ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE
ENGINEERING AND TECHNOLOGY

ESTIMATION OF STRESS - STRENGTH RELIABILITY FOR A NON - IDENTICAL - COMPONENT - STRENGTHS SYSTEM BASED ON UPPER RECORD VALUES

M.Sc. THESIS

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REKOR DEĞERLERİNE DAYALI BENZER - OLMAYAN - BİLEŞENLERLİ SİSTEM İÇİN STRES - DAYANIKLILIK GÜVENİLİRLİĞİNİN TAHMİNİ

YÜKSEK LİSANS TEZİ

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TEMMUZ 2017

Tau Raphael RASETHUNTSA, a M.Sc. student of ITU Graduate School of Science Engineering and Technology 509141208 successfully defended the thesis entitled "ESTIMATION OF STRESS - STRENGTH RELIABILITY FOR A NON - IDENTICAL - COMPONENT - STRENGTHS SYSTEM BASED ON UPPER RECORD VALUES", which he/she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

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Date of Submission : **05 June 2017**
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To my spouse Pulane Patricia Lehloara and our unborn children

FOREWORD

As my time in Turkey comes to an end, I cannot help but be in awe at how much I have learnt in my short time at Istanbul Technical University, and more so at the fact that a human can achieve beyond his own expectations. ’The kingdom of God is indeed within every man’ or as Stefan Zweig puts it, ’the more a man limits himself, the nearer he is on the other hand to what is limitless (∞)’. I would like to thank my supervisor Dr Mustafa Nadar for introducing me to the topic which led to this work and for taking me as his student when it seemed like coming to Turkey was a futile venture and everything was gloom. I am grateful to have been taught by Prof. Nalan Antar because though she delivered her Partial Differential Equations course in Turkish, surprisingly I learnt a lot from her classes. Special thanks to Burak Duman, Tugay Taşçı, and Ahmet Alperen Ölgev for their help with turkish and making me feel like their kin. I would also like to thank my family for their support and my spouse for giving me a reason to look forward to the future. Last but not least I thank the Turkish government for sponsoring my education and I hope they continue to take more students from the poorest countries which the world looks down on.

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ABBREVIATIONS

cps	**:** coverage probabilities
CI	**:** Asymptotic confidence interval
CR	**:** Bayesian credible interval
HPD	**:** Highest probability density
MSE	**:** Mean squared error
MLE	**:** Maximum likelihood estimation
MCMC	**:** Markov chain Monte Carlo
SE	**:** Squared error
UMVUE	**:** Uniformly minimum unbiased estimator

SYMBOLS

R_{s,k_1,k_2}	: Reliability parameter
$\bar{L}_{CI}$	: Average asymtotic inteval lengths
$\bar{L}_{CR}$	: Average Bayesian credible interval lengths
$\hat{R}^{L}_{s,k_1,k_2}$	: Lindley aprroximate of R_{s,k_1,k_2}
$\hat{R}_{s,k_1,k_2}$	: MLE of R_{s,k_1,k_2}
$\hat{R}^{MC}_{s,k_1,k_2}$	: MCMC estimate of R_{s,k_1,k_2}
Kw-G	: Kumaraswamy family of generalized distributions
$G(.)$	: Cumulative distribution function of baseline distribution
$F(.)$	: Cumulative distribution function
$Gamma(a,b)$	: Gamma distribution with shape parameter a, and rate parameter b

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SUMMARY

Since its inception in 1956, the stress-strength model has produced hundreds of papers and even now researchers are flocking to take advantage of this simple yet rewarding model. Each paper produced has always tried to fill gaps in the literature or modify the model to suit desired applications, and the present work is no different. Motivated by the lack of literature on multicomponent stress-strength models based on record values, this thesis is an attempt to produce more realistic stress-strength models by deviating from the much studied traditional way of assuming identical strengths for system components. The thesis considers the estimation of stress-strength reliability in a multi-component system with non-identical component strengths based on upper record values from the family of Kumaraswamy generalized distributions. In frequentist estimation, the maximum likelihood estimator (MLE) of the reliability, its asymptotic distribution and asymptotic confidence intervals are constructed. Bayes estimates under symmetric (squared error) and asymmetric (LINEX) loss functions using conjugate prior distributions are derived and corresponding highest probability density (HPD) credible intervals are also constructed. In Bayesian estimation, Lindley approximation and the Markov Chain Monte Carlo (MCMC) method are employed due to lack of explicit forms. For the first time using records, the uniformly minimum variance unbiased estimator (UMVUE) for the multicomponent system reliability parameter is derived for a common and known shape parameter of the stress and strength variates distributions. Comparisons of the performance of the estimators are carried out using Monte Carlo simulations, the mean squared error (MSE), bias, credible sets and coverage probabilities. The similarity in the definitions of both upper and lower record values implies that the present work may be regarded as covering the case of lower record values. Finally, the prevalent natural occurrence of record type data in practice, especially in life tests and industrial tests, leads to a demonstration being presented on how the proposed model may be utilized in materials science and engineering with the analysis of high-strength steel fatigue life data. The example also serves to show that the model may be applied in comparisons problems. The thesis is concluded with possible future considerations for improving the stress-strength model.

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ÖZET

Stres-dayanıklılık modelinde, güvenilirlik, X gücüne sahip bir nesnenin, üzerine Y tarafından uygulanan belli miktarda bir baskıya dayanmasıdır. Matematiksel olarak stres-dayanıklılık güvenilirliği böyle tanımlanır:

$$R = P(X > Y) \tag{1}$$

1956 yılında Bernbaun tarafından ortaya atıldığından bu yana, stres-dayanıklılık modeli hakkında yüzlerce sayfa yazı yazılsa da araştırmacılar halen basit ama değerli olan bu modelden faydalanmak için uğraşmaktadırlar. Üretilen her bir çalışma daima ya literatürdeki boşlukları doldurmaya ya da istenilen uygulamalara cevap verebilmek için modeli değiştirmeye yönelik olmuştur ve burada yapılan çalışmaların da bundan bir farkı yoktur. Temel model $R = P(X > Y)$, iki veya daha fazla bileşenden oluşan bir sistem durumunda genişletilebilir. Bu sistemin, k-dan-s: G çıkışlı sistemi olarak bilinen, ortak bir strese sahip, bağımsız ve aynı şekilde dağıtılan (i.i.d.) güç bileşenlerinden oluştuğu varsayılmaktadır. Sistem, k, $(1 \leq s \leq k)$ 'dan çıktığında işlev görüyor ve bileşenler strese dayanabiliyor. Matematiksel olarak model ağşadaki gibi tanımlanır:

$$\begin{aligned} R_{s,k} &= P(\text{en az s } (X_1,...,X_k)\ Y'\text{den daha buyuk}) \\ &= \sum_{j=s}^{k} \binom{k}{j} \int_0^\infty [1-F(x)]^j [F(x)]^{k-j} dG(x) \end{aligned} \tag{2}$$

$(X_1,...,X_k)$ i.i.d'dir, cdf $F(.)$ ve Y ortak stresdir ve rastgele güçlükler cdf $G(.)$'dir. Çok bileşenli stres-dayanıklılık güvenilirliği ile ilgili mevcut literatürün çokluğununda i.i.d rastgele değişkenler olarak varsayıldığı dikkat çekmektedir. Bununla birlikte, bir sistemin bileşenlerinin yapıları farklı olduğunda, dayanıklılık değişkeni üzerindeki benzerlik varsayımı geçerli değildir. Örneğin mekanikte istenen mekanik özelliklerin elde edilmesi için ısıyla işlemden geçirme, söndürme veya soğutma madde üzerinde çeşitli çatlamalara neden olabilir. Kaynak veya sert lehimleme gibi birleştirme işlemleri, kaynak alanındaki döküm hatalarına, ve bitişik ısı etkilenen bölgelerdeki çatlaklara neden olabilir ve sonuçta bileşenler farklılaşır. En azından, sistem bileşenlerini benzer olmayan rastgele güçlülüğe sahip sayan bir model doğal olarak daha gerçekçi bir fikir gibi görünmektedir. Tamamen farklı olasılık dağılımlarını takip eden güç değişkenlerine sahip bir model bu açıdan daha cazip. Öyleyse, k_1 bileşenlerin 1. tip ve kalan $k_2 = k - k_1$ bileşenleri 2. tip olan k bileşenlerini içeren bir sistem olduğunu varsayalım. i. tip türünün bileşenleri için rasgele bileşen güçlerinin dağılım fonksiyonu olarak F_i, $i = 1,2$ olsun. Tüm bileşenlerin bir dağıtım fonksiyonu H ile ortak bir gerilime Y maruz bırakıldığını varsayalım. Sistem, k bileşenleri işlevinin

dışında s olduğu sürece çalışır. Sözü edilen model için (2), buna göre modifiye edilebilir.

$$R_{s,k_1,k_2} = \sum_{j_1=s_1}^{k_1} \sum_{j_2=s_2}^{k_2} \left(\prod_{i=1}^{2} \binom{k_i}{j_i} \right) \int_0^\infty \prod_{i=1}^{2} \left([1 - F_i(x)]^{j_i} [F_i(x)]^{k_i - j_i} \right) dH(x) \tag{3}$$

Bu fikir, birden fazla kategori bileşen türüne kadar genişletilebilir. $k_1 = 0$ ise (3) (2) da bilinen s -out-of-k sistem güvenilirlik modeline indirgenir

$$R_{s,k_2} = \sum_{j_2=s_2}^{k_2} \binom{k_2}{j_2} \int_0^\infty [1 - F_2(x)]^{j_2} [F_2(x)]^{k_2 - j_2} dH(x).$$

