde $2-norm$unun 3-boyutlu üniter bir matrisle soldan çarpım durumunda değişmez olduğu ispatlandı. 3-boyutlu matrisler için tanımlanmış $2-normu$ ile $1-normu$ ve $\infty-normu$ arasındaki ilişkiyi gösteren bir eşitsizlik daha kanıtlandı. Son olarak bu bölümde verilen tanımları açıklayacak ve üçüncü boyutun yeni tanımlar üzerindeki etkilerini gösterecek şekilde bazı açıklayıcı örnekler verildi.

Bu örneklerde, ilk olarak 3-boyutlu matrislerin determinantının nasıl hesaplanacağı ve 3-D bir matrisin hemen hemen tekil olma durumu gösterildi. Daha sonra, 3-boyutlu bir matrisin tersinin nasıl alınacağı açıklandı. Ayrıca, 3-boyutlu bir matrisin kondisyon sayısı vektörünün nasıl hesaplanacağı açık bir şekilde gösterildi. Böylece verilen örneklerle 3-boyutlu matrisler için yapılmış olan yeni tanımların nasıl kullanılacağına açıklık getirildi.
İkinci olarak, öncelikle literatürde tanımlanmış olan ve bu çalışmada da kullanılan 3-boyutlu matris normlarının tanımları ilgili çalışmada sunulduğu şekilde verildi. Literatürde 3-boyutlu matris norm eşitsizliklerini gösteren herhangi bir çalışma olmadığından dolayı bu açığı kapatmak için bu tezde 3-boyutlu matris norm eşitsizlikleri sunuldu. Bunun sonucunda 3-boyutlu matrislerdeki norm eşitsizlikleri 2-boyutlu matris norm eşitsizliklerine benzer şekilde ispatlandı. Yani, nasıl ki bazı kaynaklarda 2-boyutlu matris norm eşitsizliklerinin ispatlarında, 2-boyutlu matrislerin vektörlere indirgenmesinden sonra vektör norm eşitsizlikleri kullanılıyorsa, bizim kanıtlarımızda da 3-boyutlu bir $A \in \mathbb{C}^{m\times n\times s}$ matrisi 2-boyutlu matrislere indirgendi ve 2-boyutlu matris norm eşitsizlikleri ile gerekli yerlerde vektör norm eşitsizlikleri kullanılarak kanıtlar yapıldı. İspatlardaki diğer önemli bir nokta ise boyutları $m\times 1\times s$ olan bir 3-boyutlu $A \in \mathbb{C}^{m\times 1\times s}$ matrisinin 2-boyutlu bir matris olan $A \in \mathbb{C}^{m\times s}$ şeklinde davrandığının kabul edilmiş olmasıdır. Bu fikre dayanarak yapılmış olan ispatlar sonucunda elde edilen 3-boyutlu matris norm eşitsizliklerinde bulunan katsayılar bir katsayılar tablosu halinde eşitsizliklerin kolayca kullanılabilmesi için sunulmuştur. Daha sonra 3-boyutlu matris normlarının ve ilgili eşitsizliklerin önemini ve kullanışlılığını göstermek amacıyla matematiksel finanstan alınmış, gerçek ve simülasyon verilerini içeren bir örnek verildi. Yani, kanıtlanan 3-boyutlu matris norm eşitsizliklerinin, simülasyon sonucu elde edilen ve gerçek veriler kullanılarak hesaplanmış 3-boyutlu matris norm değerleri için sağlanmış olduğu gösterildi. İspatlanan bu norm eşitsizlikleri kullanılarak matrisin boyutuna bağlı olarak daha optimal aralıklar elde edilebileceğinden bahsedildi. Bunların yanı sıra, stokastik diferansiyel denklemlerle yapılan çeşitli analizler Milstein ve Stokastik Runge Kutta (SRK) yöntemleri ile yapılmış olup 3-boyutlu matris norm eşitsizliklerini sağlamıştır. Bu iki yöntemin yakınsama hızlarından ve yöntemlerden bağımsız olarak 3-boyutlu matris normlar eşitsizliklerinin çalıştığı gösterildi. Ayrıca, literatür taramalarımız sonucunda 3-boyutlu matris norm eşitsizliklerinin ilk olarak bu çalışmada yer aldığını düşünmekteyiz.
Son olarak, matrislerin kullanım alanları ile ilgili araştırma yaparken matrislerin oyun teorisinin temel kısımlarında kullandığını gördük. Bunun üzerine oyun teorisinde matrislerin kullanılmasına rağmen matris normlarının kullanılmaması dikkatimizi çekti. Böylece iki kişilik sıfır toplamlı bir oyunun getiri matrisinin $1-\mathit{normu}$ ve $\infty-\mathit{normu}$ kullanılarak matris normları oyun teorisi ile ilk kez bir araya getirilmiş oldu. Bu bir araya getirme işlemi iki kişilik sıfır toplamlı matris oyunlarının çözümleri ve bu tür oyunların kurulması ile ilgili yeni bir yaklaşım ortaya atılarak sağlandı. Özellikle, herhangi bir iki kişilik sıfır toplamlı matris oyununun herhangi bir denklem çözmeden nasıl kolay ve hızlı bir şekilde yaklaşık olarak çözülebileceği gösterildi. Öncelikle, ortaya atılan metoda temel atmak amacıyla iki önsav sunuldu ve kanıtları yapıldı. Bu önsavlarda oyun değeri için alt ve üst sınırlar oluşturan eşitsizlikler verildi. Bu eşitsizlikler getiri matrisinin 1 ve ∞ normları ile oyun değerini içeren bir sabit olan k'ya bağlı bir şekilde elde edildi. Ayrıca bu sonuçlarda yer alan eşitsizliklerin herhangi bir iki kişilik sıfır toplamlı matris oyunlarında kullanılmak üzere genellemeleri yapıldı. İspatları genellerken kullanılmak üzere getiri matrisinin

satırsal ve sütunsal olmak üzere iki farklı indüs matris tanımı yapıldı. Daha sonra, önsavlarda temelleri atılan bu yöntem geliştirildi ve verilen sonuçlardaki k sabitinin içindeki oyun değerinden kurtulduk. Bu sabit içerisinde bulunan oyun değerinden kurtulmak için bazı varsayımlarda bulunuldu ve böylece yeni sonuçlar elde edildi. Bu yeni sunulan sonuçlardaki eşitsizlikler sadece $1-norm$u ve $\infty-norm$una bağlı eşitsizlikler olup, bu sonuçlar yeni teoremler şeklinde sunuldu ve detaylı bir şekilde ispatlandı. Bunların yanı sıra, iki kişilik sıfır toplamlı matris oyunlarının getiri matrislerinin herhangi bir öteleme durumundaki hali incelendi. Bunun sonucu olarak bu tarz oyunların getiri matrislerinde herhangi bir öteleme yapılması durumunda oyun değerinin öteleme miktarı kadar değiştiğini ve karma stratejiler kümesinin ise aynı kaldığı gösterildi. Ayrıca ilk olarak 2×2 boyutlu iki kişilik sıfır toplamlı matris oyunlarının sırasıyla en büyük ve en küçük elemanları, p_{max} ve p_{min}, için getiri matrisinin normlarına bağlı olarak alt ve üst sınırlar verildi ve gerekli ispatlar yapıldı. Daha sonra bu yaklaşımın genellemesi yapıldı ve $m \times n$ boyutlu bir matris oyunu için aynı sınırlar sunuldu ve ispatlandı. Bunlara ek olarak, karma stratejiler kümesinin sırasıyla en büyük ve en küçük elemanları, yani p_{max} ve p_{min}, arasındaki ilişkiyi gösteren min-max teoremi verildi ve detaylı bir şekilde kanıtlandı. Min-Max teoreminin sonucunda, oyun çözümlerinde ve kurulumlarında p_{max} ya da p_{min}'den biri kullanılmak üzere diğeri için daha optimal sınırlar elde edilebileceği gösterildi. Böylece herhangi bir iki kişilik sıfır toplamlı matris oyununu çözerken oyun değeri için daha iyi sınırlar elde etmek mümkün hale getirildi. Son olarak, yeni yaklaşımımızın tutarlığını göstermek üzere bazı test örnekler verildi. Bu örneklerin yanı sıra, gerçek bir askeri problemin simülasyonu sonucu elde edilmiş ve iki kişilik sıfır toplamlı bir oyun olarak incelenmiş bir oyun hiçbir denklem çözülmeksizin yaklaşık olarak çözüldü. Bu çözümü yapmak için öncelikle ilgili çalışmada verilmiş denklemler kullanılarak oyunun getiri matrisini oluşturuldu ve bu matrisin 1 ve ∞ normları, bu çalışmada verilen teoremlerde kullanmak amacıyla hesaplandı. Ortaya attığımız bu yeni yaklaşımla yapılan çözümün sonucunda elde edilen ve yaklaşık oyun değeri adı verilen değer, v_{app}, ile ilgili makalede iki kişilik sıfır toplamlı bir oyunu çözmek için kullanılan bir yöntemle hesaplanmış gerçek oyun değeri karşılaştırıldı. Yapılan bu karşılaştırma sonucunda, yaklaşık oyun değerinin bilinen yöntemlerle hesaplanmış gerçek değerine çok yakın olduğu görüldü. Böylece oyun çözümü için gerekli sürenin yeni yaklaşımla daha kısa olacağı açıklandı. Son olarak, yeni yöntemin nasıl kullanılacağını özetleyen bir akış şeması verilerek yöntemin kullanışına açıklık getirildi. Çalışmamızın bu kısmının yani, matris normlarıyla oyun teorisinin bir araya getirildiği kısmın, alanında ilk defa yapılmış bir çalışma olduğuna inanmaktayız.

Tezin son bölümünde ise öncelikle 3-boyutlu matrisler için sunulan temel tanımların ve özelliklerin fayda ve sonuçlarından bahsedildi. Daha sonra 3-boyutlu matris normları için sunulan eşitsizliklerin potansiyel kullanım alanlarından örnek verilip, literatüre yaptığı katkılar sunuldu. En son olarak ise, oyun teorisi ile matris normlarınım birleştiritilmesi ile oluşturulan, iki kişilik sıfır toplamlı bir oyunun nasıl daha hızlı ve kolay bir şekilde yaklaşık olarak çözülebileceğini gösteren yeni bir yaklaşım sunuldu ve bu yaklaşımın oyun teorisine nasıl bir katkı sağladığı anlatıldı.

1. INTRODUCTION

In this chapter, we give a brief history of matrices. We also present the basics of 2-dimensional matrices and the matrix norms. Moreover, we give some studies as examples in order to show the usage of the matrices.

1.1 History of Matrices

The mathematicians established different type of systems to deal with linear equations. The structure of these systems changed by the time and took its eventual form. British mathematician Arthur Cayley gave the basic information about the matrices and their notations. After that, in 1857, he improved his idea of matrices in the previous paper and presented them in the paper entilted "A Memoir on the Theory of Matrices" to the world of mathematics. This paper is recognized as the origin of the modern matrix analysis and linear algebra.

In the course of time, the matrix theory has made significant progress for the 2-dimensional matrices and involved into different areas of science. As illustrations, the study of Ignatova and Styczynski may be an example for the usage of the matrices in electrical engineering [1]. Besides, Ni et. al investigated the risk matrices in [2]. As we see from the examples, the 2-dimensional matrices adapted to the different kind of theories.

Recently, the 2-dimensional matrices are extended to the third or higher dimensions. There are different studies about the 3-dimensional matrices and hypermatrices in the literature. As we see that the theory of 3-dimensional matrices improves and finds new application area by the time.

Another important property of the matrices is their norms. The matrix norms has very common usage from mathematics to statistic, from physics to engineering. A matrix norm is a special number, which is obtained by using $m.n$ number, special to the $m \times n$ matrix. The matrix norm inequalities give the special relationship between these norms. Matrix norms are used in many different fields by the time. For example, Zielke showed the relationships between matrix norms and their condition numbers [3]. Li adapted the matrix norms to relative perturbation theory [4]. Moreover, Whitaker et al. used the matrix norm for learning anormalus features via sparse coding in 2015 [5]. Today, as the theory of 3-dimensional matrices improves, the 2-dimensional matrix norms are extended to the 3-dimension matrices as the natural consequence of this improvement. For example, Duran and İzgi, in 2014, defined the 3-dimensional matrix norms [6]. İzgi compherensively defined and investigated the 3-dimensional matrix norms and proved some theoretical results on 3-D norms in 2015 ([7] and references therein). He also exhibited the real data applications of the 3-D norms by performing the simulations, which are based on the numerical solution of stochastic differential equations, for the stock market. In 2017, in this thesis, İzgi and Özkaya showed the 3-dimensional matrix norms are equivalent and proved the relationships between these norms [8]. Furthermore, they gave some applications for the 3-dimensional matrix norms in [8]. In 2018, İzgi and Özkaya presented some basic definitions and propositions for the 3-dimensional matrices [9]. In addition to these, we demonstrated some important inequalities for 3-dimensional matrix norms.

Beside these developments in 2-dimensional matrix theory, we, İzgi and Özkaya, realized that there is no usage of the 2-dimensional matrix norms in the game theory when they studied on the matrices. The matrix norms are not introduced to the game theory even though the matrices are used in the theory according to the literature. Therefore, we improved a novel methodology, that includes the 2-dimensional matrix norms of the payoff matrix, to solve and create two person zero sum matrix games in 2018 [10]. Thus, we brought a new point of view and introduced the matrix norms to the game theory with the combination of the 2-dimensional matrix norms and the theory. Moreover, we solved a simulation of a real life military problem and other problems with their novel approach and introduced the matrix norms to the game theory. We accelerated the game solution process since they solve the game without solving any equation.

Basically, in the thesis, we tried to complete the loose end of the 3-dimensional matrices such as some basic concept of the 3-dimensional matrix theory such as determinant and condition vectors, 3-dimensional matrix inversion. Although the 3-D matrix norms were defined in the past, their relationships between each other was not demonstrated. Therefore, we showed the relationships between the 3-D matrix norms in order to fill up the gap in the 3-D matrix theory. Additionally, we presented some examples to make the definitions clear. We also illustrated the 3-D norm inequalities with an example of financial mathematics. Furthermore, we have realized that the matrix norms are not used in the game theory even if the matrices are used in order to show the payoffs of each player. For this reason, we also introduced the matrix norms, $1-norm$ and $\infty-norm$ of the payoff matrix, to the game theory for a zero sum game solution and creation. In addition to this, we developed a new methodology, based on the 2-dimensional matrix norms, to solve a zero sum matrix game. This methodology decreases the computational cost, which is another purpose of this study. Also we solve a simulation of a real life military problem, which is a zero sum game, with the new approach without solving any equations.

The remainder of this thesis is organized as follows, in the first chapter we mention the history of 2-dimensional matrices and their application areas in the literature. We also give one of the important properties of the matrices, which is the norms of a matrix. In the second chapter, we present the fundamental concept of 3-D matrices and basic definitions for them such as 3-D matrix inversion, determinant and condition number vector. We also proved some important properties in this chapter. Moreover, we illustrate the new definitions with some examples. Furthermore, we extend the 2-dimensional matrix norms inequalities to the 3-dimensional matrix norm inequalities and we prove them. We present the relationships between these inequalities as a coefficient table. Additionally, we illustrate the usage of the 3-D norm inequalities with the results of a mathematical finance problem. In the third chapter, we present some new approaches for the game theory. We bound the game value of a zero sum matrix game with $1-norm$ and $\infty-norms$ of the payoff matrix. Moreover, we give upper and lower boundaries for the greatest and the smallest element of the mixed strategy set. In addition to this, we demostrate the relationship between these elements. Furthermore, we find the approximate game value of a zero sum matrix game, which is a simulation of a real life military problem, without solving any equations. In the last chapter, we present the conclusions and contributions of each chapter.