$k_1 = 0$ ise ve $k_2 = s = 1$ ise, (3) (1) $R = P(X > Y)$ temel modele indirgenir

$$R = \int_0^\infty [1 - F_2(x)][F_2(x)] dH(x)$$

Bugüne kadar stres-dayanıklılık güvenilirliği tahmininde yapılan çalışmaların çoğunun, tam veya sansürlü örneklerin kullanılmasını gerektirdiğini ve rekor değerleri ile çok şey yapılmadığını ve daha fazlasını belirtmek gerekir, özellikle de rekorlarla çok bileşenli sistem güvenilirliğinin tahmini. Endüstriyel stres testleri gibi bazı çalışmalarda, tüm gözlemler dikkate alınmaz, ancak ölçümler sıralı yapılabilir ve yalnızca önceki değerlerden daha büyük veya daha düşük değerler kaydedilir. Bu tür veriler rekor değerleri olarak bilinir. Yapılan ölçümlerin sayısı bu nedenle tam numune boyutundan küçüktür. Bu, tüm numunenin yok edilebileceği yıkıcı testlerde çok önemli olabilir. Rekor değeri verileri doğal olarak çeşitli bağlamlarda ve pratik durumlarda ortaya çıkar. Meteorolojik analiz, spor ve atletizm olayları, petrol ve madencilik anketleri gibi örnekler verilebilir. Ayrıca, bazı hidrolojik ve maddi test verileri doğal olarak rekor tipinde olduğu fark edilmiştir. Bu çalışmada, rekor değerlerine dayalı çok bileşenli stres-dayanıklılık modelleri sistem bileşenlerini benzer (identical) kabul etmeyerek, daha gerçekçi stres-dayanıklılık modellerinin üretilmesi amaçlanmıştır. Bu modelleme, genelleştirilmiş Kumaraswamy dağılım ailesi için gerçekleştirilmiş ve çok bileşenli sistemdeki stres-dayanıklılık güvenilirliğinin tahmini çeşitli istatistiksel yaklaşımlarla değerlendirilmiştir. Klasik istatistiksel tahmin yöntemleriyle stres-dayanıklılık güvenilirliğin en çok olabilirlik tahmin edicisi (MLE), asimptotik dağılım ve asimptotik güven aralıkları oluşturulmuştur. Diğer yandan, Bayesci tahmin yöntemleriyle, simetrik (karesel hata) ve asimetrik (LINEX) kayıp fonksiyonları altında, eşlenik önsel dağılımlar kullanılarak stres-dayanıklılık güvenilirliğini tahmin edicisi ve Bayes en yüksek olasılık yoğunluğu (HDP) güven aralıkları elde edilmiştir. Bayes tahmin yönteminde, Lindley yaklaşımı ve Markov zinciri Monte Carlo (MCMC) yöntemi, açık formüllerin eksikliği nedeniyle kullanılmıştır. Bu çalışmada yeni olan, bileşenlerin aynı dağılımdan ancak benzer olmayan parametre varsayımı altında rekor değerlerine dayalı, çok bileşenli sistem güvenilirliği parametresi için düzgün en küçük varyanslı yansız tahmin edicisi (UMVUE) elde edilmiş olmasıdır. Tahmin edicilerin karşılaştırmaları Monte Carlo simülasyonları, karesel ortalama hata, bias, credible sets ve kapsama olasılıkları kullanılarak gerçekleştirilmiştir. Üst ve alt rekor değerlerinin tanımlarındaki benzerlik, mevcut çalışmanın alt rekor değerler içinde elde edilebilir. Son olarak, pratikteki rekor türü verilerinin yaygın doğal oluşumu, özellikle de yaşam sınamalarında ve endüstriyel sınamalarda, yüksek mukavemetli çelik yorma yaşam verilerinin analiziyle, bu çalışmada önerilen modelden malzeme bilimi ve mühendisliğinde nasıl faydalanılabileceği gösterilmiştir. Bu örnek ayrıca modelin karşılaştırma problemlerinde uygulanabileceğini de göstermektedir.

Tezin aşağıdaki şekilde düzenlenmiştir. 2. bölümde, R_{s,k_1,k_2}' daki MLE ve Bayesçi tahminciler, yaygın ve bilinmeyen bir şekil parametresi α için türetilmiştir. Bayesian tahmini altında, önceki davada SE ve LINEX kayıp fonksiyonları altında Lindley yaklaşımı ve MCMC yöntemi kullanılmıştır. İlgili asimtotik aralıklarla birlikte HPD güvenilir aralıkları da oluşturulmuştur. 3. bölümde, α bilindiğinde R_{s,k_1,k_2} 'lık MLE ve Bayesçi tahmincileri türetilir. Yaklaşık, kesin ve HPD güvenilir aralıkları oluşturulmuştur. Ek olarak, rekor değerlerine dayalı R_{s,k_1,k_2} UMVUE sunulmuştur. 4. bölümde tam örnekleri kullanarak R_{s,k_1,k_2} MLE ve UMVUE türetilmiş ve kayıtların durumu ile karşılaştırılmıştır 5. bölümde, tahmin edicileri karşılaştırmak için sayısal denemeler ve Monte Carlo simülasyonları gerçekleştirilmiştir. 5.1. alt bölümde, modelin nasıl kullanılacağına dair bir illüstrasyon, yüksek mukavemetli yorulma ömrü verisinin analizi ile gösterilmiştir. Tez, 5.2. alt bölümde stres-dayanıklılık modelinin gelişimine yönelik olası gelecek yönelik çalışma önerileriyle tez sonuçlandırılmıştır.

1. INTRODUCTION

In a stress-strength model, reliability refers to the ability of an object with a strength X to withstand a certain amount of stress Y exerted on it. If at some point the stress Y exceeds the strength X, the object will cease to function properly. Due to the inherent uncertainties in the constituents of the environment in which the system lives such as pressure, temperature or humidity as well as the uncertainties in the object's strength (or resistance) due to factors such as material composition or design style, X and Y are assumed to be random in nature. It is therefore reasonable to quantify this quantity, denoted by R, probabilistically as $R = P(X > Y)$. In other words, reliability is defined as the probability that the object has enough strength to withstand the stress. This idea was first introduced by [1] and later developed by [2]. Since then hundreds of papers have been published on this simple model alone. Despite its simplicity, the stress-strength model is very useful and arises frequently in different branches of science and engineering such as life testing and clinical trials [3]. The basic model $R = P(X > Y)$ may be extended to the case of a system made up of two or more components. Reliability in such a multicomponent stress-strength model was first developed by [4]. This system, known as an s-out-of-k: G system, is assumed to be made up of k independent and identically distributed (i.i.d.) strength components with a common stress. The system is considered to be functioning as long as s out of k, $(1 \leq s \leq k)$, components can withstand the stress. Mathematically the model is defined as

$$\begin{aligned} R_{s,k} &= P(\text{at least s of } (X_1, \ldots, X_k) \text{ exceed } Y) \\ &= \sum_{j=s}^{k} \binom{k}{j} \int_0^\infty [1 - F(x)]^j [F(x)]^{k-j} dG(x) \end{aligned} \tag{1.1}$$

where the $(X_1, \ldots, X_k)$ are the i.i.d. random strengths with cdf $F(.)$ and Y is the common stress variate with cdf $G(.)$.

1.1 What Has Been Done So Far

Estimation of reliability in a multicomponent system using (1.1) for various distributional assumptions on the strengths and stress variates has been extensively covered in literature. [5] studied system reliability when the $(X_1,...,X_k)$ follow the absolutely continuous multivariate exponential distribution while Y follows an independent exponential distribution. [6] studied the classical and Bayesian estimation of reliability in a multicomponent system assuming the Weibull distribution for stress and strength variates. The Generalized exponential, Burr type III, Log-logistic, and Inverse Reyleigh distributions cases were considered by [7], [8], [9] and [10] respectively among many others. The most recent works include the use of bivariate distributions by [11], [12]. The system components therein are constructed by a pair of s-independent elements $(X_1,Y_1),(X_2,Y_2),...,(X_k,Y_k)$ following the Marshal-Olkin Bivariate Weibull or bivariate Kumaraswamy distributions with a common stress T acting on all the components. A comprehensive review of the literature on classical and bayesian estimation of stress-strength reliability and some of its parallels is provided by [13], [14] and [15]. It is worth noting that much of existing literature on multicomponent stress-strength reliability assumes strength variates to be i.i.d. random variables. However, when the structures of the components of a system are different, the identicality assumption on strength variates is not practical, [15]. For instance, in mechanics, heat treating to obtain desired mechanical properties can cause various types of cracking upon quenching or cooling. Joining operations such as welding or brazing can result in casting defects in the weld area as well as cracks in the adjacent heat-affected zones, ([16]) and this ultimately makes the components different, however slight the difference may be. A model which at least considers system components to have non-identical random strengths naturally seems a more realistic idea. A model with strength variates following completely different probability distributions is more appealing in this regard. However, mathematical tractability of resulting expressions has proven daunting and progress is often stalled. Progress in this direction would yield more realistic models.

1.2 Model Description

Suppose that there is a system consisting of k components of which k_1 of the components are of type 1, k_2 are of type 2,..., and the remaining $k_n = k - \sum_{i=1}^{n} k_i$ components are of type n. Let F_i , $i = 1,2,...,n$ be the distribution function of the random component strengths for components of the i-th type. Assume that all components are exposed to a common stress Y with a distribution function H. For the aforementioned model, (1.1) can be modified accordingly to be

$$R_{s,k_1,...,k_n} = \sum_{j_1=0}^{k_1} \cdots \sum_{j_n=0}^{k_n} \left(\prod_{i=1}^{n} \binom{k_i}{j_i} \right) \int_0^\infty \prod_{i=1}^{n} \left([1 - F_i(x)]^{j_i} [F_i(x)]^{k_i - j_i} \right) dH(x) \quad (1.2)$$

where summation ranges over all possible combinations $(j_1, j_2, ..., j_n)$ with $0 \leq j_i \leq k_i$ for $i = 1,2,...,n$ such that $s \leq \sum_{i=1}^{n} j_i \leq k$. In the present work, the case of a system with two types of components is investigated. The system is regarded as working if s_1 and s_2 components of types 1 and 2 respectively can withstand the common stress. In this case, the model (1.2) can be written as follows

$$R_{s,k_1,k_2} = \sum_{j_1=s_1}^{k_1} \sum_{j_2=s_2}^{k_2} \left(\prod_{i=1}^{2} \binom{k_i}{j_i} \right) \int_0^\infty \prod_{i=1}^{2} \left([1 - F_i(x)]^{j_i} [F_i(x)]^{k_i - j_i} \right) dH(x) \quad (1.3)$$

where summation ranges over all possible pairs (j_1, j_2) with $0 \leq j_1 \leq k_1$ and $0 \leq j_2 \leq k_2$ such that $s \leq j_1 + j_2 \leq k$,see [15]. This idea can be extended to more than two category types of components. Furthermore, one can assume each component to be of a different type. This idea is demonstrated later on for a much simplified version of the model (1.3). When $k_2 = 0$ then (1.3) reduces to the well known s-out-of-k system reliability model in (1.1)

$$R_{s,k_1} = \sum_{j_1=s_1}^{k_1} \binom{k_1}{j_1} \int_0^\infty [1 - F_1(x)]^{j_1} [F_1(x)]^{k_1 - j_1} dH(x).$$

If $k_2 = 0$ and $k_1 = s = 1$, then (1.3) reduces to the fundamental model $R = P(X > Y)$

$$R = \int_0^\infty [1 - F_1(x)][F_1(x)] dH(x)$$

The model (1.3) is not new to statistical literature. [17] and [18] investigated the MLE and Bayesian estimation of R_{s,k_1,k_2} assuming the Weibull and exponential distributions on the strength and stress variates respectively. Hassan et al. [19] considered various estimation methods for R_{s,k_1,k_2} when the non-identical component strengths and stress variates follow the exponentiated Pareto distribution. One of the objectives of this article is to improve the inference methods of the model given recent considerable development in the stress-strength reliability models. It is also worth pointing out that most of the work done so far on estimation of stress-strength reliability assumes the use of complete or censored samples and a lot has not been done with record values, ([20]), and more so estimation of multicomponent system reliability with records.

1.3 What are Record Values?

In some studies, such as industrial stress tests, not all observations are considered but measurements may be made sequentially and only values larger (or smaller) than all previous ones are recorded. Such data is known as record data. The number of measurements made is therefore smaller than the complete sample size. This can be crucial in destructive sampling where all the sample may be destroyed. Record value data arise naturally in a variety of contexts and practical situations. [21] gives examples such as meteorological analysis, sports and athletics events, and oil and mining surveys. Furthermore, some hydrological and material-testing data have been noticed to be naturally of records type by [22]. Formally, if X_1, X_2,... is an infinite sequence of i.i.d. random variables from a continuous distribution with cdf F and pdf f, then an observation X_j is called an upper record value (or simply record) if its value exceeds that of all previous observations. Thus, X_j is a record if $X_j > X_i$ for every $i < j$. A similar definition can be given for lower record values. The main concept of records was first presented by [23] and detailed theory and methods of statistical inference based on records was later developed by [24], [25], as well as [26]. Estimation of $R = P(X > Y)$ based on records was studied by [27], [3], [28], [29], [**?**] and recently by [30] among many others.

1.4 Kumaraswamy Generalized Family of Distributions

The past decades have seen an enormous increase in the interest to develop new and more flexible statistical distributions. The spark in interest has been motivated mainly by an apparent need to find models that are a better fit to our modern real data which is often characterized by high to moderate degrees of skewness and kurtosis. New distributions are discovered almost daily and they are developed to model various kinds of data from fields such as biology, economics, reliability engineering and many others. Recently there has been a further increased interest in defining new families of continuous distributions by introducing additional shape parameters to already existing parent distributions. These new classes of distributions so generated, contain not only new distributions which provide better fits, but also contain existing distributions as special sub-models as well. Thus this enables one to study a variety of distributions in one go with a single representation. From [31]'s idea of a class of beta generalized distributions and a distribution for double bounded random processes introduced by [32], [33] constructed an interesting new family of generalized distributions, the Kumaraswamy generalized (*Kw*-*G*) distributions. This class is defined as follows: For any parent distribution function $G(x)$, the cumulative distribution function (cdf) of the *Kw*-*G* distribution is given by

$$F(x) = 1 - \{1 - G(x)^{\alpha}\}^{\beta}, \tag{1.4}$$

where $\alpha, \beta > 0$ are additional shape parameters to the G distribution which introduce skewness and vary tail weights.Its probability density function (pdf) is given by

$$f(x) = \alpha\beta g(x) G(x)^{\alpha-1}\{1 - G(x)^{\alpha}\}^{\beta-1} \tag{1.5}$$

where

$$g(x) = \frac{d}{dx}G(x) \tag{1.6}$$

If X is a random variable having the pdf (1.5), it will be denoted by $X \sim Kw\text{-}G(\alpha, \beta)$. It is also imperative to note that the parameters of the underlying baseline distribution, G, are assumed known in the model. Furthermore, in order to avoid complications with mathematical tractability of resulting pdf formulas when using (1.5), caution must be exercised in choosing a baseline distribution. Since its inception nearly seven years ago, the *Kw*-*G* distribution has received a considerable amount of attention from the statistical community, with over 282 citations to date. Its versatility and effectiveness in a variety of situations has been portrayed in numerous papers. From modelling the number of millions of revolutions reached by ball bearings before fatigue failure by [34], to modelling the number of absences among shift-workers in a steel industry by [35]. The introduction of the two parameters α and β allows the *Kw*-*G* distribution to assume a wide range of shapes. Which is an ideal ability in data fitting and modeling. According to [33], because of its tractability, the *Kw*-*G* distribution can be effective even if the data are censored and one of its major benefits is its ability of fitting skewed data that can not be properly fitted by existing distributions. [36] demonstrated this ability by applying the Kumaraswamy Weibull distribution to failure data. The *Kw-G* family of distributions includes as special models the Kumaraswamy distribution as well as the Beta-generalized distribution by Eugene among several others. It is also obvious that for $\alpha = \beta = 1$, $Kw\text{-}G \equiv G$. A physical intepretation of (1.4) given by [33] when α and β are positive integers is as follows. Consider a system formed by β independent series components and that each compomponent is made up of α parallel independent subcomponents. The system fails if any of the β components fails and each component fails if all of the α subcomponents fail. The time to failure distribution of the entire system has precisely the *Kw-G* distribuiton. A recent and lucid account of literature on the applications of *Kw-G* distributions to date can be found on the doctoral thesis by [37].

2. ESTIMATION OF R_{s,k_1,k_2} FOR UNKNOWN α I

2.1 Maximum Likelihood Estimation of R_{s,k_1,k_2}

From the total of k system components in the model (1.3), let the first k_1 type 1 component strengths follow a *Kw-G* distribution with parameters α and $\beta = \beta_1$, while the remaining $k_2 = k - k_1$ type 2 component strengths follow a *Kw-G* distribution with parameters α and $\beta = \beta_2$. Assume further that Y follows a *Kw-G* distribution with parameters α and β_3 independently. The respective distribution functions are

$$\left.\begin{aligned} F_1(x) &= 1 - \{1 - G(x)^{\alpha}\}^{\beta_1} \\ F_2(x) &= 1 - \{1 - G(x)^{\alpha}\}^{\beta_2} \\ H(x) &= 1 - \{1 - G(x)^{\alpha}\}^{\beta_3} \end{aligned}\right\} \tag{2.1}$$

Substitution of (2.1) into (1.3) yields :

$$\begin{aligned} R_{s,k_1,k_2} &= \sum_{j_1=s_1}^{k_1}\sum_{j_2=s_2}^{k_2}\left(\prod_{l=1}^{2}\binom{k_l}{j_l}\right)\int_0^{\infty}\prod_{l=1}^{2}\left([1-G(x)^{\alpha}]^{j_l\beta_l}[1-\{1-G(x)^{\alpha}\}^{\beta_l}]^{k_l-j_l}\right) \\ &\quad \times \beta_3\alpha g(x)G(x)^{\alpha-1}[1-G(x)^{\alpha}]^{\beta_3-1}\,dx, \text{ set } t = 1-G(x)^{\alpha} \\ &= \sum\nolimits_{\mathbf{k}}^{\mathbf{s}}\int_0^1 u^{\left(\sum_{i=1}^{2}(j_i+i_i)\beta_i+\beta_3-1\right)}du \\ &= \sum\nolimits_{\mathbf{k}}^{\mathbf{s}}\left(\frac{\beta_3}{p\beta_1+q\beta_2+\beta_3}\right) \end{aligned} \tag{2.2}$$

where

$$\sum\nolimits_{\mathbf{k}}^{\mathbf{s}} \equiv \sum_{j_1=s_1}^{k_1}\sum_{j_2=s_2}^{k_2}\left(\prod_{l=1}^{2}\binom{k_l}{j_l}\right)\sum_{i_1=1}^{k_1-j_1}\sum_{i_2=1}^{k_1-j_1}\left(\prod_{l=1}^{2}\binom{k_l-j_l}{i_l}(-1)^{i_l}\right) \tag{2.3}$$

and

$$p = j_1 + i_1,\ q = j_2 + i_2 \tag{2.4}$$

It is noted that the reliability expression (2.2) is independent of the common parameter α. Furthermore, it can also be shown that under the same setup, R_{s,k_1,k_2} assumes a similar form for other lifetime distributions such as Weibull, Rayleigh, Gompertz, Burr Type III and several other distributions commonly used to model reliability data. The main goal of this paper is to estimate R_{s,k_1,k_2} using upper records data from the *Kw-G* distribution. In order to find the MLE of R_{s,k_1,k_2}, we first need to obtain the MLEs of the parameters α,β_1, β_2, and β_3. So let $R_1,\dots,R_{n_1},P_1,\dots,P_{n_2}$ and $S_1,\dots,S_m$ be independent random samples of upper records of sizes n_1, n_2 and m from the distributions $Kw\text{-}G(\alpha,\beta_1)$, $Kw\text{-}G(\alpha,\beta_2)$ and $Kw\text{-}G(\alpha,\beta_3)$ with relializations $\underline{r}=(r_1,\dots,r_{n_1})$, $\underline{p}=(p_1,\dots,p_{n_1})$, and $\underline{s}=(s_1,\dots,s_m)$ respectively. The respective likelihood functions of the observed samples of records as given by [25] are

$$\begin{aligned}\mathscr{L}_1(\alpha,\beta_1|\underline{r}) =& f(r_{n_1})\prod_{i=1}^{n_1-1}\frac{f(r_i)}{1-F(r_i)},\quad 0<r_1<\dots<r_{n_1}<\infty\\ =&(\alpha\beta_1)^{n_1}g(r_{n_1})G(r_{n_1})^{\alpha-1}\{1-G(r_{n_1})^{\alpha}\}^{\beta_1-1}\prod_{i=1}^{n_1-1}\frac{g(r_i)G(r_i)^{\alpha-1}}{1-G(r_i)^{\alpha}}\\ =&(\alpha\beta_1)^{n_1}\{1-G(r_{n_1})^{\alpha}\}^{\beta_1}\prod_{i=1}^{n_1}\frac{g(r_i)G(r_i)^{\alpha-1}}{1-G(r_i)^{\alpha}}\end{aligned}\tag{2.5}$$

$$\begin{aligned}\mathscr{L}_2(\alpha,\beta_2|\underline{p}) =& f(p_{n_2})\prod_{i=1}^{n_2-1}\frac{f(p_i)}{1-F(p_i)},\quad 0<p_1<\dots<p_{n_2}<\infty\\ =&(\alpha\beta_2)^{n_2}\{1-G(p_{n_2})^{\alpha}\}^{\beta_2}\prod_{i=1}^{n_2}\frac{g(p_i)G(p_i)^{\alpha-1}}{1-G(p_i)^{\alpha}}\end{aligned}\tag{2.6}$$

$$\begin{aligned}\mathscr{L}_3(\alpha,\beta_3|\underline{s}) =& f(s_m)\prod_{i=1}^{m-1}\frac{f(s_i)}{1-F(s_i)},\quad 0<s_1<\dots<s_m<\infty\\ =&(\alpha\beta_3)^{m}\{1-G(s_m)^{\alpha}\}^{\beta_3}\prod_{i=1}^{m}\frac{g(s_i)G(s_i)^{\alpha-1}}{1-G(s_i)^{\alpha}}\end{aligned}\tag{2.7}$$