1.2 2-Dimensional Matrices

Basic Definitions and Properties

In this section, we present some basic definitions for the 2-dimensional matrices in the literature. After Arthur Cayley published the fundamental facts about the matrix theory, the mathematicians put new definitions and propositions upon the theory. There are several different definitions of a matrix.

Definition 1 (Matrix) A matrix is a rectangular array of numbers. The numbers in the array are said to be the entries in the matrix [11].

Definition 2 (Column and Row Matrix) A matrix with only one column is called a column vector or column matrix and a matrix with only one row is said to be row vector or row matrix [11].

The entry that occurs in row i and column j of a matrix A will be denoted by a_{ij}. A general $m \times n$ matrix is in the form

$$A = \begin{bmatrix} a_{11} & . & a_{1n} \\ . & . & . \\ a_{n1} & . & a_{mn} \end{bmatrix}$$

Definition 3 (Equal Matrices) If two matrices A and B in the same shape is called equal if $a_{ij} = b_{ij}$ for all i, j [12].

Proposition 1 (Addition and Substraction) Let A and B be $m \times n$ matrices, then $A \pm B = a_{ij} \pm b_{ij}$ for all i, j [13].

Proposition 2 (Multiplication) Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$ be two matrices, then $C = AB \in \mathbb{C}^{m\times p}$. The entries of the matrix C is $c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}$. Moreover, let α be a scalar, then $\alpha A = \alpha a_{ij}$ for all i, j [14].

Definition 4 (Trace) Let A be a $m \times m$ square matrix, then trace of A is the sum of the entries on the main diagonal of A and it is denoted as $tr(A)$ or $trace(A)$ [11].

Definition 5 (Determinant) The $n \times n$ matrix $A = [a_{ij}]$, the determinant of A is defined to be a scalar $det(A) = \sum_p \sigma(p) a_{1p_1} a_{2p_2} ... a_{np_n}$ where the sum is taken over the $n!$ permutation $p = (p_1, p_2, ..., p_n)$ of $(1, 2, ..., n)$ where $\sigma(p)$ is the sign function of the permutation [13].

Definition 6 (Identity Matrix) The $n \times n$ matrix with 1's on the main diagonal and 0's elsewhere is called the identity matrix of order n. For every $m \times n$ matrix A, $AI_n = A$ and $I_m A = A$ [13].

Definition 7 (Inverse Matrix) The given square matrices $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ that satisfy the condition $AB = I_n$ and $BA = I_n$ is called the inverse of A and is denoted by $B = A^{-1}$. An invertible matrix is said to be nonsingular and a square matrix with no inverse is called singular matrix [13].

Definition 8 (Transpose) Given a matrix $A \in \mathbb{C}^{m\times n}$, its conjugate transpose is the $n \times m$ matrix A^* given by $[A^*]_{ij} = [A]_{ji}$, $1 \leq i \leq m$ and $1 \leq j \leq n$ [15].

Definition 9 (Unitary Matrix) A square matrix $Q \in \mathbb{C}^{m\times m}$ is unitary if $Q^* = Q^{-1}$, i.e, if $Q^*Q = I$ [16].

Definition 10 (Hermitian Matrix) Let $A = [a_{ij}]$ be a square matrix. Then, A is said to be hermitian matrix whenever $A = A^*$ [13].

Definition 11 (Spectral Radius) Let $A = [a_{ij}]$ be a square matrix. Then, the number $\rho(A) = \max\limits_{\lambda \in \sigma(A)} |\lambda|$ is called the spectral radius of A, where λ is the eigenvalues of A [13].

Definition 12 (Condition Number) Let $A = [a_{ij}]$ be a square and nonsingular matrix. Then, the condition number of the matrix A is defined as $cond(A) = ||A|| \cdot ||A^{-1}||$ [13].

Proposition 3 (Properties of Condition Number) Let $A \in \mathbb{C}^{m\times m}$ be 2-dimensional matrix. Then,

1. For any matrix A, $cond(A) \geq 1$
2. For identity matrix, $Cond(A) = 1$
3. For any matrix A and scalar α, $cond(\alpha A) = cond(A)$

Proposition 4 (Properties of Transpose) If A and B are two matrices of the same shape, and if α is a scalar, then each of the following statements is true [13].

1. $(A+B)^* = A^* + B^*$

2. $(\alpha A)^* = \bar{\alpha} A^*$

These kind of definitions and propositions can be found with more details in any linear algebra book. In the next section, we recall an important property of a matrix.

1.3 2-Dimensional Matrix Norms

An $m \times n$ matrix can be viewed as a vector in an mn-dimensional spaces: each of the mn entries of the matrix is an independent coordinate. Therefore, any mn-dimensional norm can be used for measuring the size of such a matrix [16]. The matrix norms are used in different field of science from past to present. For example, Zielke (1988) showed the relatioship between matrix norms and condition number of the matrices [3]. Li (1998) used the $Frobenius - norm$ in relative perturbation theory [4]. Moreover, Wilkinson, in 2005, applied the matrix norms to find two different boundaries for noise variances [17]. Furhetmore, De Maio and Carotenuto invastigated the two cost function that includes either $Frobenius - norm$ or spectral norm of a hermitian matrix in their joint work [18]. As we see from the examples, the matrix norms has a wide usage area in different branches of science. We now continue with some definitions and propositions for 2-dimensional matrix norms:

Definition 13 (Matrix Norm) Let $f : \mathbb{C}^{m\times n} \to \mathbb{C}$ is a matrix norm if the following three properties hold [15]:

1. $f(A) \geq 0$

2. $f(A) = 0$ if and only if $A = 0$

3. $f(A+B) \leq f(A) + f(B)$

4. $f(\alpha A) = \alpha f(A)$

where $A, B \in \mathbb{C}^{m\times n}$, $\alpha \in \mathbb{C}$. The function $f(A)$ is usually denoted as $||A||$ in the literature.

The most frequently used matrix norms are defined as follows [13]:

- Frobenius norm, $||A||_F = \sum_{i,j} |a_{ij}|^2$.
- $2 - norm$, $||A||_2 = \sqrt{\lambda_{max}}$ where λ_{max} is the greatest eigenvalue of the matrix A.
- $1 - norm$, $||A||_1 = \max\limits_j \sum_i |a_{ij}|$, the largest absolute column sum.
- $\infty - norm$, $||A||_\infty = \max\limits_i \sum_j |a_{ij}|$, the largest absolute row sum.

On the other hand, the above matrix norms are equaivalent and the relationships between these matrices are summarized in the below table.
Table 2.1 gives the values of the function $f_{ab}(m,n)$ such that $||A||_p \le f_{ab}(m,n)||A||_q$ where A is m-by-n matrix.

Table 1.1 : Coefficients table of the 2-D norm inequalities..

p\q	1	2	F	∞
1	1	$\sqrt{m}$	$\sqrt{m}$	m
2	$\sqrt{n}$	1	1	$\sqrt{m}$
F	$\sqrt{n}$	$\sqrt{min(m,n)}$	1	$\sqrt{m}$
∞	n	$\sqrt{n}$	$\sqrt{n}$	1

2. 3-DIMENSIONAL MATRICES

In this chapter, we present the basic principals and important properties of 3-dimensional matrices. In addition to these, we state and prove the 3-D matrix norm inequalities.

2.1 Basic Principals of 3-D Matrices

As we see in the previous chapter, the 2-dimensional matrix theory has showns a significant development in the last centuries. Nowadays, the theory of 2-dimensional matrces is developed and some definitions are presented for the matrices in the third or higher dimensions. There are various type of definitions made for higher dimensional matrices by mathematicians. In this chapter, we give some basic definitions and state some propositions with the corresponding proofs of these propositions. We construct and improve the new definitions, which are based on the 2-dimensional matrices, for 3-dimensional matrices. Moreover, we represent Cauchy-Schwarz inequality and some other inequalities for the 3-dimensional matrices norms [9]. Furthermore, we present the 3-D matrix norm inequalities and prove them. Then, we give a table that shows the relationships between these norms [8].
First of all, it is important to emphasis that we refer k^{th} section of 3-dimensional matrix $A \in \mathbb{C}^{m \times n \times s}$ with A^k throughout the paper. So that, a 3-D matrix A can be written as $A = \bigcup\limits_{k=1}^{s} A^k$.

Definition 14 (Multiplication) Let $A \in \mathbb{C}^{m \times n \times s}$ and $B \in \mathbb{C}^{n \times p \times s}$ be 3-dimensional matrices, then $C = AB \in \mathbb{C}^{n \times p \times s}$ and the entries of the matrix C is obtained as $c_{ij}^k = \sum_{t=1}^{n} a_{it}^k b_{tj}^k$.

Definition 15 (Determinant Vector) Let $A \in \mathbb{C}^{m \times m \times s}$ be a 3-D matrix and $det(A^k)$ be the determinant of the k^{th} section of A. The vector $det(A) \in \mathbb{C}^{1 \times 1 \times s}$ is called the determinant vector of the 3-dimensional matrix A whose entiries are $det(A^k)$, where $k = 1, \ldots, s$.

Definition 16 (Singular and Almost Singular Matrix) A 3-dimensional matrix $A \in \mathbb{C}^{m\times m\times s}$ is said to be singular if $det(A^k) = 0$ for all $k = 1,...,s$. On the other hand, the matrix A is called almost singular if $det(A^k) = 0$ for countable k.

Definition 17 (Identity Matrix) Let $I \in \mathbb{C}^{m\times m\times s}$ be a 3-dimensional matrix whose entries, for all k, are $a^k_{ij} = 1$ whenever $i = j$, and $a^k_{ij} = 0$ while $i \neq j$. Then the matrix I is called the 3-dimensional identity matrix.

Definition 18 (Inverse Matrix) Let $A \in \mathbb{C}^{m\times m\times s}$ be a 3-dimensional matrix. Then, A^{-1} is called the inverse matrix of A when $AA^{-1} = \bigcup_{k=1}^{s} (A^k)(A^k)^{-1} = I$ where I is the 3-dimensional identity matrix.

Definition 19 (Transpose) Let $A \in \mathbb{C}^{m\times n\times s}$ be a 3-dimensional matrix and $A^* = \bigcup_{k=1}^{s} (A^k)^*$. Then the matrix $A^* \in \mathbb{C}^{n\times m\times s}$ is called the conjugate transpose of the matrix A.

Definition 20 (Unitary Matrix) Let $U \in \mathbb{C}^{m\times m\times s}$ be 3-dimensional matrix, and U^* be the conjugate transpose of the matrix U. If $U^*U = \bigcup_{k=1}^{s} (U^k)^*U^k = I$ or in other words if $(U^k)^* = (U^k)^{-1}$ for all $k = 1,...,s$, then the matrix U is called the unitary matrix.

Definition 21 (Hermitian Matrix) Let $A \in \mathbb{C}^{m\times m\times s}$ be a 3-dimensional matrix. If $A^* = A$, then the matrix A is called Hermitian matrix.

Definition 22 (Spectral Radius) Let $A \in \mathbb{C}^{m\times m\times s}$ be a 3-dimensional matrix, then the spectral radius of the matrix A is the greatest eigenvalue among the eigenvalues of all sections of the 3-D matrix A and denoted by $\rho(A)$.

We now describe the condition number vector for 3-dimensional matrices which is the generalization of the condition number in 2-dimensional matrices [13, 16].

Definition 23 (Condition Number Vector) Let $A \in \mathbb{C}^{m\times m\times s}$ be a 3-dimensional matrix. Then, the condition number vector $Cond(A) \in \mathbb{C}^{1\times 1\times s}$ is defined as $Cond(A[:,:,k]) = ||A^k||.||(A^k)^{-1}||$ for $k = 1,...,s$ where A^k is the k^{th} section of 3-D matrix A. The colon ":" refers to the all elements in the corresponding places.

Definition 24 (Ill-Conditioned and Well-Conditioned Matrices) Let $A \in \mathbb{C}^{m\times m\times s}$ be a 3-dimensional matrix. The matrix is called ill-conditioned if $Cond(A[:,:,k]) >> 1$ at least one k. Otherwise, the matrix is said to be well-conditioned.

We define the basic concepts of 3-D matrices so far. We now present and prove some propositions and inequalities for the 3-dimensional matrices.

Proposition 5 (Properties of the Condition Number Vector) Let $A \in \mathbb{C}^{m\times m\times s}$ be a 3-dimensional matrix. Then,

1. $Cond(A) = Cond(A[:,:,k]) \geq 1, \forall k = 1,...,s.$
2. $Cond(I) = Cond(I[:,:,k]) = 1$ for all k where $I \in \mathbb{C}^{m\times m\times s}$ is the 3-D identity matrix.
3. $Cond(\alpha A) = Cond(A)$ where $\alpha \in \mathbb{R}$.

Proof

1. $Cond(A) = Cond(A[:,:,k]) = ||A^k|| \cdot ||(A^k)^{-1}|| \geq ||A^k(A^k)^{-1}|| = ||I^k|| = 1$ for all $k = 1,...,s$ where $I^k \in \mathbb{C}^{m\times m}$.
2. $Cond(I) = Cond(I[:,:,k]) = ||I^k|| \cdot ||(I^k)^{-1}|| = 1$ for all k.
3. $Cond(\alpha A) = Cond(\alpha A[:,:,k]) = ||\alpha A^k|| \cdot ||(\alpha A^k)^{-1}|| = |\alpha| \cdot ||A^k|| \cdot |\alpha^{-1}| \cdot ||(A^k)^{-1}|| = ||A^k|| \cdot ||(A^k)^{-1}|| = Cond(A[:,:,k]) = Cond(A)$ for all k. ■

Proposition 6 (Cauchy-Schwarz Inequality) Let $A \in \mathbb{C}^{m\times n\times s}$, and $B \in \mathbb{C}^{n\times p\times s}$ be a 3-dimensional matrices. Then, $||AB|| \leq ||A|| \cdot ||B||$ holds.

Proof. Let $A \in \mathbb{C}^{m\times n\times s}$, $B \in \mathbb{C}^{n\times p\times s}$, and $x \in \mathbb{C}^{p\times 1\times s}$ be the 3-dimensional matrices. Then, we can write $||AB||$ with the corresponding induced matrix norms as, $||AB|| = \sup\limits_{||x||\neq 0} \frac{||ABx||}{||x||} = \sup\limits_{||Bx||\neq 0} \left(\frac{||ABx||}{||Bx||}\frac{||Bx||}{||x||}\right)$ where $ABx \in \mathbb{C}^{m\times 1\times s}$ and $Bx \in \mathbb{C}^{n\times 1\times s}$. We obtain $\sup\limits_{||Bx||\neq 0} \left(\frac{||ABx||}{||Bx||}\frac{||Bx||}{||x||}\right) \leq \sup\limits_{||Bx||\neq 0} \left(\frac{||ABx||}{||Bx||}\right)\left(\sup\limits_{||x||\neq 0} \frac{||Bx||}{||x||}\right) = ||A|| \cdot ||B||$ by using the induced norms for 2-dimensional matrices. Thus, $||AB|| \leq ||A|| \cdot ||B||$ is obtained.■

Proposition 7 *Let* $A \in \mathbb{C}^{m\times n\times s}$, *and* $x \in \mathbb{C}^{n\times 1\times s}$. *Then,* $||Ax||_2 \leq ||Ax||_F \leq ||A||_F||x||_F \leq \sqrt{r}||A||_F||x||_2$, *where* $r = min(n,s)$, $||\cdot||_F$ *and* $||\cdot||_2$ *are Frobenius-norm and 2-norm of 3-dimensional matrices,respectively.*

Proof. We consider the 3-dimensional matrix $Ax \in \mathbb{C}^{m\times 1\times s}$ as the 2-dimensional matrix $Ax \in \mathbb{C}^{m\times s}$. Then, the proof can be easily completed by using the fact $||M||_2 \leq ||M||_F \leq \sqrt{t}||M||_2$ for 2-dimensional matrix norms, where $M \in \mathbb{C}^{m\times n}$ is a 2-dimensional matrix and $t = min(m,n)$ [6].