The joint likelihood function of $(\alpha,\beta_1,\beta_2,\beta_3)$ based on the observed record values $\underline{r}$, $\underline{p}$ and $\underline{s}$ is therefore given by

$$\begin{aligned}\mathscr{L} =& \prod_{i=1}^{3}\mathscr{L}_i\\ =&\alpha^{(n_1+n_2+m)}\beta_1^{n_1}\beta_2^{n_2}\beta_3^{m}\{1-G(r_{n_1})^{\alpha}\}^{\beta_1}\{1-G(p_{n_2})^{\alpha}\}^{\beta_2}\{1-G(s_m)^{\alpha}\}^{\beta_3}\\ &\times\prod_{i=1}^{n_1}\frac{g(r_i)G(r_i)^{\alpha-1}}{1-G(r_i)^{\alpha}}\prod_{i=1}^{n_2}\frac{g(p_i)G(p_i)^{\alpha-1}}{1-G(p_i)^{\alpha}}\prod_{i=1}^{m}\frac{g(s_i)G(s_i)^{\alpha-1}}{1-G(s_i)^{\alpha}}\end{aligned}\tag{2.8}$$

and the corresponding joint log-likelihood function, denoted by l, is written as

$$\begin{aligned}l =&(n_1+n_2+m)\ln\alpha+n_1\ln\beta_1+n_2\ln\beta_2+m\ln\beta_3\\ &+\beta_1\ln[1-G(r_{n_1})^{\alpha}]+\beta_2\ln[1-G(p_{n_2})^{\alpha}]+\beta_3\ln[1-G(s_m)^{\alpha}]\\ &+\sum_{i=1}^{n_1}\ln S(r_i)+\sum_{i=1}^{n_2}\ln T(p_i)+\sum_{i=1}^{m}\ln U(s_i).\end{aligned}\tag{2.9}$$

where

$$S(r_i) = \left(\frac{g(r_i)G(r_i)^{\alpha-1}}{1-G(r_i)^{\alpha}}\right),$$
$$T(p_i) = \left(\frac{g(p_i)G(p_i)^{\alpha-1}}{1-G(p_i)^{\alpha}}\right), \text{ and}$$
$$U(s_i) = \left(\frac{g(s_i)G(s_i)^{\alpha-1}}{1-G(s_i)^{\alpha}}\right).$$

The maximum likelihood estimators of β_i ; $i = 1,2,3$ and α denoted by $\hat{\beta}_i$ and $\hat{\alpha}$ respectively, are found as follows

$$\frac{\partial l}{\partial \beta_1} = \frac{n_1}{\hat{\beta}_1} + \ln\{1-G(r_{n_1})^{\alpha}\} = 0$$

$$\therefore \quad \hat{\beta}_1 = -\frac{n_1}{\ln\left[1-G(r_{n_1})^{\hat{\alpha}}\right]} \tag{2.10}$$

and similarly

$$\hat{\beta}_2 = -\frac{n_2}{\ln\left[1-G(p_{n_2})^{\hat{\alpha}}\right]}, \tag{2.11}$$

and

$$\hat{\beta}_3 = -\frac{m}{\ln\left[1-G(s_m)^{\hat{\alpha}}\right]}. \tag{2.12}$$

The maximum likelihood estimator of the common parameter α on the other hand is the solution to the non-linear equation

$$\frac{\partial l}{\partial \alpha} = \frac{(n_1+n_2+m)}{\alpha} - \frac{\beta_1 G(r_{n_1})^{\alpha}\ln G(r_{n_1})}{(1-G(r_{n_1})^{\alpha})} - \frac{\beta_2 G(p_{n_2})^{\alpha}\ln G(p_{n_2})}{(1-G(p_{n_2})^{\alpha})}$$
$$- \frac{\beta_3 G(s_m)^{\alpha}\ln G(s_m)}{(1-G(s_m)^{\alpha})} + \sum_{i=1}^{n_1}\left[\frac{\ln G(r_i)}{1-G(r_i)^{\alpha}}\right] + \sum_{i=1}^{n_2}\left[\frac{\ln G(p_i)}{1-G(p_i)^{\alpha}}\right] \tag{2.13}$$
$$+ \sum_{i=1}^{m}\left[\frac{\ln G(s_i)}{1-G(s_i)^{\alpha}}\right] = 0$$

Therefore, $\hat{\alpha}$ can be obtained as a solution to the equation $\lambda(\alpha) = \alpha$, where

$$\lambda(\alpha) = -(n_1+n_2+m)\left[\frac{n_1 G(r_{n_1})^{\alpha}\ln G(r_{n_1})}{\ln[1-G(r_{n_1})^{\alpha}](1-G(r_{n_1})^{\alpha})}\right.$$
$$+ \frac{n_2 G(p_{n_2})^{\alpha}\ln G(p_{n_2})}{\ln[1-G(p_{n_2})^{\alpha}](1-G(p_{n_2})^{\alpha})} + \frac{m G(s_m)^{\alpha}\ln G(s_m)}{\ln[1-G(s_m)^{\alpha}](1-G(s_m)^{\alpha})} \tag{2.14}$$
$$\left. + \sum_{i=1}^{n_1}\left[\frac{\ln G(r_i)}{1-G(r_i)^{\alpha}}\right] + \sum_{i=1}^{n_2}\left[\frac{\ln G(p_i)}{1-G(p_i)^{\alpha}}\right] + \sum_{i=1}^{m}\left[\frac{\ln G(s_i)}{1-G(s_i)^{\alpha}}\right]\right]^{-1}$$

It is clear that $\hat{\alpha}$ is a fixed point of the equation $\lambda(\alpha) = \alpha$ and can therefore be obtained via an iterative scheme as follows

$$\lambda(\alpha_i) = \alpha_i \tag{2.15}$$

Where α_i is the i-th iterate of $\hat{\alpha}$. The iterative procedure will be halted when the quantity $|\alpha_{i+1} - \alpha_i|$ is sufficiently small. Thus, by the invariance property of maximum likelihood estimation, the MLE of R_{s,k_1,k_2} based on upper record values from the class of Kumaraswamy generalized distributions is given by

$$\hat{R}_{s,k_1,k_2} = \sum\nolimits_{\mathbf{k}}^{\mathbf{s}} \frac{\hat{\beta}_3}{(j_1+i_1)\hat{\beta}_1 + (j_2+i_2)\hat{\beta}_2 + \hat{\beta}_3} \tag{2.16}$$

Theorem 1. *The maximum likelihood estimates of* β_1*,* β_2*,* β_3*, and* α *as given by* (2.10)*,*(2.11)*,*(2.12)*, and* (2.13) *respectively are unique.*

Proof. See [28] for a proof in the fundamental model $R = P(X > Y)$ case using upper record values from the Kumaraswamy distribution. The proof can easily be extended to the model in the present work. □

2.1.1 Asymptotic Confidence Interval For R_{s,k_1,k_2} I

In this subsection we derive the asymptotic distribution of $\hat{\boldsymbol{\theta}} = \left(\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3, \hat{\alpha}\right)$ and from this, the asymptotic distribution of R_{s,k_1,k_2} is derived. We later construct an asymptotic confidence interval based on the asymptotic distribution of R_{s,k_1,k_2}. The expected Fisher information matrix of $\boldsymbol{\theta} = (\beta_1, \beta_2, \beta_3, \alpha)$ is given by $\boldsymbol{\Phi}(\boldsymbol{\theta}) = E\left(\mathbf{I}(\boldsymbol{\theta})\right)$, where $\mathbf{I}(\boldsymbol{\theta}) = \left[I_{ij}(\boldsymbol{\theta})\right] = \left[-\frac{\partial^2 \ell}{\partial\theta_i \partial\theta_j}\right]$ for $i, j = 1,2,3,4$ is the observed information matrix. $I_{11} = \frac{n_1}{\beta_1^2}$, $I_{22} = \frac{n_2}{\beta_2^2}$, $I_{33} = \frac{m}{\beta_3^2}$, and $I_{12} = I_{13} = I_{21} = I_{23} = I_{31} = I_{32} = 0$. What is required now is to find I_{44}.

$$\begin{aligned}
\frac{\partial^2 l}{\partial \alpha^2} =& \frac{-(n_1+n_2+m)}{\alpha^2} - \frac{\beta_1 G(r_{n_1})^\alpha (\ln G(r_{n_1}))^2}{(1-G(r_{n_1})^\alpha)^2} - \frac{\beta_2 G(p_{n_2})^\alpha (\ln G(p_{n_2}))^2}{(1-G(p_{n_2})^\alpha)^2} \\
&- \frac{\beta_3 G(s_m)^\alpha (\ln G(s_m)^\alpha)^2}{(1-G(s_m))^2} + \sum_{i=1}^{n_1} \frac{G(r_i)^\alpha (\ln G(r_i))^2}{(1-G(r_i)^\alpha)^2} \\
&+ \sum_{i=1}^{n_2} \frac{G(p_i)^\alpha (\ln G(p_i))^2}{(1-G(p_i)^\alpha)^2} + \sum_{i=1}^{m} \frac{G(s_i)^\alpha (\ln G(s_i))^2}{(1-G(s_i)^\alpha)^2}.
\end{aligned}$$

In order to determine the expression $E\left(\frac{\partial^2 l}{\partial \alpha^2}\right)$, let

$$\mu(G(r_{n_1})^\alpha) = \frac{\beta_1 G(r_{n_1})^\alpha (\ln G(r_{n_1}))^2}{(1-G(r_{n_1})^\alpha)^2} = \frac{\beta_1 G(r_{n_1})^\alpha \left(\frac{1}{\alpha}\ln G(r_{n_1})^\alpha\right)^2}{(1-G(r_{n_1})^\alpha)^2}. \tag{2.17}$$

The pdf of the n-th upper record R_n is given by

$$f_{R_n}(r_n) = \frac{f(r_n)}{(n-1)!}\left[-\ln\left(1-F(r_n)\right)\right]^{n-1}, \tag{2.18}$$

see [25]. Defining $Y(R_{n_1}) = G(R_{n_1})^{\alpha}$, a simple transformation yields the pdf of Y as

$$f_Y(y) = \frac{\beta^{n_1}}{(n_1-1)!}(1-y)^{\beta-1}\left[-\ln(1-y)\right]^{n_1-1} \tag{2.19}$$

Therefore,

$$\begin{aligned} E\left(\mu(G(r_{n_1})^{\alpha})\right) = & \int_0^1 \mu(y) f_Y(y) dy \\ = & \frac{\alpha\beta^{n_1}}{(n_1-1)!}\int_0^1 y(1-y)^{\beta_1-3}(\ln y)^2\left[\ln\left(\frac{1}{1-y}\right)\right]^{n-1} dy \\ = & \frac{\beta_1^{n_1}}{\alpha^2}\left[\sum_{k=0}^{\infty}\frac{1}{(k+1)}\left(\frac{1}{(\beta_1+k-1)^{n_1}}-\frac{1}{(\beta_1+k)^{n_1}}\right)\sum_{j=1}^{k}\frac{1}{j}\right], \end{aligned} \tag{2.20}$$

the integral (2.20) was evaluated by [28] and tables of integrals used can be found in [38]. Similarly we obtain

$$E\left(\mu(G(p_{n_2})^{\alpha})\right) = \frac{\beta_2^{n_2}}{\alpha^2}\left[\sum_{k=0}^{\infty}\frac{1}{(k+1)}\left(\frac{1}{(\beta_2+k-1)^{n_2}}-\frac{1}{(\beta_2+k)^{n_2}}\right)\sum_{j=1}^{k}\frac{1}{j}\right], \tag{2.21}$$

and

$$E\left(\mu(G(s_m)^{\alpha})\right) = \frac{\beta_3^{m}}{\alpha^2}\left[\sum_{k=0}^{\infty}\frac{1}{(k+1)}\left(\frac{1}{(\beta_3+k-1)^{m}}-\frac{1}{(\beta_3+k)^{m}}\right)\sum_{j=1}^{k}\frac{1}{j}\right] \tag{2.22}$$