Proposition 8 *Let $||\cdot||$ be a 3-dimensional matrix norm and $\rho(A)$ be the spectral radius of a matrix $A \in \mathbb{C}^{m\times m\times s}$. Then, $\rho(A) \leq ||A||$ satisfies.*

Proof Let λ^k_{max} be the greatest eigenvalue of the k^{th} section in the matrix $A \in \mathbb{C}^{m\times m\times s}$ and $v^k \neq 0$ be the corresponding eigenvector. Then, $|\lambda^k_{max}|.||v^k|| = ||\lambda^k_{max}v^k|| = ||A^kv^k|| \leq ||A^k||.||v^k||$ by the fact $A^kv^k = \lambda^k_{max}v^k$. We obtain $|\lambda^k_{max}| \leq ||A^k||$ since $||v^k|| \geq 0$ for all k. Hence, the result follows by taking maximum of this inequallity over k.■

Proposition 9 *Let $A \in \mathbb{C}^{m\times n\times s}$ be a 3-D matrix and $U \in \mathbb{C}^{m\times m\times s}$ be a 3-D unitary matrix. Then, $||UA||_2 = ||A||_2$ and $||A||_2 = \sqrt{\rho(A^*A)}$ where $||\cdot||_2$ is the 2-norm for the 3-dimensional matrices. Moreover, if $A \in \mathbb{C}^{m\times m\times s}$ is a hermitian matrix then $||A||_2 = \rho(A)$ where $\rho(A)$ is the spectral radius of the 3-D matrix A.*

Proof. First, we prove the $2-norm$ of any 3-D matrix A is invariant under left-handside multiplication by a unitary matrix as $||UA||_2 = \sqrt{\max_k\{\max_\lambda\{|((UA)^*)^k(UA)^k - \lambda_k I| = 0\}\}} = \sqrt{\max_k\{\max_\lambda\{(A^k)^*(U^k)^*U^kA^k - \lambda_k I\}| = 0\}\}} = \sqrt{\max_k\{\max_\lambda\{|(A^k)^*A^k - \lambda_k I| = 0\}\}} = ||A||_2$. On the other hand, we have $||A||_2^2 = \max_k \left\{|(A^{(k)})^*A^{(k)} - \lambda^k_{max}I| = 0\right\}$, where λ^k_{max} is the largest eigenvalue of the k^{th} section of the 3-D matrix AA^*, by the definition of 2-norm for 3-D matrices [7]. Therefore, it is clear that we can obtain $||A||_2 = \sqrt{\rho(A^*A)}$ as a natural result of the spectral radius of A^*A. Now, let us assume that $\rho(A) = \lambda$ be the spectral radius of a 3-D hermitian matrix A. Hence, $||A||_2 = \sqrt{\rho(A^*A)} = \sqrt{\rho(A^2)} = \sqrt{\lambda^2} = \rho(A)$.■

Proposition 10 *Let $A \in \mathbb{C}^{m\times n\times s}$ be a 3-dimensional matrix, then $||A||_2 \leq \sqrt{||A||_1||A||_\infty}$ holds for the 3-D matrix norms.*

Proof We know that $||A||_2^2 = \rho(A^*A)$, where $\rho(A)$ is the spectral radius of 3-D matrix A, by Proposition 5. Moreover, by Proposition 4 and the norm properties, we get

$\rho(A^*A) \leq ||A^*A||_\infty \leq ||A^*||_\infty ||A||_\infty = ||A|||_1||A||_\infty$ since $||A^*||_\infty = ||A||_1$. Therefore, $||A||_2^2 \leq ||A||_1||A||_\infty$. Thus, $||A||_2 \leq \sqrt{||A||_1||A||_\infty}$ is obtained. ■

Proposition 11 *Frobenius norm for 3-D matrices is invariant under the multiplication by unitary matrices.*

Proof Let $A \in \mathbb{C}^{m\times n\times s}$ be a 3-dimensional matrix and $U \in \mathbb{C}^{m\times m\times s}$, and $V \in \mathbb{C}^{n\times n\times s}$ be the unitary matrices.

We first prove the frobenius norm is invariant for the left-handside multiplication by unitary matrix U. To do so, we write $||UA||_F^2 = trace\big((UA)^*(UA)\big) = trace(A^*U^*UA) = trace(A^*A) = ||A||_F^2$. On the other hand, we know that $||A||_F = trace(A^*A) = trace(AA^*)$ by the definition. For the right-handside multiplication by a unitary matrix within the light of this fact, we have $||AV||_F^2 = trace\Big((AV)(AV)^*\Big) = trace(AVV^*A^*) = trace(A^*A) = ||A||_F^2$.■

Proposition 12 *Let $U \in \mathbb{C}^{m\times m\times s}$ be a 3-D unitary matrix and $A \in \mathbb{C}^{m\times m\times s}$ be a 3-D hermitian matrix, then the conjugate of the similarity transformation of A is UAU^{-1}.*

Proof The similarity transformation of A is UAU^{-1}. Its conjugate can be obtained as follows: $(UAU^{-1})^* = \Big((UA)(U^{-1})\Big)^* = (U^{-1})^*(UA)^* = \bigcup_{k=1}^{m}\Big[\Big((U^k)^{-1}\Big)^*\Big((A^k)^*(U^k)^*\Big)\Big] = \bigcup_{k=1}^{m}\Big[\Big((U^k)^*\Big)^*(A^k)^*(U^k)^{-1}\Big] = \bigcup_{k=1}^{m}\Big[U^kA^k(U^k)^{-1}\Big]$ $= UAU^{-1}$ since $(U^k)^* = (U^k)^{-1}$ and $A^* = A$.■

2.2 Applications of 3-D Matrices

In this section, we exemplify the basic concepts of 3-dimensional matrices such as determinant and condition number vector for 3-D matrices. We also illustrate the 3-dimensional matrix inversion. We need to set a notation for 3-dimensional matrices before the illustrations. Throughout the section, we will denote any 3-D matrix $A \in \mathbb{C}^{m\times n\times s}$ as $A = [A^1, A^2, ..., A^k, ..., A^s]$ where $A^k \in \mathbb{C}^{m\times n}$ represents the k^{th} section of the 3-D matrix A for all k.

Example 1. (Determinant Vector) Let $A \in \mathbb{C}^{3\times3\times3}$ be a 3-D matrix and A^k denote the k^{th} section of the matrix A where $k = 1,2,3$.

$$A = \left[A^1 = \begin{bmatrix} 1 & -3 & 5 \\ -9 & 6 & -2 \\ 7 & 8 & 0 \end{bmatrix}, A^2 = \begin{bmatrix} 5 & 12 & 4 \\ 0 & -21 & 3 \\ 19 & 2 & 0 \end{bmatrix}, A^3 = \begin{bmatrix} 2 & 4 & 6 \\ 6 & 12 & 18 \\ -2 & 7 & 20 \end{bmatrix}\right]$$

In order to evaluate $det(A)$, we firstly calculate the determinants of each section seperately as $det(A^1) = -512$, $det(A^2) = 2250$ and $det(A^3) = 0$. Then, we allocate the determinants of each section in the corresponding places at the determinant vector. Finally, we obtain the determinant vector as:

$$det(A) = \begin{bmatrix} -512 \\ 2250 \\ 0 \end{bmatrix}$$

where $det(A) \in C^{1\times1\times3}$. Moreover, the 3-dimensional matrix A is an *almost singular* since $det(A^3) = 0$.

Example 2. (Matrix Inversion) Let $A \in \mathbb{C}^{3\times3\times3}$ be a 3-D matrix and A^k denote the k^{th} section of the matrix A where $k = 1,2,3$.

$$A = \left[A^1 = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 2 & 1 \\ 8 & 1 & 6 \end{bmatrix}, A^2 = \begin{bmatrix} 3 & 7 & 1 \\ 4 & 9 & 2 \\ 6 & 8 & -3 \end{bmatrix}, A^3 = \begin{bmatrix} 2 & 7 & 8 \\ 1 & 1 & 1 \\ 5 & 4 & 3 \end{bmatrix} \right]$$

Then, A^{-1} is calculated as follows. We firstly invert all the sections by using the inversion method for 2-dimensional matrices and gather these matrices under a inverse matrix of A.

$$\begin{aligned} A^{-1} = \Bigg[(A^1)^{-1} &= \begin{bmatrix} 11/15 & -3/5 & -4/15 \\ 32/15 & -6/5 & -13/15 \\ -4/3 & 1 & 2/3 \end{bmatrix}, (A^2)^{-1} = \begin{bmatrix} 43/17 & 29/17 & 5/17 \\ 24/17 & -15/17 & -2/17 \\ -22/17 & 18/17 & -1/17 \end{bmatrix}, \\ (A^3)^{-1} &= \begin{bmatrix} -1/4 & 11/4 & -1/4 \\ 1/2 & -17/2 & 3/2 \\ -1/4 & 27/4 & -5/4 \end{bmatrix} \Bigg] \end{aligned}$$

Example 3. (Condition Number Vector) Let $A \in \mathbb{C}^{2\times2\times3}$ be 3-dimensional matrix as in the following form.

$$A = \left[A^1 = \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}, A^2 = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}, A^3 = \begin{bmatrix} 10 & -21 \\ 15 & 8 \end{bmatrix} \right]$$

We first calculate the inverse of the 3-D matrix A.

$$A^{-1} = \left[(A^1)^{-1} = \begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix}, (A^2)^{-1} = \begin{bmatrix} 2/7 & -1/7 \\ 1/14 & 3/14 \end{bmatrix}, (A^3)^{-1} = \begin{bmatrix} 8/395 & 21/395 \\ -3/79 & 2/79 \end{bmatrix} \right]$$

Secondly, we calculate the $1-norm$ of the each section for A as $||A^1||_1 = 9$, $||A^2||_1 = 6$, $||A^3||_1 = 29$ where A^k denote the k^{th} section of the matrix A where $k = 1,2,3$. Then, we evaluate the $1-norm$ of the each section of the inverse matrix $||(A^1)^{-1}||_1 = 11$, $||(A^2)^{-1}||_1 = 0.3571$, and $||(A^3)^{-1}||_1 = 0.078$. Finally, we obtain the condition number vector, $Cond(A) = Cond(A[:,:,k])$ for all k, as: $Cond(A[:,:,1]) = ||A^1||_1||(A^1)^{-1}||_1 = 99$, $Cond(A[:,:,2]) = ||A^2||_1||(A^2)^{-1}||_1 = 2.1426$ $Cond(A[:,:,3]) = ||A^3||_1||(A^3)^{-1}||_1 = 2.262$. Hence, we obtain the condition number vector below,

$$Cond(A) = \begin{bmatrix} 99 \\ 2.1426 \\ 2.262 \end{bmatrix}$$

The matrix is *ill-conditioned* since the each entry of the vector $Cond(A) \in \mathbb{C}^{1\times1\times3}$ is much greater than 1. We aim to present norm inequalities, based on the 2-dimensional matrix norms, for 3-dimensional matrices and prove them in the next section. Moreover, we show the usefulness of these inequalities for 3-D matrices obtain from simulations and real data applications. However, we firstly bring back the definitions of the 3-D matrix norms before we present the 3-dimensional matrix norms inequalties.

2.3 3-Dimensional Matrix Norms in Literature

We come up the definitions of the 3-dimensional matrix norm for the first time in 2014 [6]. In 2015, İzgi and his collabrator give an application of these 3-D matrix norms [19]. Moreover, in the same year, İzgi compherensively studied about 3-D matrix norms in [7]. He also applied these norms to examine a financial mathematics problem in his paper with his collaborator [6]. We recall the definition of these 3-D norms as follows:

Definition 25 *A 3-dimensional matrix norm $||\cdot||$ is a function from m-by-n-by-s complex matrices into $\mathbb{R}$ that satisfies the following properties:*

- $||A|| \geq 0||$ *and* $||A|| = 0|$ *if and only if* $A = 0$,
- $||\alpha A|| = |\alpha|||A||$ *, for a scalar* α,
- $||A+B|| \leq ||A|| + ||B||$, *where A and B are matrices in m-by-n-by-s dimensional space.*

Definition 26 *The* $1-norm$ *and* $\infty-norm$ *of* $A \in \mathbb{C}^{m\times n\times s}$ *are defined as follows:*

$$||A||_1 = \max_{1\leq j\leq n} \sum_{k=1}^{s} \sum_{i=1}^{m} |a_{ij}^{(k)}| = \textit{the largest absolute block-column sum.}$$

$$||A||_\infty = \max_{1\leq i\leq m} \sum_{k=1}^{s} \sum_{j=1}^{n} |a_{ij}^{(k)}| = =\textit{the largest absolute row-column sum.}$$

Definition 27 *The* $p-norm$ *of* $A \in \mathbb{C}^{m\times n\times s}$ *is defined as follows:*

$$||A||_p = \left(\sum_{k=1}^{s} \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}^{(k)}|^p \right)^{1/p}$$

Definition 28 *The* $Frobenius-norm$ *of* $A \in \mathbb{C}^{m\times n\times s}$ *is defined as follows:*

$$||A||_F = \sqrt{\sum_{k=1}^{s} \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}^{(k)}|^2}$$

Definition 29 *The* $2-norm$ *of* $A \in \mathbb{C}^{m\times n\times s}$ *is defined as follows:*
$||A||_2 = \max_{k=1,\dots,s} (\max_{||x||_2=1} ||A^{(k)}x||_2) = \sqrt{\lambda_{max}^k}$ *where* λ_{max}^k *is the largest eigenvalue of* $(A^k)^*A^k$ *for all* k. *Moreover, it can be represent as* $||A||_2^2 = \max_{k=1,\dots,s} (\lambda_{max}^k)$ *where* $\lambda_{max}^k = \max(|(A^k)^*A^k - \lambda_k I| = 0.$

2.4 3-D Matrix Norm Inequalities

The 3-D matrix norm inequalities are as important as the 2-dimensional matrix norm inequalities. They show the relationships between each other. In this section, we firstly state the 3-dimensional matrix norm inequalities. Then, we comprehensively prove each of the inequalities [8].

Let $A \in \mathbb{C}^{m\times n\times s}$ be a 3-D matrix, then the 3-matrix norms inequalities are as follows:

1. $\frac{1}{s\sqrt{n}}||A||_\infty \leq ||A||_2 \leq s\sqrt{m}||A||_\infty$

2. $\frac{1}{s\sqrt{m}}||A||_1 \leq ||A||_2 \leq s\sqrt{n}||A||_1$

3. $\frac{1}{s\sqrt{n}}||A||_\infty \leq ||A||_F \leq s\sqrt{m}||A||_\infty$

4. $\frac{1}{ms^2}||A||_1 \leq ||A||_\infty \leq ns^2||A||_1$

5. $\frac{1}{s\sqrt{m}}||A||_1 \leq ||A||_F \leq s\sqrt{n}||A||_1$

6. $\frac{1}{\sqrt{r_2 r_3}}||A||_2 \leq ||A||_F \leq \sqrt{r_1 r_3}||A||_2$ where $r_1 = min(m,s)$, $r_2 = min(n,s)$ and $r_3 \leq s$.

We basically used the similiar approach, that is used to prove the 2-dimensional matrix norm inequalities, to prove 3-D matrix norm inequalities. To make it clear, a 2-dimensional matrix is firstly induced to the vector. Then, the proofs are completed by using the norm inequalities for the vector in some linear algebra books. Therefore, we firstly induce the 3-D matrices to the 2-dimensional matrices. Then, we use the 2-dimensional matrix norm inequalities, which are already known, in our proofs [16]. Throughout all proofs, the following matrices are assumed as: $A \in \mathbb{C}^{m\times n\times s}$,
$x \in \mathbb{C}^{n\times 1\times s}$, $y \in \mathbb{C}^{s\times 1}$, $Ax \in \mathbb{C}^{m\times 1\times s}$ and $Axy \in \mathbb{C}^{m\times 1}$.