Therefore,

$$\begin{aligned} I_{44} = & E\left(\frac{\partial^2 l}{\partial\alpha^2}\right) \\ = & \frac{(n_1+n_2+m)}{\alpha^2} + \left[E\left(\mu(G(r_{n_1})^{\alpha})\right) + E\left(\mu(G(p_{n_2})^{\alpha})\right) + E\left(\mu(G(s_m)^{\alpha})\right)\right] \\ & - \left[\sum_{i=1}^{n_1} E\left(\mu(G(r_i)^{\alpha})\right) + \sum_{i=1}^{n_2} E\left(\mu(G(p_i)^{\alpha})\right) + \sum_{i=1}^{m} E\left(\mu(G(s_i)^{\alpha})\right)\right] \end{aligned} \tag{2.23}$$

with all the expectations as given in (2.20), (2.21), (2.22). Following a similar technique as above, expressions for I_{12}, I_{13} ,I_{14}, I_{21} , I_{31} , and I_{41} are also obtained as follows

$$I_{14} = I_{41} = \frac{\beta_1^{n_1}}{\alpha}\sum_{k=1}^{\infty}\frac{1}{k}\left[\frac{1}{(\beta_1+k)^{n_1}}-\frac{1}{(\beta_1+k-1)^{n_1}}\right], \tag{2.24}$$

$$I_{24} = I_{42} = \frac{\beta_2^{n_2}}{\alpha}\sum_{k=1}^{\infty}\frac{1}{k}\left[\frac{1}{(\beta_2+k)^{n_2}}-\frac{1}{(\beta_2+k-1)^{n_2}}\right], \tag{2.25}$$

and

$$I_{34} = I_{43} = \frac{\beta_3^{m}}{\alpha}\sum_{k=1}^{\infty}\frac{1}{k}\left[\frac{1}{(\beta_3+k)^{m}}-\frac{1}{(\beta_3+k-1)^{m}}\right] \tag{2.26}$$

Theorem 2. *As $n_1, n_2, m \to \infty$, $\frac{n_1}{m} \to q_1$ and $\frac{n_2}{m} \to q_2$,$0 < q_1 < 1, 0 < q_2 < 1$, then*

$$\begin{bmatrix} \sqrt{n_1}\left(\hat{\beta}_1 - \beta_1\right) \\ \sqrt{n_2}\left(\hat{\beta}_2 - \beta_2\right) \\ \sqrt{m}\left(\hat{\beta}_3 - \beta_3\right) \\ \sqrt{m}\left(\hat{\alpha} - \alpha\right) \end{bmatrix} \to N_4\left(0, \mathbf{A}^{-1}(\boldsymbol{\theta})\right)$$

where $\mathbf{A}(\boldsymbol{\theta})$ and $\mathbf{A}^{-1}(\boldsymbol{\theta})$ are symmetric matrices such that

$$\begin{aligned} \mathbf{A}(\boldsymbol{\theta}) &= \begin{pmatrix} u_{11} & 0 & 0 & u_{14} \\ & u_{22} & 0 & u_{24} \\ & & u_{33} & u_{34} \\ & & & u_{44} \end{pmatrix}, \text{ and} \\ \mathbf{A}^{-1}(\boldsymbol{\theta}) &= \frac{1}{|\mathbf{A}(\boldsymbol{\theta})|} \begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ & v_{22} & v_{23} & v_{24} \\ & & v_{33} & v_{34} \\ & & & v_{44} \end{pmatrix} \end{aligned} \tag{2.27}$$

Here, the stress sample size m is assumed to be greater than the two strength samples sizes n_1 and n_2. If otherwise, the formulae can always be readjusted accordingly. The entries of each of the matrices are

$$\begin{gathered} u_{11} = \lim_{n_1,n_2,m\to\infty} \frac{1}{n_1} I_{11},\; u_{14} = u_{41} = \lim_{n_1,n_2,m\to\infty} \frac{\sqrt{q_1}}{n_1} I_{14}, \\ u_{24} = u_{42} = \lim_{n_1,n_2,m\to\infty} \frac{\sqrt{q_2}}{n_2} I_{24},\; u_{34} = u_{43} = \lim_{n_1,n_2,m\to\infty} \frac{1}{m} I_{34} \\ u_{22} = \lim_{n_1,n_2,m\to\infty} \frac{1}{n_2} I_{22},\; u_{33} = \lim_{n_1,n_2,m\to\infty} \frac{1}{m} I_{33},\; u_{44} = \lim_{n_1,n_2,m\to\infty} \frac{1}{m} I_{44} \\ u_{12} = u_{13} = u_{21} = u_{23} = u_{31} = u_{32} = 0 \\ v_{11} = -\frac{u_{24}^2}{\beta_3^2} - \frac{u_{34}^2}{\beta_2^2} + \frac{u_{44}}{\beta_2^2\beta_3^2}, \quad v_{12} = -\frac{u_{14}u_{24}}{\beta_3^2}, \quad v_{13} = -\frac{u_{14}u_{34}}{\beta_2^2}, \quad v_{14} = \frac{u_{14}}{\beta_2^2\beta_3^2} \\ v_{22} = -\frac{u_{14}^2}{\beta_3^2} - \frac{u_{34}^2}{\beta_1^2} + \frac{u_{44}}{\beta_1^2\beta_3^2}, \quad v_{23} = -\frac{u_{24}u_{34}}{\beta_1^2}, \quad v_{24} = -\frac{u_{24}}{\beta_1^2\beta_3^2}, \\ v_{33} = -\frac{u_{14}^2}{\beta_2^2} - \frac{u_{24}^2}{\beta_1^2} + \frac{u_{44}}{\beta_1^2\beta_2^2}, \quad v_{34} = \frac{u_{34}}{\beta_1^2\beta_2^2}, \\ v_{44} = \frac{1}{\beta_1^2\beta_2^2\beta_3^2}, \quad \text{and} \\ |\mathbf{A}(\boldsymbol{\theta})| = -\frac{\left(\beta_1^2 u_{14}^2 + \beta_2^2 u_{24}^2 + \beta_3^2 u_{34}^2 - u_{44}\right)}{\beta_1^2\beta_2^2\beta_3^2} \end{gathered} \tag{2.28}$$

Proof. The proof of the theorem follows from the asymptotic normality of MLE, details of the proof can be found in [39]. □

Theorem 3. *As* $n_1, n_2, m \to \infty$, $\dfrac{n_1}{m} \to q_1$ *and* $\dfrac{n_2}{m} \to q_2$ *then*

$$\sqrt{m}\left(\hat{R}_{s,k_1,k_2} - R_{s,k_1,k_2}\right) \to \mathbf{N}(0,\sigma^2) \tag{2.29}$$

where

$$\begin{aligned}\sigma^2 =& \frac{1}{|\mathbf{A}(\boldsymbol{\theta})|}\left[\left(\frac{\partial R_{s,k_1,k_2}}{\partial \beta_1}\right)^2 v_{11} + 2\frac{\partial R_{s,k_1,k_2}}{\partial \beta_1}\frac{\partial R_{s,k_1,k_2}}{\partial \beta_2} v_{12} + \left(\frac{\partial R_{s,k_1,k_2}}{\partial \beta_2}\right)^2 v_{22}\right.\\ &\left. + 2\frac{\partial R_{s,k_1,k_2}}{\partial \beta_1}\frac{\partial R_{s,k_1,k_2}}{\partial \beta_3} v_{13} + 2\frac{\partial R_{s,k_1,k_2}}{\partial \beta_2}\frac{\partial R_{s,k_1,k_2}}{\partial \beta_3} v_{23} + \left(\frac{\partial R_{s,k_1,k_2}}{\partial \beta_3}\right)^2 v_{33}\right].\end{aligned}$$

Proof. Using Theorem 2 and applying the delta method, (see [39]), the asymptotic distribution of $R_{s,k_1,k_2} = g(\hat{\boldsymbol{\theta}})$ can be written as follows

$$\sqrt{m}\left(\hat{R}_{s,k_1,k_2} - R_{s,k_1,k_2}\right) \to \mathbf{N}(0,\sigma^2) \tag{2.30}$$

where $\sigma^2 = \mathbf{b^T A^{-1}}(\boldsymbol{\theta})\mathbf{b}$ with

$$\begin{aligned}\mathbf{b} &= \left[\frac{\partial R_{s,k_1,k_2}}{\partial \beta_1}, \frac{\partial R_{s,k_1,k_2}}{\partial \beta_2}, \frac{\partial R_{s,k_1,k_2}}{\partial \beta_3}, \frac{\partial R_{s,k_1,k_2}}{\partial \alpha}\right]^T\\ &= \left[\frac{\partial R_{s,k_1,k_2}}{\partial \beta_1}, \frac{\partial R_{s,k_1,k_2}}{\partial \beta_2}, \frac{\partial R_{s,k_1,k_2}}{\partial \beta_3}, 0\right]^T\end{aligned}$$

where the partial derivatives of R_{s,k_1,k_2} with respect to β_1, β_2, and β_3 as defined in Section 2.2.1. □

Theorem 3 can therefore be used to construct asymptotic confidence intervals of R_{s,k_1,k_2}. Using the invariance property of the MLE, the variance σ^2 is estimated using the empirical Fisher information matrix and the MLEs of β_1, β_2, β_3, and α as follows

$$u_{11} = \frac{1}{\hat{\beta}_1},\; u_{22} = \frac{1}{\hat{\beta}_2},\; u_{33} = \frac{1}{\hat{\beta}_3},$$

$$u_{14} = \frac{\sqrt{q_1}}{n_1}\frac{\hat{\beta}_1^{\,n_1}}{\hat{\alpha}}\sum_{k=1}^{\infty}\frac{1}{k}\left[\frac{1}{(\hat{\beta}_1+k)^{n_1}} - \frac{1}{(\beta_1+k-1)^{n_1}}\right]$$

$$u_{24} = \frac{\sqrt{q_2}}{n_2}\frac{\hat{\beta}_2^{\,n_2}}{\hat{\alpha}}\sum_{k=1}^{\infty}\frac{1}{k}\left[\frac{1}{(\hat{\beta}_2+k)^{n_2}} - \frac{1}{(\hat{\beta}_2+k-1)^{n_2}}\right]$$

$$u_{34} = \frac{1}{m}\frac{\hat{\beta}_3^{\,m}}{\hat{\alpha}}\sum_{k=1}^{\infty}\frac{1}{k}\left[\frac{1}{(\hat{\beta}_3+k)^{m}} - \frac{1}{(\hat{\beta}_3+k-1)^{m}}\right]$$

$$\begin{aligned}u_{44} = & \frac{(q_1+q_2+1)}{\hat{\alpha}^2} + \frac{1}{m}\left[E\left(\mu(G(r_{n_1})^{\hat{\alpha}})\right) + E\left(\mu(G(p_{n_2})^{\hat{\alpha}})\right) + E\left(\mu(G(s_m)^{\hat{\alpha}})\right)\right]\\ & - \frac{1}{m}\left[\sum_{i=1}^{n_1} E\left(\mu(G(r_i)^{\hat{\alpha}})\right) + \sum_{i=1}^{n_2} E\left(\mu(G(p_i)^{\hat{\alpha}})\right) + \sum_{i=1}^{m} E\left(\mu(G(s_i)^{\hat{\alpha}})\right)\right]\end{aligned}$$

with all the expectations as defined in (2.20), (2.21), (2.22) and with all parameters replaced by their respective MLEs. Therefore, a $100(1-\gamma)\%$ asymptotic confidence interval of R_{s,k_1,k_2} is given by

$$\left(\hat{R}_{s,k_1,k_2} - z_{1-\frac{\gamma}{2}}\frac{\hat{\sigma}}{\sqrt{m}}, \hat{R}_{s,k_1,k_2} + z_{1-\frac{\gamma}{2}}\frac{\hat{\sigma}}{\sqrt{m}}\right) \tag{2.31}$$

where z_γ is the $100\gamma-$th percentile of $\mathbf{N}(0,1)$.

2.1.2 Bootstrap Confidence Intervals

It is observed that the asymptotic confidence intervals do not perform very well for small sample sizes, Kundu and Gupta (2005) . In this subsection we construct bootstrap confidence intervals for R_{s,k_1,k_2} since an explicit pdf for R_{s,k_1,k_2} is unavailable. **(i) Boot-p Method Step 1** : From the samples $\{r_1,...,r_{n_1}\}$, $\{p_1,...,p_{n_2}\}$,and $\{s_1,...,s_m\}$, compute $\hat{\alpha},\hat{\beta}_1,\hat{\beta}_2$, and $\hat{\beta}_3$. **Step 2** : Using $\hat{\alpha}$ and $\hat{\beta}_1$ generate a bootstrap sample $\{r_1^*,...,r_{n_1}^*\}$, using $\hat{\alpha}$ and $\hat{\beta}_2$ generate a bootstrap sample $\{p_1^*,...,p_{n_2}^*\}$ and similarly from $\hat{\alpha}$ and $\hat{\beta}_3$ generate a bootstrap sample $\{s_1^*,...,s_m^*\}$. Based on $\{r_1^*,...,r_{n_1}^*\}$, $\{p_1^*,...,p_{n_2}^*\}$, and $\{s_1^*,...,s_m^*\}$ compute the bootstrap estimate of R_{s,k_1,k_2}, denoted by $\hat{R}^*_{s,k_1,k_2}$ **Step 3** : Repeat step 2, NBOOT times. **Step 4** : Let $G(r) = P(\hat{R}^*_{s,k_1,k_2} \leq r)$, be the CDF of $\hat{R}^*_{s,k_1,k_2}$.Define $\hat{R}_{Bp}(r) = G^{-1}(r)$ for each given r. The $100(1-\gamma)\%$ confidence interval of R_{s,k_1,k_2} is given by

$$\left(\hat{R}^{Bp}_{s,k_1,k_2}(\frac{\gamma}{2}), \hat{R}^{Bp}_{s,k_1,k_2}(1-\frac{\gamma}{2})\right) \tag{2.32}$$

(ii) Boot-t Method

Step 1 : From the samples $\{r_1,...,r_{n_1}\}$, $\{p_1,...,p_{n_2}\}$,and $\{s_1,...,s_m\}$, compute $\hat{\alpha},\hat{\beta}_1,\hat{\beta}_2$, and $\hat{\beta}_3$.

Step 2 : Using $\hat{\alpha}$ and $\hat{\beta}_1$generate a bootstrap sample $\{r_1^*,...,r_{n_1}^*\}$, using $\hat{\alpha}$ and $\hat{\beta}_2$ generate a bootstrap sample $\{p_1^*,...,p_{n_2}^*\}$ and similarly from $\hat{\alpha}$ and $\hat{\beta}_3$ generate a bootstrap sample $\{s_1^*,...,s_m^*\}$. Based on $\{r_1^*,...,r_{n_1}^*\}$, $\{p_1^*,...,p_{n_2}^*\}$, and $\{s_1^*,...,s_m^*\}$ compute the bootstrap estimate of R_{s,k_1,k_2}, denoted by $\hat{R}^*_{s,k_1,k_2}$. Compute the bootstrap estimate of R_{s,k_1,k_2} and the following statistic :

$$T^* = \frac{\sqrt{m}\left(\hat{R}^*_{s,k_1,k_2} - \hat{R}_{s,k_1,k_2}\right)}{\sqrt{Var\left(\hat{R}^*_{s,k_1,k_2}\right)}}$$

$Var\left(\hat{R}^*_{s,k_1,k_2}\right)$ can be computed using Theorem 2.

Step 3 : Repeat step 2 NBOOT times.

Step 4 : From the NBOOT T^*values obtained, determine the upper and lower bound of the $100(1-\gamma)\%$ confidence interval of R_{s,k_1,k_2} as follows : Let $H(x) = P(T^* \leq x)$ be the CDF of T^*. For a given r, define

$$\hat{R}^{Bt}_{s,k_1,k_2}(x) = \hat{R}_{s,k_1,k_2} + m^{-\frac{1}{2}}\sqrt{Var\left(\hat{R}_{s,k_1,k_2}\right)}H^{-1}(x)$$

The approximate $100(1-\gamma)\%$ confidence interval of R_{s,k_1,k_2} is given by :

$$\left(\hat{R}^{Bt}_{s,k_1,k_2}(\frac{\gamma}{2}), \hat{R}^{Bt}_{s,k_1,k_2}(1-\frac{\gamma}{2})\right) \tag{2.33}$$

2.2 Bayesian Estimation of R_{s,k_1,k_2}.

In this section, the Bayes estimate of R_{s,k_1,k_2} is derived under the assumption that the parameters α, β_1, β_2, and β_3 themselves are random variables, see [40] and [41] for more details on the bayesian approach to parameter estimation. Consider the likelihood functions for β_1, β_2, and β_3 based on upper record values from the Kw-G distribution in equations (2.5), (2.6), and (2.7). From these functions, it can be deduced that the suggested conjugate family of prior distributions for $\boldsymbol{\theta} = (\beta_1, \beta_2, \beta_3, \alpha)$ is the gamma distribution. So, it is assumed that $\beta_1 \sim Gamma(\delta_1, \gamma_1)$, $\beta_2 \sim Gamma(\delta_2, \gamma_2)$,$\beta_3 \sim Gamma(\delta_3, \gamma_3)$, and $\alpha \sim Gamma(\delta_4, \gamma_4)$. The pdfs are given by

$$\begin{aligned}
\pi(\beta_1) =& \frac{\gamma_1^{\delta_1}\beta_1^{\delta_1-1}e^{-\gamma_1\beta_1}}{\Gamma(\delta_1)}, \quad \beta_1 > 0,\ \delta_1, \gamma_1 > 0 \\
\pi(\beta_2) =& \frac{\gamma_2^{\delta_2}\beta_2^{\delta_2-1}e^{-\gamma_2\beta_2}}{\Gamma(\delta_2)}, \quad \beta_2 > 0,\ \delta_2, \gamma_2 > 0 \\
\pi(\beta_3) =& \frac{\gamma_3^{\delta_3}\beta_3^{\delta_3-1}e^{-\gamma_3\beta_3}}{\Gamma(\delta_3)}, \quad \beta_3 > 0,\ \delta_3, \gamma_3 > 0, \text{ and} \\
\pi(\alpha) =& \frac{\gamma_4^{\delta_4}\alpha^{\delta_4-1}e^{-\gamma_4\alpha}}{\Gamma(\delta_4)}, \quad \alpha > 0,\ \delta_4, \gamma_4 > 0
\end{aligned} \tag{2.34}$$

respectively. The joint prior distribution function of $\boldsymbol{\theta}$ is given by

$$f(\boldsymbol{\theta}) = \pi(\alpha)\pi(\beta_1)\pi(\beta_2)\pi(\beta_3)$$

and the joint posterior distribution function of $\boldsymbol{\theta}$ is given by

$$\begin{aligned}
\pi(\boldsymbol{\theta}\,|data) &= \frac{\mathscr{L}(\boldsymbol{\theta}\,|data)f(\boldsymbol{\theta})}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \mathscr{L}(\boldsymbol{\theta}\,|data)f(\boldsymbol{\theta})d\boldsymbol{\theta}} \\
&= \frac{M_1(\underline{r};\alpha)M_2(\underline{r};\alpha)M_3(\underline{r};\alpha)\beta_1^{n_1+\delta_1-1}\beta_2^{n_2+\delta_2-1}\beta_3^{m+\delta_3-1}\alpha^{n_1+n_2+m+\delta_4-1}}{\Gamma(n_1+\delta_1)\Gamma(n_2+\delta_2)\Gamma(m+\delta_3)I_0(\underline{r},\underline{p},\underline{s})} \\
&\quad \times e^{-\beta_1(\gamma_1+\lambda_1)}e^{-\beta_2(\gamma_2+\lambda_2)}e^{-\beta_3(\gamma_3+\lambda_3)}e^{-\alpha\gamma_4}
\end{aligned} \tag{2.35}$$

where

$$I_0(\underline{r},\underline{p},\underline{s}) = \int_0^\infty \frac{\alpha^{n_1+n_2+m-1}M_1(\underline{r};\alpha)M_2(\underline{p};\alpha)M_3(\underline{s};\alpha)e^{-\alpha\gamma_4}}{(\gamma_1+\lambda_1)^{n_1+\delta_1}(\gamma_2+\lambda_2)^{n_2+\delta_2}(\gamma_3+\lambda_3)^{m+\delta_3}}d\alpha \tag{2.36}$$

$$\begin{aligned}
M_1(\underline{r};\alpha) &= \prod_{i=1}^{n_1}\left(\frac{g(r_i)G(r_i)^{\alpha-1}}{1-G(r_i)^\alpha}\right), \\
M_2(\underline{p};\alpha) &= \prod_{i=1}^{n_2}\left(\frac{g(p_i)G(p_i)^{\alpha-1}}{1-G(p_i)^\alpha}\right), \\
M_3(\underline{s};\alpha) &= \prod_{i=1}^{m}\left(\frac{g(s_i)G(s_i)^{\alpha-1}}{1-G(s_i)^\alpha}\right),
\end{aligned} \tag{2.37}$$

with

$$\lambda_1 = -\ln\{1-G(r_{n_1})^\alpha\},\ \lambda_2 = -\ln\{1-G(p_{n_2})^\alpha\},\ \text{and } \lambda_3 = -\ln\{1-G(s_m)^\alpha\}$$

Under the Square Error loss function, the estimate of $R_{s,k_1,k_2} = g(\boldsymbol{\theta})$ is the mean of the posterior function in (2.35), which can be written as a ratio of two integrals as follows.

$$\begin{aligned}
E\left[g(\boldsymbol{\theta})\,|data\right] &= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty g(\boldsymbol{\theta})\pi(\boldsymbol{\theta}\,|data)d\boldsymbol{\theta} \\
&= \frac{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty g(\boldsymbol{\theta})\mathscr{L}(\boldsymbol{\theta}\,|data)f(\boldsymbol{\theta})d\boldsymbol{\theta}}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \mathscr{L}(\theta\,|data)f(\boldsymbol{\theta})d\boldsymbol{\theta}}
\end{aligned} \tag{2.38}$$

It is difficult or perhaps impossible to get an explicit analytic expression for (2.38). Numerical methods such as, (i) Lindley approximation, as well as (ii) Markov Chain Monte Carlo(MCMC) method can be used. An alternative to Lindley's method is an approximation method of a slightly higher order by [42] which has been used by [28]. Only Lindley approximation will be considered in the present work.