Proof. (#1) Let $v \in \mathbb{C}^n$ be a vector and $M \in \mathbb{C}^{m\times n}$ be a 2-dimensional matrix, then we know $\frac{1}{\sqrt{n}}||M||_\infty \leq ||M||_2 \leq \sqrt{m}||M||_\infty$ and $||v||_\infty \leq ||v||_2 \leq \sqrt{n}||v||_\infty$. We will use these inequalities by making the required adaptions during the proof.

$||A||_\infty = \sup\limits_{x\neq 0}\frac{||Ax||_\infty}{||x||_\infty} = \sup\limits_{x\neq 0}\left(\frac{\sup\limits_{y\neq 0}\frac{||Axy||_\infty}{||y||_\infty}}{||x||_\infty}\right) = \sup\limits_{x\neq 0}\left(\sup\limits_{y\neq 0}\frac{||Axy||_\infty}{||x||_\infty||y||_\infty}\right)$ where $Axy \in \mathbb{C}^{m\times 1}$.

By using $||y||_2 \leq \sqrt{s}||y|_\infty$ and $||x||_2 \leq \sqrt{n}||x||_\infty$ we have,

$\leq \sup\limits_{x\neq 0}\left(\sup\limits_{y\neq 0}\frac{||Axy||_\infty}{\frac{||x||_2}{\sqrt{n}}\frac{||y||_2}{\sqrt{s}}}\right) = \sup\limits_{x\neq 0}\left(\sqrt{ns}.\sup\limits_{y\neq 0}\frac{||Axy||_\infty}{||x||_2||y||_2}\right) = \sup\limits_{x\neq 0}\left(\sqrt{ns}\frac{\sup\limits_{y\neq 0}\frac{||Axy||_\infty}{||y||_2}}{||x||_2}\right)$.

On the other hand, since $||y||_\infty \leq ||y||_2$, we obtain,

$\leq \sup\limits_{x\neq 0}\left(\sqrt{ns}\frac{\sup\limits_{y\neq 0}\frac{||Axy||_\infty}{||y||_\infty}}{||x||_2}\right) = \sup\limits_{x\neq 0}\left(\sqrt{ns}\frac{||Ax||_\infty}{||x||_2}\right)$.

We also have $||Ax||_\infty \leq \sqrt{s}||Ax||_2$, then we reach,

$\leq \sup\limits_{x\neq 0}\left(\sqrt{ns}\sqrt{s}\frac{||Ax||_2}{||x||_2} = s\sqrt{n}||A||_2\right)$. Thus, $||A||_\infty \leq s\sqrt{n}||A||_2$ is obtained.

On the other part, in order to find an upper bound for $||A||_2$:

$||A||_2 = \sup\limits_{x\neq 0}\frac{||Ax||_2}{||x||_2} = \sup\limits_{x\neq 0}\left(\frac{\sup\limits_{y\neq 0}\frac{||Axy||_2}{||y||_2}}{||x||_2}\right) = \sup\limits_{x\neq 0}\left(\sup\limits_{y\neq 0}\frac{||Axy||_2}{||x||_2||y||_2}\right)$ where $Axy \in \mathbb{C}^{m\times 1}$.

By using $||y||_\infty \leq ||y|_2$ and $||x||_\infty \leq \sqrt{s}||x||_2$, we get,

$\leq \sup\limits_{x\neq 0}\left(\sup\limits_{y\neq 0}\frac{||Axy||_2}{\frac{||x||_\infty}{\sqrt{s}}||y||_\infty}\right) = \sup\limits_{x\neq 0}\left(\sqrt{s}\sup\limits_{y\neq 0}\frac{||Axy||_2}{||x||_\infty||y||_\infty}\right) = \sup\limits_{x\neq 0}\left(\sqrt{s}\frac{\sup\limits_{y\neq 0}\frac{||Axy||_2}{||y||_\infty}}{||x||_\infty}\right)$.

And since $||y||_2 \leq \sqrt{s}||y||_\infty$, we obtain,

$\leq \sup\limits_{x\neq 0}\left(\sqrt{s}\frac{\sup\limits_{y\neq 0}\frac{||Axy||_2}{||y||_2}}{||x||_\infty}\right) = \sup\limits_{x\neq 0}\left(s\frac{||Ax||_2}{||x||_\infty}\right)$.

Moreover, we have$||Ax||_2 \leq \sqrt{m}||Ax||_\infty$, therefore we reach,

$\leq \sup\limits_{x\neq 0}\left(s\sqrt{m}\frac{||Ax||_\infty}{||x||_\infty}\right) = s\sqrt{m}||A||_\infty$. Hence, $||A||_2 \leq s\sqrt{m}||A||_\infty$ is obtained.

Consequently, if we combine the two inequalities we have obtained, we get

$\frac{1}{s\sqrt{n}}||A||_\infty \leq ||A||_2 \leq s\sqrt{m}||A||_\infty$.■

Proof. **(#2)** Let $v \in \mathbb{C}^n$ be a vector and $M \in \mathbb{C}^{m\times n}$ be a 2-dimensional matrix. Remember that $\frac{1}{\sqrt{m}}||M||_1 \leq ||M||_2 \leq \sqrt{n}||M||_1$ and $||v||_2 \leq ||v||_1 \leq \sqrt{n}||v||_2$.

$||A||_1 = \sup\limits_{x\neq 0} \frac{||Ax||_1}{||x||_1} = \sup\limits_{x\neq 0} \left(\frac{\sup\limits_{y\neq 0} \frac{||Axy||_1}{||y||_1}}{||x||_1} \right) = \sup\limits_{x\neq 0} \left(\sup\limits_{y\neq 0} \frac{||Axy||_1}{||x||_1||y||_1} \right)$ where $Axy \in \mathbb{C}^{m\times 1}$.

By taking into consideration these $||y||_2 \leq ||y|_1$ and $||x||_2 \leq \sqrt{s}||x||_1$, we write,

$\leq \sup\limits_{x\neq 0} \left(\sqrt{s} \sup\limits_{y\neq 0} \frac{||Axy||_1}{||x||_2||y||_2} \right)$.

And we know $||y||_1 \leq \sqrt{n}||y||_2$, so,

$\leq \sup\limits_{x\neq 0} \left(\sqrt{s}\sqrt{s} \frac{\sup\limits_{y\neq 0} \frac{||Axy||_1}{||y||_1}}{||x||_2} \right) = \sup\limits_{x\neq 0} \left(s \frac{||Ax||_1}{||x||_2} \right)$.

When we finally use $\frac{1}{\sqrt{m}}||Ax||_1 \leq ||Ax||_2$, we get,

$\sup\limits_{x\neq 0} \left(s\sqrt{m} \frac{||Ax||_2}{||x||_2} \right) = s\sqrt{m} \sup\limits_{x\neq 0} \frac{||Ax||_2}{||x||_2} = s\sqrt{m}||A||_2$.

Thus, $||A||_1 \leq s\sqrt{m}||A||_2$ is obtained. In order to find an upper bound,

$||A||_2 = \sup\limits_{x\neq 0} \frac{||Ax||_2}{||x||_2} \leq \sup\limits_{x\neq 0} \left(\sup\limits_{y\neq 0} \frac{||Axy||_2}{||x||_2||y||_2} \right) = \sup\limits_{x\neq 0} \left(\sqrt{sn} \frac{\sup\limits_{y\neq 0} \frac{||Axy||_2}{||y||_1}}{||x||_1} \right)$ since $||x||_1 \leq \sqrt{n}||x||_2$ and $||y||_1 \leq \sqrt{s}||y||_2$. Then, by using $||y||_2 \leq ||y|_1$, we have,

$= \sup\limits_{x\neq 0} \left(\sqrt{sn} \frac{\sup\limits_{y\neq 0} \frac{||Axy||_2}{||y||_2}}{||x||_1} \right) = \left(\sup\limits_{x\neq 0} \left(\sqrt{sn}.\frac{||Ax||_2}{||x||_1} \right) \right)$.

Then we use $||Ax||_2 \leq \sqrt{s}||Ax||_1$ and obtain,

$\leq \sup\limits_{x\neq 0} \left(\sqrt{s}\sqrt{sn} \frac{||Ax||_1}{||x||_1} \right) = s\sqrt{n} \sup\limits_{x\neq 0} \frac{||Ax||_1}{||x||_1} = s\sqrt{n}||A||_1$.

As a result of the boundaries we found, we obtain $\frac{1}{s\sqrt{m}}||A||_1 \leq ||A||_2 \leq s\sqrt{n}||A||_1$.■

Proof. **(#3)** Let $v \in \mathbb{C}^n$ be a vector and $M \in \mathbb{C}^{m\times n}$ be a 2-dimensional matrix. We know that $\frac{1}{\sqrt{n}}||M||_\infty \leq ||M||_F \leq \sqrt{m}||M||_\infty$ and $||v||_\infty \leq ||v||_F \leq \sqrt{n}||v||_\infty$ since $2-norm$ is equal to $Frobenius-norm$ in vectors.

$$\begin{aligned}
||A||_\infty &= \sup_{x\neq 0} \frac{||Ax||_\infty}{||x||_\infty} = \sup_{x\neq 0} \left(\frac{\sup\limits_{y\neq 0} \frac{||Axy||_\infty}{||y||_\infty}}{||x||_\infty} \right) \leq \sup_{x\neq 0} \left(\sqrt{ns} \sup_{y\neq 0} \frac{||Axy||_\infty}{||x||_F||y||_F} \right) \\
&= \sup_{x\neq 0} \left(\sqrt{ns} \frac{\sup\limits_{y\neq 0} \frac{||Axy||_\infty}{||y||_F}}{||x||_F} \right) \leq \sup_{x\neq 0} \left(\frac{\sup\limits_{y\neq 0} \frac{||Axy||_\infty}{||y||_\infty}}{||x||_F} \right) = \sup_{x\neq 0} \left(\sqrt{ns} \frac{||Ax||_\infty}{||x||_F} \right) \\
&\leq \sup_{x\neq 0} \left(\sqrt{ns}\sqrt{s} \frac{||Ax||_F}{||x||_F} \right) = s\sqrt{n} \sup_{x\neq 0} \frac{||Ax||_F}{||x||_F} = s\sqrt{n}||A||_F.
\end{aligned}$$

Therefore, $||A||_\infty \leq s\sqrt{n}||A||_F$ is obtained. On the other hand,

$$\begin{aligned}
||A||_\infty &= \sup_{x\neq 0}\frac{||Ax||_F}{||x||_F} = \sup_{x\neq 0}\left(\frac{\sup\limits_{y\neq 0}\frac{||Axy||_F}{||y||_F}}{||x||_F}\right) \leq \sup_{x\neq 0}\left(\sup_{y\neq 0}\frac{||Axy||_F}{\frac{||x||_\infty}{\sqrt{s}}||y||_\infty}\right) \\
&= \sup_{x\neq 0}\left(\sqrt{s}\sup_{y\neq 0}\frac{||Axy||_F}{||x||_\infty||y||_\infty}\right) = \sup_{x\neq 0}\left(\sqrt{s}\frac{\sup\limits_{y\neq 0}\frac{||Axy||_F}{||y||_\infty}}{||x||_\infty}\right) \leq \sup_{x\neq 0}\left(s\frac{\sup\limits_{y\neq 0}\frac{||Axy||_F}{||y||_F}}{||x||_\infty}\right) \\
&= \sup_{x\neq 0}\left(s\frac{||Ax||_F}{||x||_\infty}\right) \leq \sup_{x\neq 0}\left(s\sqrt{m}\frac{||Ax||_\infty}{||x||_\infty}\right) = s\sqrt{m}\sup_{x\neq 0}\frac{||Ax||_\infty}{||x||_\infty} = s\sqrt{m}||A||_\infty.
\end{aligned}$$

Hence, $||A||_F \leq s\sqrt{m}||A||_\infty$. As a consequence, $\frac{1}{s\sqrt{n}}||A||_\infty \leq ||A||_F \leq s\sqrt{m}||A||_\infty$.■

Proof. (#4) Let $v \in \mathbb{C}^n$ be a vector and $M \in \mathbb{C}^{m\times n}$ be a 2-dimensional matrix. We have $\frac{1}{m}||M||_1 \leq ||M||_\infty \leq n||M||_1$ and $||v||_1 \leq ||v||_\infty \leq n||v||_1$. After making the suitable changes in these inequalities, the proof is presented as follows:

$$\begin{aligned}
||A||_1 &= \sup_{x\neq 0}\frac{||Ax||_1}{||x||_1} = \sup_{x\neq 0}\left(\frac{\sup\limits_{y\neq 0}\frac{||Axy||_1}{||y||_1}}{||x||_1}\right) \leq \sup_{x\neq 0}\left(\sup_{y\neq 0}\frac{||Axy||_1}{\frac{||x||_\infty}{s}\frac{||x||_\infty}{s}}\right) \\
&= \sup_{x\neq 0}\left(s^2\sup_{y\neq 0}\frac{||Axy||_1}{||x||_\infty||y||_\infty}\right) = \sup_{x\neq 0}\left(s^2\frac{\sup\limits_{y\neq 0}\frac{||Axy||_1}{||y||_\infty}}{||x||_\infty}\right) \leq \sup_{x\neq 0}\left(s^2\frac{\sup\limits_{y\neq 0}\frac{||Axy||_1}{||y||_1}}{||x||_\infty}\right) \\
&= \sup_{x\neq 0}\left(s^2\frac{||Ax||_1}{||x||_\infty}\right) \leq \sup_{x\neq 0}\left(s^2 m\frac{||Ax||_\infty}{||x||_\infty}\right) = s^2 m\sup_{x\neq 0}\frac{||Ax||_\infty}{||x||_\infty} = s^2 m||A||_\infty.
\end{aligned}$$

Thus, we have $||A||_1 \leq ms^2||A||_\infty$. We now find upper bound,

$$\begin{aligned}
||A||_\infty &= \sup_{x\neq 0}\frac{||Ax||_\infty}{||x||_\infty} = \sup_{x\neq 0}\left(\frac{\sup\limits_{y\neq 0}\frac{||Axy||_\infty}{||y||_\infty}}{||x||_\infty}\right) \leq \sup_{x\neq 0}\left(\sup_{y\neq 0}\frac{||Axy||_\infty}{\frac{||x||_1}{n}||y||_1}\right) \\
&= \sup_{x\neq 0}\left(n\sup_{y\neq 0}\frac{||Axy||_\infty}{||x||_1||y||_1}\right) = \sup_{x\neq 0}\left(n\frac{\sup\limits_{y\neq 0}\frac{||Axy||_\infty}{||y||_1}}{||x||_1}\right) \leq \sup_{x\neq 0}\left(ns\frac{\sup\limits_{y\neq 0}\frac{||Axy||_\infty}{||y||_\infty}}{||x||_1}\right) \\
&= \sup_{x\neq 0}\left(ns\frac{||Ax||_\infty}{||x||_1}\right) \leq \sup_{x\neq 0}\left(ns^2\frac{||Ax||_1}{||x||_1}\right) = ns^2\sup_{x\neq 0}\frac{||Ax||_1}{||x||_1} = ns^2||A||_1.
\end{aligned}$$

As a result, we have $||A||_\infty \leq ns^2||A||_1$. Finally, the result follows after combining the two inequality we have found.■

Proof. (#5) Let $v \in \mathbb{C}^n$ be a vector and $M \in \mathbb{C}^{m\times n}$ be a 2-dimensional matrix. We know $\frac{1}{\sqrt{m}}||M||_1 \leq ||M||_F \leq \sqrt{n}||M||_1$ and $||v||_2 \leq ||v||_1 \leq \sqrt{n}||v||_2$ and $||y||_F = ||y||_2$. The proof is completed in the light of these facts.

$$\begin{aligned}
||A||_1 &= \sup_{x\neq 0}\frac{||Ax||_1}{||x||_1} = \sup_{x\neq 0}\left(\frac{\sup_{y\neq 0}\frac{||Axy||_1}{||y||_1}}{||x||_1}\right) \leq \sup_{x\neq 0}\left(\sup_{y\neq 0}\frac{||Axy||_1}{\frac{||x||_F}{\sqrt{s}}||y||_F}\right)\\
&= \sup_{x\neq 0}\left(\sqrt{s}\sup_{y\neq 0}\frac{||Axy||_1}{||x||_F||y||_F}\right) = \sup_{x\neq 0}\left(\sqrt{s}\frac{\sup_{y\neq 0}\frac{||Axy||_1}{||y||_F}}{||x||_1}\right) \leq \sup_{x\neq 0}\left(s\frac{\sup_{y\neq 0}\frac{||Axy||_1}{||y||_1}}{||x||_F}\right)\\
&= \sup_{x\neq 0}\left(s\frac{||Ax||_1}{||x||_F}\right) \leq \sup_{x\neq 0}\left(s\sqrt{m}\frac{||Ax||_F}{||x||_F}\right) = s\sqrt{m}\sup_{x\neq 0}\frac{||Ax||_F}{||x||_F} = s\sqrt{m}||A||_F.
\end{aligned}$$

Thus, $||A||_1 \leq s\sqrt{m}||A||_F$.