2.2.1 Lindley Approximation

In this subsection we approxiamate R_{s,k_1,k_2} using the famous Lindley approximation method, see Lindley (1980) for details. For a vector of parameters $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3, \alpha)$ and a function $W(\boldsymbol{\beta})$ consider the integral

$$E\left[W(\boldsymbol{\beta})\,|\text{data}\right] = \frac{\int_0^\infty W(\boldsymbol{\beta}) e^{\ell(\boldsymbol{\beta})+\rho(\boldsymbol{\beta})} d\boldsymbol{\beta}}{\int_0^\infty e^{\ell(\boldsymbol{\beta})+\rho(\boldsymbol{\beta})} d\boldsymbol{\beta}} \tag{2.39}$$

where $\ell(\boldsymbol{\beta})$ is the log-likelihood function of $\boldsymbol{\beta}$ and $\rho(\boldsymbol{\beta})$ is the nanural logarithm of the prior density of $\boldsymbol{\beta}$. For sufficiently large sample sizes n_1,n_2, and m , using Lindley's method, the Bayes estimate of R_{s,k_1,k_2} is given by

$$\begin{aligned} E\left[W(\boldsymbol{\beta})\,|\text{data}\right] =& w + \frac{1}{2}\sum_i\sum_j (w_{ij} + 2w_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_i\sum_j\sum_k\sum_l \tau_{ijk}\sigma_{ij}w_l \Big|_{\boldsymbol{\beta}=\hat{\boldsymbol{\beta}}} \\ &+ \text{terms of order } n^{-2} \text{ or smaller.} \end{aligned} \tag{2.40}$$

Where $\boldsymbol{\beta} = (\theta_1, ..., \theta_m)$ $i,j,k,l = 1,...,m$, $\hat{\boldsymbol{\beta}}$ is the MLE of β, $w = w(\boldsymbol{\beta})$, $w_i = \dfrac{\partial w}{\partial \theta_i}$, $w_{ij} = \dfrac{\partial^2 w}{\partial\theta_i\partial\theta_j}$, $\tau_{ijk} = \dfrac{\partial^3 \tau}{\partial\theta_i\partial\theta_j\partial\theta_k}$, $\rho_j = \dfrac{\partial\rho}{\partial\theta_j}$, and $\sigma_{ij} = (i,j)$-th element of the inverse of the matrix $\left[-\tau_{ij}\right]$ with all the parameters $\boldsymbol{\beta} = (\theta_1, \theta_2, \theta_3, \eta)$ replaced by their respective MLEs . Therefore in our case $\boldsymbol{\beta} = (\theta_1, \theta_2, \theta_3, \eta)$, Lindley's approximation gives

$$\begin{aligned} E\left[w(\boldsymbol{\beta})\,|\text{data}\right] \;=\; & w + (w_1u_1 + w_2u_2 + w_3u_3 + u_5 + u_6) + \frac{1}{2}[\mathscr{A}(w_1\sigma_{11} + w_2\sigma_{12} + w_3\sigma_{13}) \\ & + B(w_1\sigma_{21} + w_2\sigma_{22} + w_3\sigma_{23}) + \mathscr{C}(\mathscr{A}(w_1\sigma_{31} + w_2\sigma_{32} + w_3\sigma_{43}) \\ & + \mathscr{D}(w_1\sigma_{41} + w_2\sigma_{42} + w_3\sigma_{43})] \end{aligned}$$

where

$$u_i = \rho_1\sigma_{i1} + \rho_2\sigma_{i2} + \rho_3\sigma_{i3} + \rho_4\sigma_{i4}, \quad i = 1,2,3$$

$$u_5 = w_{12}\sigma_{12} + w_{13}\sigma_{13} + w_{23}\sigma_{23}$$

$$u_6 = \frac{1}{2}(w_{11}\sigma_{11} + w_{22}\sigma_{22} + w_{33}\sigma_{33})$$

$$\begin{aligned} \mathscr{A} =& \tau_{111}\sigma_{11} + 2\tau_{121}\sigma_{12} + 2\tau_{131}\sigma_{13} + 2\tau_{141}\sigma_{14} + 2\tau_{231}\sigma_{23} \\ & + 2\tau_{241}\sigma_{24} + 2\tau_{341}\sigma_{34} + \tau_{221}\sigma_{22} + \tau_{331}\sigma_{33} + \tau_{441}\sigma_{44} \\ \mathscr{B} =& \tau_{112}\sigma_{11} + 2\tau_{122}\sigma_{12} + 2\tau_{132}\sigma_{13} + 2\tau_{142}\sigma_{14} + 2\tau_{232}\sigma_{23} \\ & + 2\tau_{242}\sigma_{24} + 2\tau_{342}\sigma_{34} + \tau_{222}\sigma_{22} + \tau_{332}\sigma_{33} + \tau_{442}\sigma_{44} \end{aligned}$$

$$\mathscr{C} = \tau_{113}\sigma_{11} + 2\tau_{123}\sigma_{12} + 2\tau_{133}\sigma_{13} + 2\tau_{143}\sigma_{14} + 2\tau_{233}\sigma_{23} + 2\tau_{243}\sigma_{24} + 2\tau_{343}\sigma_{34} + \tau_{223}\sigma_{22} + \tau_{333}\sigma_{33} + \tau_{443}\sigma_{44}$$

$$\mathscr{D} = \tau_{114}\sigma_{11} + 2\tau_{124}\sigma_{12} + 2\tau_{134}\sigma_{13} + 2\tau_{144}\sigma_{14} + 2\tau_{234}\sigma_{23} + 2\tau_{244}\sigma_{24} + 2\tau_{344}\sigma_{34} + \tau_{224}\sigma_{22} + \tau_{334}\sigma_{33} + \tau_{444}\sigma_{44}$$

In our case, $w(\boldsymbol{\beta}) = R_{s,k_1,k_2}$

$$\rho_1 = \frac{\delta_1 - 1}{\beta_1} - \gamma_1, \quad \rho_2 = \frac{\delta_2 - 1}{\beta_2} - \gamma_2, \quad \rho_3 = \frac{\delta_3 - 1}{\beta_3} - \gamma_3, \quad ,\rho_4 = \frac{\delta_4 - 1}{\alpha} - \gamma_4$$

$$\tau_{11} = -\frac{n_1}{\beta_1^2}, \quad \tau_{22} = -\frac{n_2}{\beta_2^2}, \quad \tau_{33} = -\frac{m}{\beta_3^2}$$

$$\tau_{14} = \tau_{41} = -\frac{G(r_{n_1})^{\alpha} \ln G(r_{n_1})}{(1 - G(r_{n_1})^{\alpha})}, \quad \tau_{24} = \tau_{42} = -\frac{G(p_{n_2})^{\alpha} \ln G(p_{n_2})}{(1 - G(p_{n_2})^{\alpha})},$$

$$\tau_{34} = \tau_{43} = -\frac{G(s_m)^{\alpha} \ln G(s_m)}{(1 - G(s_m)^{\alpha})}$$

$$\begin{aligned}\tau_{44} = &-\frac{(n_1 + n_2 + m)}{\alpha^2} - \frac{\beta_1 G(r_{n_1})^{\alpha} (\ln G(r_{n_1}))^2}{(1 - G(r_{n_1})^{\alpha})^2} - \frac{\beta_2 G(p_{n_2})^{\alpha} (\ln G(p_{n_2}))^2}{(1 - G(p_{n_2})^{\alpha})^2} \\ &- \frac{\beta_3 G(s_m)^{\alpha} (\ln G(s_m))^2}{(1 - G(s_m)^{\alpha})^2} + \sum_{i=1}^{n_1} \left[\frac{G(r_i)^{\alpha} (\ln G(r_i))^2}{(1 - G(r_i)^{\alpha})^2}\right] + \sum_{i=1}^{n_2} \left[\frac{G(p_i)^{\alpha} (\ln G(p_i))^2}{(1 - G(p_i)^{\alpha})^2}\right] \\ &+ \sum_{i=1}^{m} \left[\frac{G(s_i)^{\alpha} (\ln G(s_i))^2}{(1 - G(s_i)^{\alpha})^2}\right]\end{aligned}$$