In order to find the upper boundary:

$$\begin{aligned}
||A||_F &= \sup_{x\neq 0}\frac{||Ax||_F}{||x||_F} = \sup_{x\neq 0}\left(\frac{\sup_{y\neq 0}\frac{||Axy||_F}{||y||_F}}{||x||_F}\right) \leq \sup_{x\neq 0}\left(\sup_{y\neq 0}\frac{||Axy||_F}{\frac{||x||_1}{\sqrt{n}}\frac{||y||_1}{\sqrt{s}}}\right)\\
&= \sup_{x\neq 0}\left(\sqrt{ns}\sup_{y\neq 0}\frac{||Axy||_F}{||x||_1||y||_1}\right) = \sup_{x\neq 0}\left(\sqrt{ns}\frac{\sup_{y\neq 0}\frac{||Axy||_F}{||y||_1}}{||x||_1}\right) \leq \sup_{x\neq 0}\left(\sqrt{ns}\frac{\sup_{y\neq 0}\frac{||Axy||_F}{||y||_F}}{||x||_1}\right)\\
&= \sup_{x\neq 0}\left(\sqrt{ns}\frac{||Ax||_F}{||x||_1}\right) \leq \sup_{x\neq 0}\left(s\sqrt{n}\frac{||Ax||_1}{||x||_1}\right) = s\sqrt{n}\sup_{x\neq 0}\frac{||Ax||_1}{||x||_1} = s\sqrt{n}||A||_1.
\end{aligned}$$

Then, we get $||A||_F \leq s\sqrt{n}||A||_1$. Finally, after combining the inequalities for upper and lower boundaries, the result follows.■

Proof. (#6) Let $v \in \mathbb{C}^n$ be a vector, $M \in \mathbb{C}^{m\times n}$ be a 2-dimensional matrix and $r_M = Rank(M)$. We have the following inequalities $||M||_2 \leq ||M||_F \leq \sqrt{r_m}||M||_2$ and $||v||_2 = ||v||_F \leq \sqrt{r}||v||_2$ where $r \leq n$. In order to find the lower bound:

$||A||_2 = \sup_{x\neq 0}\frac{||Ax||_2}{||x||_2} = \sup_{x\neq 0}\left(\sup_{y\neq 0}\frac{||Axy||_2}{||x||_2||y||_2}\right)$,

We know that $||x|_F \leq \sqrt{r_1}||x||_2$ and $||y||_F \leq \sqrt{r_3}$ where $r_1 = min(m,s)$, $r_3 \leq s$ then,

$\leq \sup_{x\neq 0}\left(\sup_{y\neq 0}\frac{||Axy||_2}{\frac{||x||_F}{\sqrt{r_1}}\frac{||y||_F}{\sqrt{r_3}}}\right) = \sup_{x\neq 0}\left(\sqrt{r_1r_3}\frac{\sup_{y\neq 0}\frac{||Axy||_2}{||y||_F}}{||x||_F}\right)$, and since $||y||_F = ||y||_2$;

$= \sup_{x\neq 0}\left(\sqrt{r_1r_3}\frac{\sup_{y\neq 0}\frac{||Axy||_2}{||y||_2}}{||x||_F}\right) = \sup_{x\neq 0}\left(\sqrt{r_1r_3}\frac{||Ax||_2}{||x||_F}\right)$ As a final step, since $||Ax||_2 \leq ||Ax||_F$,

$\leq \sup_{x\neq 0}\left(\sqrt{r_1r_3}\frac{||Ax||_F}{||x||_F}\right) = \sqrt{r_1r_3}\sup_{x\neq 0}\frac{||Ax||_F}{||x||_F} = \sqrt{r_1r_3}||A||_F$ Consequently, we obtain $||A||_2 \leq \sqrt{r_1r_3}||A||_F$. On the other side, we find the upper boundary as follows:

$||A||_F = \sup_{x\neq 0}\frac{||Ax||_F}{||x||_F} \leq \sup_{x\neq 0}\left(\sup_{y\neq 0}\frac{||Axy||_F}{||x||_F||y||_F}\right)$, and since $||x||_2 \leq ||x||_F$ and $||y||_2 = ||y||_F$;

$\leq \sup_{x\neq 0}\left(\sup_{y\neq 0}\frac{||Axy||_F}{||x||_2||y||_2}\right) = \sup_{x\neq 0}\left(\frac{\sup_{y\neq 0}\frac{||Axy||_F}{||y||_2}}{||x||_2}\right)$; we know $||y||_F \leq \sqrt{r_3}||y||_2$;

$= \sup_{x\neq 0}\left(\sqrt{r_3}\frac{\sup_{y\neq 0}\frac{||Axy||_F}{||y||_F}}{||x||_2}\right) = \sup_{x\neq 0}\left(\sqrt{r_3}\frac{||Ax||_F}{||x||_2}\right)$; since $||Ax||_F \leq \sqrt{r_2}||A||_2$ where $r_2 = min(m,s)$

$\leq \sup_{x\neq 0}\left(\sqrt{r_3}\sqrt{r_2}\frac{||Ax||_2}{||x||_2}\right) = \sqrt{r_3 r_2}||A||_2$. The result follows by gathering together the two inequalities for upper and lower boundaries.■

Let $A \in \mathbb{C}^{m\times n\times s}$ be a 3-D matrix. The coefficients for the 3-D matrix norm inequalities can be obtained easily with the Table 2.1. Let $||A||_a \leq N_{ab}||A||_b$ and $N_{ab}(m,n,s)$ is obtained from the table:

Table 2.1 : Coefficient Table of the 3-D Matrix Norm Inequalities.

a\b	1	2	F	∞
1	1	$s\sqrt{m}$	$s\sqrt{m}$	ms^2
2	$s\sqrt{n}$	1	$\sqrt{r_2 r_3}$	$s\sqrt{m}$
F	$s\sqrt{n}$	$\sqrt{r_1 r_3}$	1	$s\sqrt{m}$
∞	ns^2	$s\sqrt{n}$	$s\sqrt{n}$	1

2.5 Applicatios of the 3-Dimensional Matrix Norm Inequalities

In this section, we demonstrate that the 3-D norm inequalities we have proved hold for the 3-D matrix norm values of the simulation and real data. Table 2.2 is obtained by the 3-D matrix norm values which are obtained by the result of analysis of stochastic differential equation (SDE) for stock market. The first column of Table 2.2 includes the 3-D norm values which are obtained by simulations of the interest rate that are updated within %[-2,2] randomly at each step using Milstein method. The second column consists of the 3-D norm values which are obtained by the result of the analysis of the real interest rate by Stochastic Runge-Kutta (SRK) method [6,7]. Firstly, we make a

Table 2.2 : 3-Dimensional Matrix Norm Obtained by Simulations

Norm	Milstein	SRK
$1-norm$	3.9863	3.6732E+07
$2-norm$	2.8489	2.8541E+06
$Inf-norm$	6.4641	1.1129E+10
$Fro-norm$	4.0322	4.3077E+07

similiar table as Table 2.1 for the coefficients of the 3-D matrix norm inequalities of the 3-D matrix $M \in \mathbb{C}^{1000\times 101\times 280}$. Therefore, we know that $m = 1000$, $n = 101$ and

$s = 280$ since we defined a 3-D matrix as $A^{m\times n\times s}$. Lastly, we can easily obtain Table 2.3 as shown below. A similiar table can be obtained for the other 3-D matrices in [19].

Table 2.3 : Matrix Norm Inequalities Coffiecient Table for $M \in \mathbb{C}^{1000\times 101\times 280}$

a\b	1	2	F	∞
1	1	8854.38	8854.38	64009000
2	2542.62	1	159.85	8854.38
F	2542.62	280	1	8854.38
∞	28280	2542.62	2542.62	1

As a final step, we can summarize the inequalities for 3-D matrix norm values which are obtained for different methods in Table 2.4:

Table 2.4 : Inequalities for the 3-D Matrix Norm Value obtained by Milstein and SRK Methods

#	Milstein	SRK
1	$2.30E-03 \leq 2.849 \leq 5.72E+04$	$1.39E+06 \leq 2.85E+06 \leq 2.83E+13$
2	$4.50E-04 \leq 2.849 \leq 1.12E+04$	$1.44E+04 \leq 2.85E+06 \leq 2.94E+11$
3	$2.30E-03 \leq 4.032 \leq 5.72E+04$	$1.39E+06 \leq 4.31E+07 \leq 2.83E+13$
4	$1.42E-05 \leq 6.464 \leq 3.16E+07$	$1.55E-07 \leq 1.11E+10 \leq 2.35E+15$
5	$4.50E-04 \leq 4.032 \leq 1.12E+04$	$1.44E+04 \leq 4.31E+07 \leq 2.94E+11$
6	$1.69E-02 \leq 4.032 \leq 7.98E+02$	$1.13E+04 \leq 4.31E+07 \leq 4.56E+08$

The number of the norm inequalities.

The norm inequalities are represented with respect to the related 3-D matrix norm. Therefore, if we use a bigger size matrix, we obtain large interval. However, we can obtain more optimal intervals in the cases that we use smaller dimensional matrices. In addition to these, there are two different 3-D matrix norm values which are obtained by two different methods of numerical solution of stochastic differential equation called Milstein and SRK. The convergence rates for Milstein and SRK are 1.0 and 2.0, respectively (see [20]). Although the faster methods are prefered to analyze the SDE, we can see that these preferences do not make any difference on the 3-D matrix norm inequalities if we investigate Table 2.4. Thus, we can say that the 3-D norm inequalities hold independent of the methods.

3. THE MATRIX NORMS IN THE GAME THEORY

To the best of our knowledge, the matrix norms are not used even though the matrices are used in the game theory. Therefore, we basically aim to combine the matrix norms and the game theory in this chapter. For this reason, we focus on solving and creating the two person zero sum matrix games by using the matrix norms of the payoff matrix. We present and prove some theorems for the game value and the maximum and minimum elements of the mixed strategy set ,p_{max} and p_{min}, by using $1-norm$ and $\infty-norm$ of the payoff matrix. We propose a methodology that approximately solves any two person zero sum matrix game without dealing with solving any equations. We also illustrate the application of our methods for some zero sum matrix game problems [10]. We start with a literature review of the game theory before presenting our approaches.

3.1 Basics of the Game Theory

The game theory might be explained as a mathematical decision theory between participants in a competitive environment [21]. The theory of game is came to exist with the study of von Neumann in the very beginning of 20th century. The improvements in this field are stepped up with the proof of minimax theorem [22]. In the study, Theory of Games and Economic Behavior by John von Neumann and Oskar Morgenstern in 1944, the fundamental principals of the game theory is presented [23]. Today, the game theory is an irreplaceable part of economic theory and mathematical finance. Especially, it gives great opportunity to analyze the financial problems. However, the usage area of this theory is not limited only with financial problems. The game theory also has a wide range of application area in real life problems. The essential purpose of the theory is to determine the optimal options from the strategy set of participants in a competitive situation such as game of draughts, military problems, criminal cases and so on. As an illustration, Kose et. al used the game theory combining with geographical information systems to answer a military decision problem in [24]. Wang et. al modeled the interaction of a MIMI radar and a jammer as a two person zero sum game in their paper [25]. Egorov and Sonin analyzed the battle for throne by using game theory in [26]. We now continue with some basic definition and important properties of the game theory.

Definition 30 (Game) The strategic form, or normal form, of a two person zero sum game is denoted by a triplet (X,Y,A), where

1. X is a nonempty set, the set of strategies of Player I

2. Y is a nonempty set, the set of strategies of Player II

3. A is a real-valued function defined on $X \times Y$

To be clear, Player I selects $x \in X$ and Player II selects $y \in Y$ at the same time and without knowing each other's choices. Then their options are made known and I wins $A(x,y)$ from Player II, where $A(x,y)$ may be anything such as liras, dollar or something else. If $A < 0$, then Player I gives $|A|$ amount to the Player II. As a summary, one's loss is the other's gain.

Definition 31 (Constant Sum Game) A two player strategic form gaime is constant sum if there exists a constant c such that for each strategy profile $a \in X \times Y$ [27].

Definition 32 (Zero Sum Game) A game is called zero sum if the sum of payoffs equals to zero for any outcome [28].

Definition 33 (Solution of a Zero Game) Let A be the payoff matrix of a game and x_i and y_j be the elements of the mixed strategy sets of the players. Then the game value v is calculated as $v = \max_i \sum_j a_{ij} y_j = \min_j \sum_i a_{ij} x_i$ for all i, j [21].

Definition 34 (Mixed Strategies) Let (X,Y,A) be a normal form game, and any set Z let $\Pi(Z)$ be the set of all probability distribution over Z. Then the set of mixed strategies for Player I(or Player II) is $S_I = \Pi(Z_I)$ (or $S_{II} = \Pi(Z_{II})$) [27].

Definition 35 (The Minimax Theorem) For every finite two person zero sum game [29],

1. there is a number v, said to be the game value,
2. there is a mixed strategy set for Player I such that I's average gain is at least v no matter what Player II does,
3. there is a mixed strategy set for Player II such that I's average loss is at most v no matter what Player I does.

If $v = 0$ then the game is said to be fair. If $v > 0$, then the game favors Player I. If the game value is negative, then the game favor Player II.

Definition 36 (Saddle Point) A point that is simultaneously a row minimum and a column maximum of the payoff matrix is called a saddle point [29].

Definition 37 (Pure Nash Equilibrium) An outcome, a combination of moves, is pure Nash equilibrium if the each move involved is the best respond to the other moves.

3.2 A New Glance to the Game Value

We primarily debate and study about the game value and its boundaries in this section. We start to achieve inequalities for the game value v with 2×2 matrix game. Then, we generalize the inequalities for $m \times n$ matrix games. However, we firstly obtain the inequalities in Lemma 1 and Lemma 2 depend on $1 - norm, \infty - norm$ of the payoff matrix and a constant k consisting of the game value v. Then, we state some new theorems and success to obtain the inequalities for the game value which include only $1 - norm$ and $\infty - norm$ of the payoff matrix [10]. Additionaly, we present the consequences of the perturbation onto the game value and mixed strategy set.
As a consequence, one may get some conditions, which may be used during the game creation process, by using these theoretical results. In order to avoid misunderstandings, we suppose the rows and columns of the payoff matrix for the Player I and Player II, respectively. It is important to indicate that we analyze the games in the view of the Player I that is the row player. One may use our approaches for Player II, as well.

Lemma 1 *Let (P,Q,A) be a finite two person zero sum game where $A \in R^{2\times 2}$ is the payoff matrix and $P = \{p_1, p_2\}$ and $Q = \{q_1, q_2\}$ are the mixed strategy sets for the* *players and v represents the game value. Then,*

$\frac{k}{||A||_\infty} \leq v \leq ||A||_1$ *when v is positive,*

$-||A||_1 \leq v \leq \frac{k}{||A||_\infty}$ *when v is negative,*

hold where $k = min\{(v(|a|+|b|)),(v(|c|+|d|))\}$.

Proof. We consider the payoff matrix as in the following form:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

For $v > 0$: Let $||A||_\infty$=$|a|+|b|$. Then, $|a|+|b| \geq |c|+|d|$. Therefore, we can write the following inequality, $\frac{|c|+|d|}{||A||_\infty} \leq 1$. Then, we obtain $\frac{v(|c|+|d|)}{||A||_\infty} \leq v$ by multiplying both side of the previous inequality by v. If we define $k = v(|c|+|d|)$, we have the first inequality as follows, $\frac{k}{||A||_\infty} \leq v$ ...(1)

On the other hand, we know that v=ap+(1-p)c $\leq |a|+|c|$ since p $\in [0,1]$. Then, we obtain the second inequality as $v \leq ||A||_1$ ...(2)

since $||A||_1 = max\{|a|+|c|,|b|+|d|\}$. We have, $\frac{k}{||A||_\infty} \leq v \leq ||A||_1$ for $v > 0$ from (1) and (2).

For $v < 0$: Let $||A||_1 = |a|+|c|$ for the payoff matrix A, above.

Case 1. While $a \leq 0$ and $c \geq 0$:

We know that, $0 \leq p \leq 1$ and $0 \leq 1-p \leq 1$, since $p \in [0,1]$. Then we have, $0 \geq ap \geq a$, since $a \leq 0$. We also get, $c \geq (1-p)c \geq 0$, since $c \geq 0$. We obtain the following by adding these inequalities and using basic algebraic arrangements, $c \geq ap+(1-p)c \geq a \geq a-c$ =$-(|a|+|c|)$=$-||A||_1$. Therefore, $v \geq -||A||_1$ where $v = ap+(1-p)c$.

Case 2. Assume $a \leq 0$ and $c \leq 0$:

We again have $0 \leq p \leq 1$ and $0 \leq 1-p \leq 1$, and we get $0 \geq ap \geq a$ since $a \leq 0$ and $0 \geq (1-p)c \geq c$ for $c \leq 0$ with the similar approach in the previous case. By using these inequalities, it is clear to obtain $0 \geq ap+(1-p)c \geq a+c$. We also have $a+c = -(-a+(-c)) = -(|a|+|c|) = -||A||_1$ by making basic algebraic tricks while keeping $a \leq 0$ and $c \leq 0$ in mind. Hence, the inequality $v \geq -||A||_1$ holds.

On the other hand, the inequality $\frac{|c|+|d|}{||A||_\infty} \leq 1$ is valid for all the cases. Then we have, $\frac{v(|c|+|d|)}{||A||_\infty} \geq v$ since $v < 0$. As a result, we have $-||A||_1 \leq v \leq \frac{k}{||A||_\infty}$ where $k = v(|c|+|d|)$. ■

The next lemma is a generalization of Lemma 2.1. In other words, the following lemma is about $m \times n$ matrix game with the game value $v \in R$.

Lemma 2 (Generalization) *Let A be a m×n real valued payoff matrix and v be the game value of a two person zero sum game. Then,*

$\frac{k}{||A|||_\infty} \leq v \leq ||A||_1$ *for positive v,*

$-||A||_1 \leq v \leq \frac{k}{||A|||_\infty}$ *for negative v,*

hold where $k = \max\limits_{1\leq i\leq m, i\neq p} \sum_{j=1}^{n} v|a_{ij}|$ *and* $||A||_\infty = \sum_{j=1}^{n} |a_{pj}|$ *for fixed p.*

Proof. We deal with the following m-by-n real valued payoff matrix

$$A = \begin{bmatrix} a_{11} & . & a_{1n} \\ . & . & . \\ a_{n1} & . & a_{mn} \end{bmatrix}$$

and we use v^+ for positive game value v and v^- for negative v in the proof.

Case 1. For v^+:

Let $||A||_\infty = \sum_{j=1}^{n} |a_{pj}|$ for a fixed p. We have $\sum_{j=1}^{n} |a_{pj}| \geq \max\limits_{1\leq i\leq m, i\neq p} \sum_{j=1}^{n} a_{ij}$ by the definition of $\infty - norm$. If we define $m = \max\limits_{1\leq i\leq m, i\neq p} \sum_{j=1}^{n} |a_{ij}|$, then we can rewrite the inequality above as $\frac{m}{||A||_\infty} \leq 1$, and also obtain $\frac{k}{||A||_\infty} \leq v^+$ where $k = v^+ m$.

Moreover, we can evaluate the game value $v^+ = \sum_{i=1}^{m} p_i a_{ij}$ for any j. Then, it is clear that $v^+ \leq \sum_{i=1}^{m} |a_{ij}|$ satisfies for all j since $p \in [0,1]$. If we take the maximum of the both side with respect to j then we get $v^+ \leq \max\limits_{j} \sum_{i=1}^{m} |a_{ij}| = ||A||_1$.

Case 2. For v^-:

We already have the following inequality for v^+, $\frac{m}{||A||_\infty} \leq 1$. However, we deal with v^- in this case. Therefore, the upper boundary for negative game value is $\frac{k}{||A||_\infty} \geq v^-$ where $k = v^- m$ and $m = \max\limits_{1\leq i\leq m, i\neq p} \sum_{j=1}^{n} |a_{ij}|$.

In order to obtain the lower boundary for v^-, we have $v^- = \sum_{i=1}^{m} p_i a_{it}$ for any t. On the other hand, we can write that $-|a_{ij}|p_i \geq -|a_{ij}|$ in general since $0 \leq p_i \leq 1$. Therefore, the inequality $v^- \geq \sum_{i=1}^{m}(-p_i|a_{it}|) \geq -\sum_{i=1}^{m} |a_{it}| \geq -||A||_1$ holds since $\sum_{i=1}^{m} |a_{it}| \leq \max\limits_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}| = ||A||_1$. Consequently, $-||A||_1 \leq v^- \leq \frac{k}{||A|||_\infty}$. ■

As we can see in the statement of the lemmas above, k depends on the game value v so far. In the following corollary we improve our approaches to get rid of the game value in k.

Corollary 1 *Let* (P,Q,A) *be a finite two person zero sum game where* $A \in R^{m\times n}$ *is the payoff matrix, v is the game value and* $m = \max\limits_{1\leq i\leq m, i\neq p} \sum_{j=1}^{n} |a_{ij}|$. *Then,*

if $v \geq 1$, $\frac{m}{||A|||_\infty} \leq v \leq ||A||_1$

if $v \leq -1$, $-||A||_1 \leq v \leq -\frac{m}{||A|||_\infty}$.

The next theorem will generalize all the approaches we used above. The essential purpose of the following theorem is to give the boundaries for the game value just by calculating the related matrix norms of the payoff matrix.

Theorem 1 (Main Theorem) *Let $A \in R^{2\times 2}$ be a payoff matrix for two person zero sum matrix game and v be the game value where*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then,
if $|v| \geq 1$, then $\frac{|c|+|d|}{||A||_\infty} \leq |v| \leq ||A||_1$,
if $|v| \leq 1$, and $v \neq 0$, then $\frac{1}{||A||_1} \leq |v| \leq \frac{||A||_\infty}{|c|+|d|}$.

Proof. We have the followings by Lemma 1: $\frac{|c|+|d|}{||A||_\infty} \leq 1$ where $||A||_\infty = |a| + |b|$, $v \leq ||A||_1$ when v is positive and $-||A||_1 \leq v$ when v is negative.
a) For $v > 0$:
i. If $v \geq 1 \Rightarrow \frac{|c|+|d|}{||A||_\infty} \leq v \leq ||A||_1$ by using the results of Lemma 1.
ii. If $0 < v \leq 1 \Rightarrow$ Let $k^{-1} = v \Rightarrow k = \frac{1}{v} \geq 1$, then we have the following by using (i),
$\frac{|c|+|d|}{||A||_\infty} \leq k \leq ||A||_1 \Rightarrow \frac{1}{||A||_1} \leq v \leq \frac{||A||_\infty}{|c|+|d|}$.
b) For $v < 0$:
i. If $v \leq -1 \Rightarrow v \leq -1 \leq -\frac{|c|+|d|}{||A||_\infty} \Rightarrow -||A||_1 \leq v \leq -\frac{|c|+|d|}{||A||_\infty}$ by Lemma 1.
ii. If $-1 \leq v < 0 \Rightarrow$ Let $t^{-1} = v \Rightarrow t = \frac{1}{v} \leq -1$.
We have the following inequality by using (i), $-||A||_1 \leq t \leq -\frac{|c|+|d|}{||A||_\infty}$. Then, we can write that $-\frac{||A||_\infty}{|c|+|d|} \leq v \leq -\frac{1}{||A||_1}$.
After making some arrangement for the inequalities we found in (a) and (b), the results follow,
$\frac{|c|+|d|}{||A||_\infty} \leq |v| \leq ||A||_1$ when $|v| \geq 1$,
$\frac{1}{||A||_1} \leq |v| \leq \frac{||A||_\infty}{|c|+|d|}$ when $|v| \leq 1$, and $v \neq 0$. ■

Before generalizing Main theorem, we give a definition that is going to be used in Generalized Main Theorem.

Definition. *Let $A \in R^{m\times n}$ be a real valued matrix, and let $||A||_\infty$ be the sum of absolute values of the h^{th} row's entries, then the matrix $B \in R^{(m-1)\times n}$ is obtained by deleting h^{th} row of the matrix A is called a row-wise induced matrix of A. Similarly, let A$\in R^{m\times n}$ be a real valued matrix, and let $||A||_1$ be the sum of absolute values of the s^{th} column's* *entries, then the matrix $B \in R^{m\times(n-1)}$ is obtained by deleting s^{th} column of the matrix A is called a column-wise induced matrix of A.*

Theorem 2 (Generalized Main Theorem) *Let A be a m×n payoff matrix and v be the game value for a two person zero sum game. Then,*
if $|v| \geq 1, then\ \frac{||B||_\infty}{||A||_\infty} \leq |v| \leq ||A||_1$
if $|v| \leq 1, and\ v \neq 0, then\ \frac{1}{||A||_1} \leq |v| \leq \frac{||A||_\infty}{||B||_\infty}$,
where B is the row-wise induced matrix of A.

Proof. The proof can be done by using Lemma 2 and Main Theorem, directly.

Remark 1 *For two person zero sum games we should investigate the game value from the view of each player. Therefore, we obtain two different inequalities for the same game value. Note that, in order to get the optimum interval for the game value, one should compare the inequalities obtained for each player.*

Moreover, Main Theorem's results may not be the best way to solve 2×2 matrix games since they can be easily solved by using well-known methods. However, one may prefer to use these inequalities for a bigger size matrix game so that they may have an idea about the approximated game value without solving any equations.

The following proposition shows us how the game value v and the mixed strategy sets change under a perturbation.

Proposition 13 *Let A be a 2×2 payoff matrix for two person zero sum game and v be a game value, then the value of the perturbated game $A+X$ with all positive entries (or $A+Y$ with all negative) is $v+x$ (or $v+y$) and the mixed strategy set is invariant, where $X(or\ Y) \in R^{2\times 2}$ with all entries x (or y), and $x = |min(A)|$ (or $y = -|max(A)|$).*

Proof. We are going to prove the proposition only for positive perturbation, the proof for negative perturbation can be done by similar approach. The probability of the strategies and game value for A are $p = \frac{d-c}{(a-b)+(d-c)}$ and $v = ap + c(1-p) = \frac{ad-bc}{(a-b)+(d-c)}$, respectively, by the equalizing strategies [7].
Let

$$A+X = \begin{bmatrix} a+x & b+x \\ c+x & d+x \end{bmatrix}$$

be the payoff matrix for perturbated game where $X \in R^{2\times 2}$ is a matrix with $x_{ij} = x$.

By equalizing strategies, we calculate the probability of the strategy as below:

$$\begin{aligned}
(a+x)p_1 + (c+x)(1-p_1) &= (b+x)p_1 + (d+x)(1-p_1) \\
ap_1 + p_1x + c(1-p_1) + x(1-p_1) &= bp_1 + xp_1 + d(1-p_1) + x(1-p_1) \\
ap_1 + c - cp_1 &= bp_1 + d - dp_1 \\
d - c &= ap_1 + dp_1 - cp_1 - bp_1 \\
p_1 &= \frac{d-c}{(a-b)+(d-c)}
\end{aligned}$$

As it can be seen $p_1 = p$. This means, the mixed strategy set is invariant under any perturbation of the payoff matrix. Now, let v_x be the game value of the pertubated matrix game $A+X$. Then, we can evaluate v_x as $v_x = (a+x)p + (c+x)(1-p) = ap + c(1-p) + x$. Hence, $v_x = v + x$.■

Corollary 2 *Any matrix game can be converted to a matrix game with fully or partially negative (or positive) entries with the same strategy set.*

According to Corollary 2, we will prove the following theorems for the payoff matrix with positive entries.

Remark 2 *Notice that we gave the lemmas and theorems for $v \neq 0$. The next corollary helps us to create a matrix game with $v = 0$. To make it clear, we firstly create a matrix game with the game value except zero using the given theorems. Lastly, we perturbate the payoff matrix and the game value by using the Proposition 13. We will illustrate this situation in Section 3.4 (see last part of Example 2).*

Corollary 3 *Any matrix game with the game value $v \neq 0$ can be converted to a matrix game with the game value $v = 0$ under the same strategies.*

Apart from the Corollary 2 and 3, we will give the next proposition and lemma for the zero sum game with the game value $v = 0$. However, we suggest to use the Corollary 2 and 3 in order to create a matrix game with $v = 0$.

Proposition 14 *Let $A \in R^{2\times 2}$ be a payoff matrix of a zero sum game and $v = 0$, then the columns (and also the rows) of A are linearly dependent.*

Proof. Let A be the payoff matrix,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then, since $v = 0$, we can write the following system by using the equalizing strategies,

$$\begin{aligned} pa + (1-p)c &= 0 \quad (3.1) \\ pb + (1-p)d &= 0 \quad (3.2) \end{aligned}$$

The coefficent matrix of this system is A^T. First, we multiply (3.1) by d, then we obtain,

$$pad + (1-p)cd = 0 \quad (3.3)$$

In addition, if we multiply (3.2) by c, then we get,

$$pbc + (1-p)cd = 0 \quad (3.4)$$

Consequently, we have the following equality from (3.3) and (3.4),

$$pad + (1-p)cd - pbc - (1-p)cd = 0 \quad (3.5)$$

Lastly, we have $p(ad-bc) + (1-p)(cd-cd) = 0$ by (3.5), since $p \neq 0$, and then $ad - bc = 0$ which is the determinant of the coefficient matrix A^T. Hence, $|A^T| = |A| = 0$ which means the columns and also the rows of the payoff matrix A are linearly dependent. ■

Lemma 3 *Let $A \in R^{m \times n}$ be a payoff matrix of the zero sum game and the game value $v = 0$ then, $-\frac{||B||_\infty}{||A||_\infty} \leq v \leq min\{||A||_1, 1 - \frac{||B||_\infty}{||A||_\infty}\}$ holds where B is the row-wise induced matrix of A.*

Proof. Let $A \in R^{m \times n}$ be a payoff matrix of the zero sum game and $v = 0$. We know that $||A||_\infty \geq ||B||_\infty$, where B is the row-wise induced matrix of A. Then, we have $1 \geq \frac{||B||_\infty}{||A||_\infty}$ or equivalently $-1 \leq -\frac{||B||_\infty}{||A||_\infty} \leq 0 = v$. Therefore, we can write $v = 0 \leq 1 - \frac{||B||_\infty}{||A||_\infty}$. We also have the following inequality from the previous theorems $v \leq ||A||_1$. Hence, the result follows by using these inequalities, $-\frac{||B||_\infty}{||A||_\infty} \leq v \leq min\{||A||_1, 1 - \frac{||B||_\infty}{||A||_\infty}\}$.■

3.3 Extrema for the Game Strategies

In this section, we try to find some boundaries for the maximum and minimum elements in the mixed strategy set. Therefore, we give some theorems about these strategies. Throughout the paper, we use p_{max} and p_{min} for the greatest and the smallest elements of the mixed strategy set of the players, respectively.

Firstly, we present the next theorem which helps us to find the lower and upper boundaries for p_{max} and p_{min}, respectively. Then, we state the min-max theorem to show the relationship between p_{max} and p_{min}. Finally, we generalize the first theorem of this section.

Theorem 3 *Let $A \in R^{2\times 2}$ be a payoff matrix with positive entries for two person zero sum game. Then,*
$p_{max} \geq L$ where $L = max\{1 - \frac{v}{||A||_1}, \frac{v}{||B||_1}\}$
$p_{min} \leq U$ where $U = min\{1 - \frac{v}{||B||_1}, \frac{v}{||A||_1}\}$
hold where B is the column-wise induced matrix of A.

Proof. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be the payoff matrix of a two person zero sum game.
As a beginning, let us suppose that $||A||_1$=a+c, so a+c $\geq$ b+d.
We also know the following equality from the method of equalizing strategy for the solution of a zero sum game, $v = ap + (1-p)c = bp + (1-p)d$.

Additionally, we can obtain $(a+c)p_{max} \geq (b+d)p_{max}$ and $(a+c)p_{min} \geq (b+d)p_{min}$ by making some algebraic arrangement in the equality above. Therefore we can write the following inequality by combining these inequalities,
$(b+d)p_{min} \leq (a+c)p_{min} \leq v \leq (b+d)p_{max} \leq (a+c)p_{max}$ ...(1)
We obtain by (1), $p_{min} \leq \frac{v}{a+c} = \frac{v}{||A||_1}$ ...(2)
and $p_{max} \geq \frac{v}{b+d} = \frac{v}{||B||_1}$ where B is the column-wise induced matrix of A.

We have 2 elements in the mixed strategy set since the game is a 2×2 matrix game, so that $p_{max} + p_{min} = 1$ ...(3)

Moreover, we get $p_{max} + p_{min} \leq \frac{v}{||A||_1} + p_{max}$ from (2) and (3). Then, we obtain, $1 - \frac{v}{||A||_1} \leq p_{max}$. Hence, $p_{max} \geq L$ where $L = max\{1 - \frac{v}{||A||_1}, \frac{v}{||B||_1}\}$.
On the other hand, we have $p_{min} \leq \frac{v}{a+c} = \frac{v}{||A||_1}$ from the inequality in (2). We know that $1 = p_{min} + p_{max} \geq \frac{v}{||B||_1} + p_{min}$. Thus, $p_{min} \leq U$ where $U = min\{1 - \frac{v}{||B||_1}, \frac{v}{||A||_1}\}$.$\blacksquare$

Now, we represent the relationships between p_{max} and p_{min} with the following theorem.

Theorem 4 (Min-Max Theorem for the Game Strategies)
Let $A \in R^{m\times n}$ be the payoff matrix. Then,
$\frac{1-p_{min}}{m-1} \leq p_{max} \leq 1-(m-1)p_{min}$ and
$1-(m-1)p_{max} \leq p_{min} \leq \frac{1-p_{max}}{m-1}$,
hold where p_{max} and p_{min} are the greatest and the smallest elements of the mixed strategy set, respectively.

Proof. In the view of the row side player we have $p_1 + \ldots + p_{min} + \ldots + p_{max} + \ldots + p_m = 1$ since $A^{m\times n} \in R$. We can rewrite this equality as, $p_1 + \ldots + p_{min} + \ldots + p_m = 1 - p_{max}$ or $p_1 + \ldots + p_{max} + \ldots + p_m = 1 - p_{min}$. Notice that we have $(m-1)$ terms on the left hand side of it. Therefore, we have $(m-1)p_{min} \leq 1 - p_{max}$ or $(m-1)p_{max} \geq 1 - p_{min}$, respectively. Hence, we have $1-(m-1)p_{min} \geq p_{max}$ and $p_{min} \leq \frac{1-p_{max}}{m-1}$. Or $p_{max} \geq \frac{1-p_{min}}{m-1}$ and $1-(m-1)p_{max} \leq p_{min}$, the result follows. $\blacksquare$

Theorem 5 *Let $A \in R^{m\times n}$ be the payoff matrix with positive entries. Then, the boundaries for p_{max} and p_{min} which are the greatest and the smallest elements of the mixed strategy set, respectively, are as follows,*
$p_{max} \geq L$ *where* $L = max\{\frac{1-\frac{|v|}{||A||_1}}{m-1}, \frac{|v|}{||B||_1}\}$
$p_{min} \leq U$ *where* $U = min\{\frac{1-\frac{|v|}{||B||_1}}{m-1}, \frac{|v|}{||A||_1}\}$
where B is the column wise induced matrix of A.

Proof. First, let $||A||_1 = a_{1k} + a_{2k} + \ldots + a_{mk}$ and $||B||_1 = a_{1t} + a_{2t} + \ldots + a_{mt}$ for fixed k and t (i.e. $k,t \leq$n), then we have,
$v = p_1a_{1t} + p_2a_{2t} + \ldots + p_ma_{mt} \leq p_{max}(a_{1t} + a_{2t} + \ldots + a_{mt}) = p_{max}||B||_1$. Hence, $p_{max} \geq \frac{v}{||B||_1}$.

Similarly, we can find an upper bound for p_{min} with the same approaches,
$v = p_1a_{1k} + p_2a_{2k} + \ldots + p_ma_{mk} \geq p_{min}(a_{1k} + a_{2k} + \ldots + a_{mk}) = p_{min}||A||_1$.
Therefore, we have $p_{min} \leq \frac{v}{||A||_1}$.

On the other hand, we know that $p_1 + p_2 + ... + p_m = 1$.
$1 = p_1 + p_2 + ... + p_{max} + ... + p_m \geq \frac{|v|}{||B||_1} + (m-1)p_{min}$ by the above inequality for p_{max}. After some arrangement, $\frac{1-\frac{|v|}{||B||_1}}{m-1} \geq p_{min}$. Therefore, $p_{min} \leq U$ where $U = min\{\frac{1-\frac{|v|}{||B||_1}}{m-1}, \frac{|v|}{||A||_1}\}$.

Lastly, we can obtain the other lower boundary for p_{max} with the same process, $1 = p_1 + p_2 + ... + p_{min} + ... + p_m \leq \frac{|v|}{||A||_1} + (m-1)p_{max}$ by using the inequality for p_{min}. Then, $p_{max} \geq \frac{1-\frac{|v|}{||A||_1}}{m-1}$. Hence, $p_{max} \geq L$ where $L = max\{\frac{1-\frac{|v|}{||A||_1}}{m-1}, \frac{|v|}{||B||_1}\}$.$\blacksquare$

3.4 Applications

In this section, we will illustrate the usage of our novel approaches for the game theory with the some test examples.

Example 1.(Game Creation) Assume that we want to create a 2×2 positive entries zero sum matrix game with the game value $v = 5$ and the mixed strategies $p_1 = p_2 = \frac{1}{2}$. Let A be the payoff matrix of the game where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Suppose that $||A||_1 = a + c$ and $||A||_\infty = a + b$. Firstly, we will use Main Theorem to obtain the first condition, $\frac{|c|+|d|}{||A||_\infty} \leq |v| \leq ||A||_1 \Rightarrow \frac{c+d}{a+b} \leq 5 \leq a + c$. Then, we have two conditions by using the inequality above,
$c + d \leq 5a + 5b$ ...(1)
$a + c \geq 5$ ...(2)
In order to obtain more conditions, we use the inequalities for maximum and minimum strategies by Theorem 5.
Case I. Let's assume that $p_{max} \geq 1 - \frac{\frac{v}{||A||_1}}{m-1}$ and $p_{min} \leq 1 - \frac{\frac{v}{||B||_1}}{m-1}$ where $m = 2$ since A is 2-by-2 matrix. By using the above inequalities we obtain,
$p_{max} \geq 1 - \frac{5}{a+c} \Rightarrow 10 \geq a + c$ ...(3)
$p_{min} \leq 1 - \frac{5}{b+d} \Rightarrow 10 \leq b + d$ ...(4)
Case II. Suppose that, $p_{max} \geq 1 - \frac{\frac{v}{||A||_1}}{m-1}$ and $p_{min} \leq \frac{v}{||A||_1}$.
We already have boundary from the previous case with the inequality (3) for p_{max}. Moreover, the inequality for p_{min} is $\frac{1}{2} \leq \frac{5}{a+c}$. Hence, we have $a + c \leq 10$ which is the exactly same with the inequality (3).

Case III. Now assume that, $p_{max} \geq \frac{v}{||B||_1}$ and $p_{min} \leq 1 - \frac{\frac{v}{||B||_1}}{m-1}$.
We have $b+d \geq 10$ by using the inequality for p_{max}, which is the same as the inequality (4). Since the inequality for p_{min} is the same as in Case I, we obtain the inequality (4) again.
Case IV. Suppose that $p_{max} \geq \frac{v}{||B||_1}$ and $p_{min} \leq \frac{v}{||A||_1}$.
By using the above inequality for p_{max} we have $b+d \geq 10$ by the assumption for p_{max}, that is same as (4). Moreover, we obtain $a+c \leq 10$ by using the inequality for p_{min}, which is exactly the same as (3).
After these analysis, for example, we can choose the payoff matrix's entries as a=4, b=7, c=6 and d=3 which hold for each cases. Hence, the payoff matrix A is

$$A = \begin{bmatrix} 4 & 7 \\ 6 & 3 \end{bmatrix}$$

Example 2.(Perturbated Game) The purpose of this example is to show how we create a matrix game with negative entries and we also create a matrix game with the game value $v = 0$ by using our approach.
Let A be the payoff matrix as in Example 1. In that example, the game value is $v = 5$, and the probabilities of the strategies are $p_1 = p_2 = 0.5$.

$$A = \begin{bmatrix} 4 & 7 \\ 6 & 3 \end{bmatrix}$$

We choose $y = -|max(A)| = -7$ by using Proposition 13 in order to obtain the negative entries for the payoff matrix. Here the matrix Y is in the following form

$$Y = \begin{bmatrix} -7 & -7 \\ -7 & -7 \end{bmatrix}$$

Then, the payoff matrix $P = A + Y$ of the perturbated game, which is obtained by using Proposition 2.6, is

$$P = \begin{bmatrix} -3 & 0 \\ -1 & -4 \end{bmatrix}$$

If we calculate the game value by Proposition 13, we get $v = -2$ and the probabilities are the same as before.

Now, we see how we create matrix game with $\bar{v} = 0$. Firstly, we need to create the payoff matrix A as in Example 1. Then, in order to create the matrix game with $\bar{v} = 0$

according to Proposition 13, we choose $\bar{y} = -v = -5$ where $v = 5$ is the game value of the matrix game A. Here, $\bar{P}$ is the payoff matrix obtained by $A + \bar{Y}$,

$$\bar{P} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$$

where $\bar{Y} \in R^{2\times 2}$ is matrix with $\bar{y}_{ij} = -5$. The determinant of $\bar{P}$ is zero which is consistent with Proposition 14. Hence, the game value $\bar{v}$ of the matrix game $\bar{P}$ is 0 and the mixed strategy set is invariant.

Example 3.(3×3 **Game**)In this example, we create a bigger size zero sum matrix game. Suppose that we want to create a 3×3 zero sum matrix game with the game value $v = 2$ and the mixed strategies $p_1 = 0.30$, $p_2 = 0.25$ and $p_3 = 0.45$. Let A be the payoff matrix of the game where

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$$

Let's assume that $||A||_1 = |a| + |d| + |g|$, $||B||_1 = |b| + |e| + |h|$, $||A||_\infty = |a| + |b| + |c|$ and $||B||_\infty = |d| + |e| + |f|$ where B is the corresponding induced matrix of the payoff matrix A. We firstly obtain some conditions by using GMT as follows,

$\frac{|d|+|e|+|f|}{|a|+|b|+|c|} \leq 2 \leq |a| + |d| + |g|$

$|d| + |e| + |f| \leq 2(|a| + |b| + |c|)$...(1)

$|a| + |d| + |g| \geq 2$...(2)

We now make the case analysis by using Theorem 5 in order to get other conditions for game creation. As we see in the previous example, the game can be created by using any of the cases. Therefore, we will use the case

$p_{max} \geq \frac{1-\frac{|v|}{||A||_1}}{m-1}$ and $p_{min} \leq \frac{1-\frac{|v|}{||B||_1}}{m-1}$ where $m = 3$ since the payoff matrix A is 3-*by*-3.

$p_{max} = 0.45 = \frac{9}{20} \geq \frac{1-\frac{2}{|a|+|d|+|g|}}{2}$. We have the following inequality by making arrangements, $20 \geq |a| + |d| + |g|$...(3)

We find another condition by using the inequality for p_{min},

$p_{min} = 0.25 = \frac{1}{4} \leq \frac{1-\frac{2}{|b|+|e|+|h|}}{2}$. We get, $4 \leq |b| + |e| + |h|$...(4)

Hence, we need to determine the entries of the payoff matrix A accoring to the inequalities (1), (2), (3) and (4). If we choose the entries as $a = 3.38$, $b = 0.45$, $c = 3.6$, $d = -1.4$, $e = 4$, $f = 1.4$, $g = 3.55$, $h = 2.35$ and $k = -0.2$. Thus, the payoff matrix is

$$A = \begin{bmatrix} 3.38 & 0.45 & 3.6 \\ -1.4 & 4 & 1.4 \\ 3.55 & 2.35 & -0.2 \end{bmatrix}$$

If we solve this game with known methods, we get the game value $v = \sum_{i=1}^{3} p_i a_{ik} \approx 2$ for any $k = 1,2,3$ and the mixed strategy set $P = \{0.45, 0.3, 0.25\}$. Consequently, we succesfully created the zero sum matrix game with our methods.

Example 4.(The real data) The intention of this example is to show that the theorems in our paper hold for any zero sum matrix game. Let the payoff matrices D and D^T be obtained from the given inequalities in [24] for defender and attacker, respectively. The game value of this game for each case is given as $v = 0.5791$ in the paper since the problem is two person zero sum game (see [24]). The payoff matrix D in [24] is

$$D = \begin{bmatrix} 0.4298 & 0.4298 & 0.9253 & 0.9253 & 0.0936 & 0.5293 \\ 0.4073 & 0.6989 & 0.4073 & 0.4804 & 0.5311 & 0.7425 \\ 0.7208 & 0.5616 & 0.5616 & 0.4726 & 0.7625 & 0.1954 \end{bmatrix}$$

Firstly, we investigate the game from the defender's side. The maximum and minimum elements of the mixed strategy set are given as $p_{max} = 0.5340$, $p_{min} = 0.1726$ in the corresponding paper.

In this case, the matrix norms of the payoff matrix D, $\bar{D}$ and $D^\star$ are $||D||_1$= 1.8942, $||D||_\infty = 3.3331$, $||\bar{D}||_1 = 1.8783$ and $||D^\star||_\infty = 3.2745$, where $\bar{D}$ and $D^\star$ are the column-wise and row-wise induced matrix of D, respectively. Then, we find the boundaries for the approximated game value w by using Generalized Main Theorem (GMT),

$\frac{1}{||D||_1} \leq |w| \leq \frac{||D||_\infty}{||D^\star||_\infty} \Rightarrow \frac{1}{1.8942} \leq |w| \leq \frac{3.3331}{3.2745} \Rightarrow 0.5279 \leq w \leq 1.0179$ ...(1)

We now find the boundaries for $p^\star_{max}$ and $p^\star_{min}$ by using Theorem 5, $m = 3$ in this case since $D \in R^{3\times 6}$.

$p^\star_{max} \geq L$ where $L = max\{\frac{1-\frac{0.5791}{1.8942}}{3-1}, \frac{0.5791}{1.8783}\} = max\{0.3471, 0.3083\}$. Hence, $p^\star_{max} \geq 0.3471$.

$p^\star_{min} \leq U$ where $U = min\{\frac{1-\frac{0.5791}{1.8783}}{3-1}, \frac{0.5791}{1.8942}\} = min\{0.2458, 0.3057\}$. Thus, $p^\star_{min} \leq 0.2458$. Therefore, we see that the inequalities above hold for the given p_{max} and p_{min}.

Secondly, we consider the game from attacker's side. We know the game value for the attacker is v =0.5791 as well.

The required matrix norms are $||D^T||_1 = 3.3331$, $||D^T||_\infty = 1.8942$, $||(D^T)^*||_\infty = 1.8783$ and $||(D^T)^{**}||_1 = 3.2745$, where $(D^T)^*$ and $(D^T)^{**}$ are the row-wise and

column-wise induced matrix of the payoff matrix D^T, respectively. By using GMT, $\frac{1}{||D^T||_1} \leq |w| \leq \frac{||D^T||_\infty}{(||D^T)^*||_\infty} \Rightarrow \frac{1}{3.3331} \leq |w| \leq \frac{1.8942}{1.8783} \Rightarrow 0.3 \leq w \leq 1.009$ ...(2)
Consequently, we obtain the optimum interval as $0.5279 \leq w \leq 1.009$ from (1) and (2) in the light of Remark 1. As a result of the fact that the actual game value $v = 0.5791$ falls into the optimum interval.

The given maximum and minimum elements of the strategy set are $p_{max} = 0.5522$ and $p_{min} = 0$ for the attacker in [24]. So, the boundaries for p_{max} and p_{min} can be found similarly by using Theorem 3.3 ($m = 6$ since $D^T \in R^{6\times 3}$).
$p^*_{max} \geq L$ where $L = max\{\frac{1-\frac{0.5791}{3.3331}}{6-1}, \frac{0.5791}{3.2745}\} = max\{0.1653, 0.1769\}$. Thus, $p^*_{max} \geq 0.1769$.
$p^*_{min} \leq U$ where $U = min\{\frac{1-\frac{0.5791}{3.2745}}{6-1}, \frac{0.5791}{3.3331}\} = min\{0.1646, 0.1737\}$. Hence $p^*_{min} \leq 0.1646$.
It is clear from the above analyses that our approaches work for a real life military problem, which is designed as a two person zero sum matrix game, as well.

It is valuable to emphasis the following remark before we calculate and compare the approximated game value v_{app} with v by using our results.

Remark 3 *It is very important to notice that the approximated game value v_{app} must fall in the optimum interval we have found for the game value, and the probabilities must be chosen by taking consideration of the inequalities for p_{max} and p_{min}. Moreover, we must obey the principal of the probability theory in order to decide the rest of the elements in the mixed strategy set. That is, total sum of the strategies must be 1.*

We firstly calculate the approximated game value $v^\star_{app}$ by using the inequalities we found for p_{max} and p_{min} for the defender. We know that $p^\star_{max} \geq 0.3471$ and $p^\star_{min} \leq 0.2458$. Firstly, we need to decide $p^{\star\star}_1$, $p^{\star\star}_{max}$ and $p^{\star\star}_{min}$ for the strategies since the player has 3 options in this case. We can determine the strategies of the game with the following steps:

1. Choose p_{max} (or p_{min}) by using the related inequality.

2. Use Min-Max Theorem to find a new interval for p_{min} (or p_{max}).

3. Choose p_{min} (or p_{max}) from the new interval.

4. Find the probability sum of the rest of the strategies and determine them one by one, arbitrarily.

5. Distribute them in any order you want by keeping Remark 3 in your mind.

6. Evaluate v_{app}.

For example, we choose the strategy scenario as $p^{\star\star}_{min} = 0.22$, $p^{\star\star}_1 = 0.38$ and $p^{\star\star}_{max} = 0.40$ according to these steps. While we attempt to evaluate $v^{\star}_{app}$, we can distribute $p^{\star\star}_1$, $p^{\star\star}_{max}$ and $p^{\star\star}_{min}$ in any order we want by keeping Remark 3 in our mind. However, the possible distributions do not make significant differences onto the game value as long as we take account the optimum interval for the game value. So that, we deal with the mixed strategy set $S^{\star} = \{0.22, 0.38, 0.40\}$ for this scenario. In accordance with Remark 3, we can use any column we want when we evaluate the approximated game value. In this perspective, as an example, we calculate the approximated game value $v^{\star}_{app}$ for the zero sum matrix game by using the first column of the payoff matrix D, $v^{\star}_{app} = (0.4298 \times 0.22) + (0.4073 \times 0.38) + (0.7208 \times 0.40) = 0.5377$. The absolute error for this scenario is $|v - v_{app}| = |0.5791 - 0.5377| = 0.041$ which is a quite small error. Moreover, one may find better approximated game value using the different column for this case or totally different scenario.

Now, we calculate the v^{*}_{app} for the attacker. We first need to determine the strategies as in the defender's case by following the steps above. We have 6 elements in the mixed strategy set for this case. Here, we choose $p^{**}_{min} = 0.05$, $p^{**}_1 = 0.15$, $p^{**}_2 = 0.15$, $p^{**}_3 = 0.15$, $p^{**}_4 = 0.11$ and $p^{**}_{max} = 0.39$. In this application, we work with $S^{*} = \{0.15, 0.39, 0.15, 0.15, 0.11, 0.05\}$ as mixed strategy set in this scenario. For instance, we prefer to use the third column of the payoff matrix D^T while we evaluate the v_{app}. Thus, $v^{*}_{app} = (0.7208 \times 0.15) + (0.5616 \times 0.39) + (0.5616 \times 0.15) + (0.4726 \times 0.15) + (0.7625 \times 0.11) + (0.1954 \times 0.05) = 0.5759$. The absolute error is $|v - v_{app}| = |0.5791 - 0.5759| = 0.003$. As in the defender's case, one may find better approximated game value by choosing different column or distribution for the strategy set.

In the corresponding paper, the authors create two different linear systems which include 6 and 3 inequalities for the attacker and defender, respectively. Then, they solve each of the systems by using linear programming method. Contrast to this, we only use $1-norm$ and $\infty-norm$ of the payoff matrix to evaluate the approximated game value without solving any equation with the methodology above. This is one of the most important advantage part of our approaches.

Finally, we summarize our approaches in Figure 3.1 which may be as a guide for the practitioners.

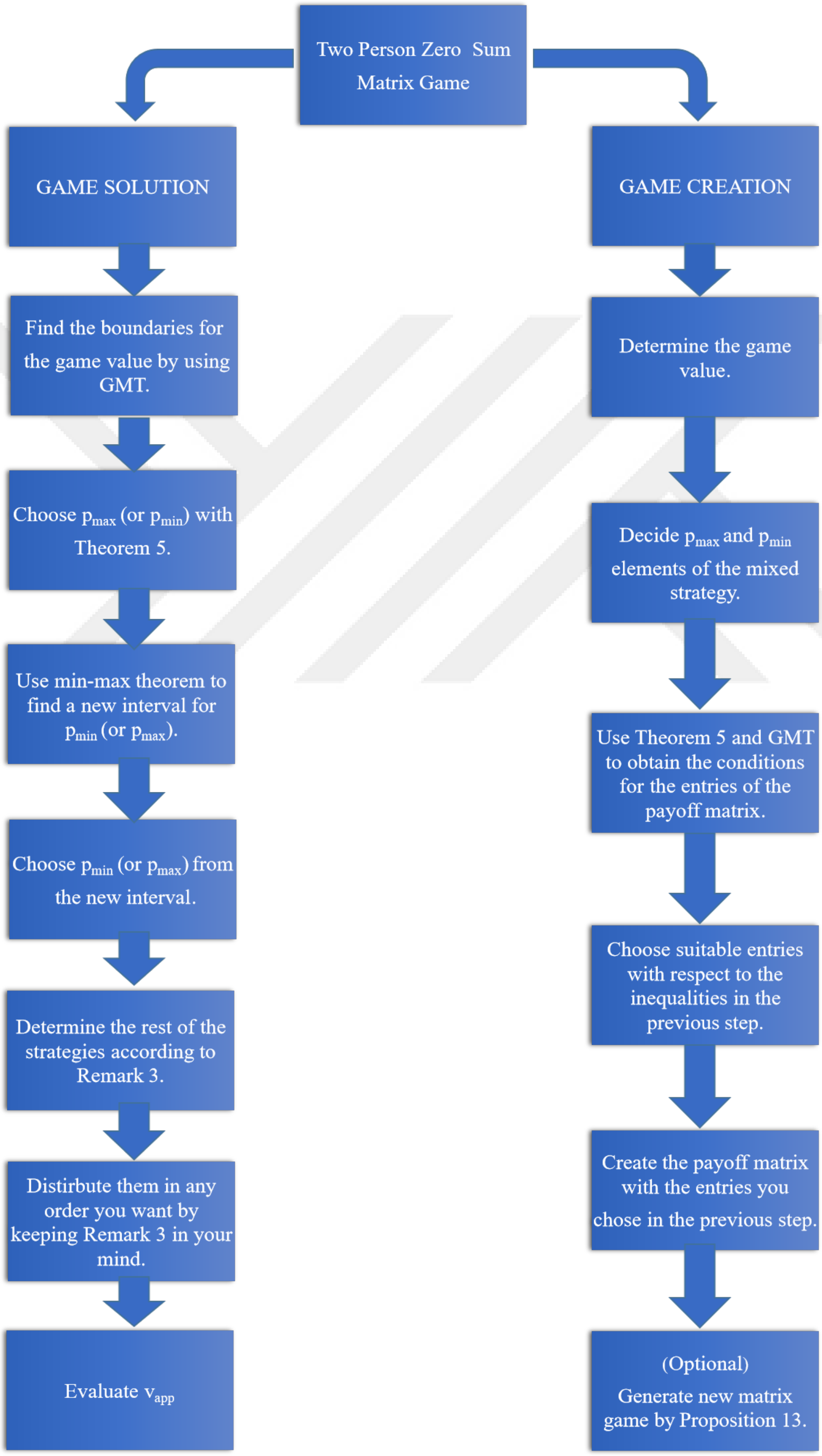

Figure 3.1 : Flowchart for the solution and creation of the two person zero sum matrix games

4. CONCLUSIONS

Firstly, we present some basic concepts for the 3-dimensional matrices in this study. In addition to this, we prove some important propositions by using the new definitions for the 3-D matrices. We define the determinant vector and condition number vectors for the 3-D matrices. We also present the singular and nonsingular 3-D matrices based on the definition of the determinant vector. Beside these, we also introduce the ill-conditoned and well-conditioned 3-dimensional matrix by using the definition of the condition number vector. Moreover, we state and prove the Cauchy-Schwarz inequality and some other inequalities about the 3-D matrix norms. Furthermore, we demonstrate that the $Frobenius-norm$ is invariant under multiplication by a unitary matrix and the $2-norm$ is invariant under left-handside multiplication by a unitary matrix. Finally, we exemplify the usage of the new and extended definitions for 3-dimensional matrices. We believe that these definitions and propositions will contribute to the development of the 3-dimensional matrix theory.

Secondly, in addition to the fundamental concept of the 3-dimensional matrices, we present the 3-dimensional matrix norm inequalities. We prove the 3-D matrix norm inequalities with a similiar approach as it is proved for the 2-dimensional matrix norm inequalities. The relationships between these 3-D norms can be clearly seen with the inequalities. Moreover, the upper and lower bound can be quickly and approximately found for any 3-dimensional matrix norm by using the related 3-D matrix norm inequalities and norms which can be easily computed such as $||\cdot||_{\infty}, ||\cdot||_1$ norms. Therefore, the computational cost for the analysis, which uses the 3-D matrix norms, decreases. Hence, the evaluation process will be completed fast. The situations ,which we have to select data depending on the norm values, may be done quickly by calculating one of the following norms $||\cdot||_{\infty}, ||\cdot||_1$, that can be easily evaluated. In the case that we need to use other norms, we can use the 3-D matrix norm inequalities and find an interval, that may help us to determine whether we need that norm or not, for those norms. However, it is important to notice that it is tedious and time-consuming to calculate $||\cdot||_F$, $||\cdot||_2$ norms even for 2-dimensional matrices, it will obviously be more time-consuming than 2-dimensional matrix norms for 3-D matrix norms since the 3-dimensional matrices have more entries. For this reason, it may be easier to find the values of those norm with a proper choices by using the norm inequalities for 3-D matrices. On the other hand, one may use the 3-dimensional matrix norm inequalities to analyze the matrix based algorithms since they require the usage of the matrix norms. One may switch from one norm to another easily. As a result of this, analysis of some simulations and algorithms may be completed with a smaller computational cost. Consequently, the matrix norms, which have a wide range of use in science, get a new point of view and dimension with the 3-dimensional matrices.

Thirdly, we introduce the matrix norms to the game theory. Even though the matrices are used in the theory of game, the matrix norms do not take a part in it. Therefore, we bring them together and present a new point of view to the game theory. Another, interesting part of this study is that the studies about game theory mostly centered upon solving a game and there are very less of paper that study on game creation in the literature. Therefore, we mainly focus on creating a zero sum matrix game in addition to solving it in this paper. In order to combine the matrix norms and the game theory, we stated and proved some theorems, which are based on $1-norm$ and $\infty-norm$ of the payoff matrix, for the game value of a two person zero sum matrix game. Hence, we present a new perspective to the solution and creation of the two person zero sum game by using the matrix norm of the payoff matrix. We state and prove some theorems for the game value so that we achieve to obtain boundaries, which are only based on $1-norm$ and $\infty-norm$ of the payoff matrix, for the game value. Moreover, we show how to obtain the game value of the pertubated matrix game with the original payoff matrix while the mixed strategy set is invariant.

Moreover, we work on the maximum and minimum elements of the mixed strategy set and exhibit some theorems. We find the lower and upper boundaries of p_{max} and p_{min} for $m\times n$ matrix game with these theorems, respectively. Additionally, we demonstrate the relationship between p_{max} and p_{min} with min-max theorem for the game strategies.

We also examine and show the consistency of our approaches with some test examples. First, we create two person zero sum matrix games by using our theorems. Clearly, it can be seen that the given inequalities and theorems may be used easily for 2×2 matrix games. However, we generalize our methods for $m \times n$ zero sum matrix game. In this case, there will be $m.n$ entries to determine, which means that the number of conditions to be determined will be more than in the case of 2×2 zero sum matrix games. So, it may be difficult to decide all conditions. However, it is not impossible to determine it all if we carefully use the inequalities for the game value v, p_{max} and p_{min}.

Then, we exhibit the applications of our method to the perturbated games and obtain their payoff matrix and the game values. Moreover, we analyze the simulation results of a real life military problem, which is designed as a two person zero sum game and is solved with linear programming method in the corresponding paper, and we find the approximated game value for this game. Comparison of our result with the actual game value shows that the approximated game value is very close to the actual one.

Although any zero sum matrix game can be solved by using linear programming or other well-known methods in the literature, our approaches may solve and obtain approximated game value without solving any linear system. If we consider a big size matrix game, the usage of linear programming methods may be tedious and time-consuming. On the other hand, one may get an approximated game value faster while the computational cost may decrease by using our approaches. In addition to simplicity of our methods' application, this is one of the most important result and advantage part of our work. We believe that this study bring a new point of view to the game solution and creation process.

Consequently, we generalized some definitions and properties of the 2-dimensional matrices to 3-dimensional matrices. Moreover, we introduced the 3-dimensional matrix norm inequalities. Moreover, we combined the game theory with the matrix norms. We believe that all these contributions of 3-dimensional matrices will help the development of the 3-dimensional matrix theory. We also believe that we brought a new perspective to the game theory with the use of matrix norms.

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