The terms σ_{ij} $i, j = 1, 2, 3, 4$ are found using the terms τ_{ij}, $i, j = 1, 2, 3, 4$. Finally

$$\tau_{111} = \frac{2n_1}{\beta_1^3}, \quad \tau_{144} = \tau_{441} = \tau_{414} = -\frac{G(r_{n_1})^{\alpha} (\ln G(r_{n_1}))^2}{(1 - G(r_{n_1})^{\alpha})^2}$$

$$\tau_{222} = \frac{2n_2}{\beta_2^3}, \tau_{424} = \tau_{244} = \tau_{442} = -\frac{G(p_{n_2})^{\alpha} (\ln G(p_{n_2}))^2}{(1 - G(p_{n_2})^{\alpha})^2}$$

$$\tau_{333} = \frac{2n_1}{\beta_1^3}, \tau_{434} = \tau_{443} = \tau_{344} = -\frac{G(s_m)^{\alpha} (\ln G(s_m))^2}{(1 - G(s_m)^{\alpha})^2}$$

$$\begin{aligned}\tau_{444} = &\frac{2(n_1 + n_2 + m)}{\alpha^3} - \frac{\beta_1 G(r_{n_1})^{\alpha} (1 + G(r_{n_1})^{\alpha}) (\ln G(r_{n_1}))^3}{(1 - G(r_{n_1})^{\alpha})^3} \\ &- \frac{\beta_2 G(p_{n_2})^{\alpha} (1 + G(p_{n_2})^{\alpha}) (\ln G(p_{n_2}))^3}{(1 - G(p_{n_2})^{\alpha})^3} - \frac{\beta_3 G(s_m)^{\alpha} (1 + G(s_m)^{\alpha}) (\ln G(s_m))^3}{(1 - G(s_m)^{\alpha})^3} \\ &+ \sum_{i=1}^{n_1} \left[\frac{G(r_i)^{\alpha} (1 + G(r_i)^{\alpha}) (\ln G(r_i))^3}{(1 - G(r_i)^{\alpha})^3}\right] + \sum_{i=1}^{n_2} \left[\frac{G(p_i)^{\alpha} (1 + G(p_i)^{\alpha}) (\ln G(p_i))^3}{(1 - G(p_i)^{\alpha})^3}\right] \\ &+ \sum_{i=1}^{m} \left[\frac{G(s_i)^{\alpha} (1 + G(s_i)^{\alpha}) (\ln G(s_i))^3}{(1 - G(s_i)^{\alpha})^3}\right].\end{aligned}$$

$$u_{12}=\frac{\partial^2 R_{s,k_1,k_2}}{\partial\beta_1\partial\beta_2}=\sum_{\mathbf{k}}^{\mathbf{s}}\left(\frac{2pq\beta_3}{(p\beta_1+q\beta_2+\beta_3)^3}\right),$$
$$u_{13}=\frac{\partial^2 R_{s,k_1,k_2}}{\partial\beta_1\partial\beta_3}=\sum_{\mathbf{k}}^{\mathbf{s}}\left(\frac{-p\left(p\beta_1+q\beta_2-\beta_3\right)}{(p\beta_1+q\beta_2+\beta_3)^3}\right),$$
$$u_{23}=\frac{\partial^2 R_{s,k_1,k_2}}{\partial\beta_2\partial\beta_3}=\sum_{\mathbf{k}}^{\mathbf{s}}\left(\frac{-q\left(p\beta_1+q\beta_2-\beta_3\right)}{(p\beta_1+q\beta_2+\beta_3)^3}\right),$$

Due to the lack of an explicit pdf for R_{s,k_1,k_2}, in order to construct the highest posterior density (HPD) credible intervals, the MCMC method is preferred to generate samples from the posterior density function (2.35). The Bayes estimate and HPD credible intervals can then be computed from these samples under the SE and LINEX loss functions.

2.2.2 MCMC Method

From (2.35), it can be deduced that the posterior distributions of β_1,β_2,β_3, and α are as follows:

$$\begin{aligned}
\beta_1|\beta_2,\beta_3,data &\sim Gamma\left(n_1+\delta_1,\gamma_1-\ln\{1-G(r_{n_1})^\alpha\}\right),\\
\beta_2|\beta_1,\beta_3,data &\sim Gamma\left(n_2+\delta_2,\gamma_2-\ln\{1-G(p_{n_2})^\alpha\}\right),\\
\beta_3|\beta_1,\beta_2,data &\sim Gamma\left(m+\delta_3,\gamma_3-\ln\{1-G(s_m)^\alpha\}\right),\\
\pi\left(\alpha|\beta_1,\beta_2,\beta_3,data\right) &\propto \alpha^{n_1+n_2+m+\delta_4-1}e^{-\alpha\gamma_4}F(data)
\end{aligned} \tag{2.41}$$

where

$$F(data)=\left(\prod_{i=1}^{n_1}\frac{g(r_i)G(r_i)^{\alpha-1}}{1-G(r_i)^\alpha}\right)\left(\prod_{i=1}^{n_2}\frac{g(p_i)G(p_i)^{\alpha-1}}{1-G(p_i)^\alpha}\right)\left(\prod_{i=1}^{m}\frac{g(s_i)G(s_i)^{\alpha-1}}{1-G(s_i)^\alpha}\right)$$

The samples for β_1,β_2, and β_3 can thus be generated easily using the gamma distribution. The posterior distribution of α on the other hand cannot be written analytically to a well known distribution and it is not possible to sample directly using standard methods. The Metropolis-Hastings method is used to generate random samples from the posterior distribution of α. Therefore, the algorithm for Gibbs sampling is as follows:

1. Start with an initial guess $\left(\beta_1^{(0)},\beta_2^{(0)},\beta_3^{(0)},\alpha^{(0)}\right)$.

2. Set $t=1$.

3. Generate $\alpha^{(t)}$ from $\pi\left(\alpha|\beta_1,\beta_2,\beta_3,data\right)$.

4. Generate $\beta_1^{(t)}$ from $Gamma\left(n_1+\delta_1, \gamma_1-\ln\{1-G(r_{n_1})^{\alpha}\}\right)$.

5. Generate $\beta_2^{(t)}$ from $Gamma\left(n_2+\delta_2, \gamma_2-\ln\{1-G(p_{n_2})^{\alpha}\}\right)$.

6. Generate $\beta_3^{(t)}$ from $Gamma\left(m+\delta_3, \gamma_3-\ln\{1-G(s_m)^{\alpha}\}\right)$.

7. Compute $R_{s,k_1,k_2}^{(t)} = \sum_{\mathbf{k}}^{\mathbf{s}} \dfrac{\beta_3^{(t)}}{p\beta_1^{(t)}+q\beta_2^{(t)}+\beta_3^{(t)}}$.

8. Set $t=t+1$.

9. Repeat steps $1-8$ T times.

The sample obtained in the above algorithm is then used to obtain the Bayes estimate of R_{s,k_1,k_2} as well as the HPD credible intervals for R_{s,k_1,k_2}. The Bayes estimate of R_{s,k_1,k_2} under the SE and LINEX loss functions is given respectively by

$$\hat{R}_{s,k_1,k_2}^{MC} = \frac{1}{T}\sum_{t=1}^{T} R_{s,k_1,k_2}^{(t)} \tag{2.42}$$

$$\hat{R}_{s,k_1,k_2}^{MCLNX} = -\frac{1}{v}\ln E\left(e^{-vR_{s,k_1,k_2}}\right) = -\frac{1}{v}\ln\left(\frac{1}{T}\sum_{t=1}^{T} e^{-vR_{s,k_1,k_2}^{(t)}}\right) \tag{2.43}$$

The $100(1-\eta)\%$ HPD credible intervals for R_{s,k_1,k_2} can be obtained by the method of [43].

3. ESTIMATION OF R_{s,k_1,k_2} FOR KNOWN α I

3.1 Maximum Likelihood Estimation And Confidence Intervals For R_{s,k_1,k_2} I

For the sake of simplicity it is assumed that $\alpha = 1$. So let $R_1,...,R_{n_1}$, $P_1,....,P_{n_2}$ and $S_1,..., S_m$ be independent random samples of upper record values of sizes n_1, n_2 and m from the distributions $Kw\text{-}G(1,\beta_1)$, $Kw\text{-}G(1,\beta_2)$ and $Kw\text{-}G(1,\beta_3)$ respectively. In this case the MLE of R_{s,k_1,k_2} is given by

$$\hat{R}_{s,k_1,k_2} = \sum_{\mathbf{k}}^{\mathbf{s}} \frac{1}{\left[p\dfrac{n_1}{m}\dfrac{\ln(1-G(s_m))}{\ln(1-G(r_{n_1}))} + q\dfrac{n_2}{m}\dfrac{\ln(1-G(s_m))}{\ln(1-G(p_{n_2}))} + 1\right]} \tag{3.1}$$

In order to construct an exact confidence interval for R_{s,k_1,k_2}, its distribution needs to be determined, and to do so one must first obtain the distribution of $\hat{R}_{s,k_1,k_2}$. Using elementary transformation techniques it can easily be shown that

$$\begin{gathered} -2\beta_1 \ln(1-G(R_{n_1})) \sim \chi^2_{2n_1},\ -2\beta_2 \ln(1-G(P_{n_2})) \sim \chi^2_{2n_2}, \quad \text{and} \\ -2\beta_3 \ln(1-G(S_{n_1})) \sim \chi^2_{2m}. \end{gathered} \tag{3.2}$$

The quantities in (3.2) are all independent of each other. Thus,

$$\frac{-2n_1\beta_3^{-1}\ln(1-G(S_m))}{-2m\beta_1^{-1}\ln(1-G(R_{n_1}))} \sim F(2m,2n_1) \tag{3.3}$$

and

$$\frac{-2\beta_3^{-1}\ln(1-G(S_m))}{-2\beta_2^{-1}\ln(1-G(P_{n_2}))} \sim F(2m,2n_2) \tag{3.4}$$

In other words,

$$\begin{aligned} &\frac{n_1}{m}\frac{\ln(1-G(S_m))}{\ln(1-G(R_{n_1}))} \sim \frac{\beta_1}{\beta_3}F(2m,2n_1), \text{ and} \\ &\frac{n_2}{m}\frac{\ln(1-G(S_m))}{\ln(1-G(P_{n_2}))} \sim \frac{\beta_2}{\beta_3}F(2m,2n_2) \end{aligned}$$

have scaled F-distributions . Thus from (2.2), we have that the distribution of $\hat{R}_{s,k_1,k_2}$ is that of

$$\sum_{\mathbf{k}}^{\mathbf{s}} \frac{1}{\left[p\dfrac{\beta_1}{\beta_3}F(2m,2n_1) + q\dfrac{\beta_2}{\beta_3}F(2m,2n_2) + 1\right]} \tag{3.5}$$

An explicit formula for the pdf of $\hat{R}_{s,k_1,k_2}$ is clearly too complex and is therefore not pursued further. We therefore conclude that the exact $(1-\gamma)100\%$ confidence interval for $\hat{R}_{s,k_1,k_2}$ is

$$\left(\sum\nolimits_{\mathbf{k}}^{\mathbf{s}} F_1, \sum\nolimits_{\mathbf{k}}^{\mathbf{s}} F_2\right) \tag{3.6}$$

where

$$\begin{aligned} F_1 &\equiv \frac{1}{\left[p\frac{\hat{\beta}_1}{\hat{\beta}_3}F_{1-\frac{\gamma}{2}}(2m,2n_1)+q\frac{\hat{\beta}_2}{\hat{\beta}_3}F_{1-\frac{\gamma}{2}}(2m,2n_2)+1\right]} \\ F_2 &\equiv \frac{1}{\left[p\frac{\hat{\beta}_1}{\hat{\beta}_3}F_{\frac{\gamma}{2}}(2m,2n_1)+q\frac{\hat{\beta}_2}{\hat{\beta}_3}F_{\frac{\gamma}{2}}(2m,2n_2)+1\right]}. \end{aligned}$$

An approximate confidence interval for R_{s,k_1,k_2} can also be derived using the Fisher information matrix. The Fisher information matrix for $\boldsymbol{\theta} = (\beta_1, \beta_2, \beta_3)$ is given by

$$\begin{aligned} I(\theta) &= -\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial \beta_1^2}\right) & E\left(\frac{\partial^2 l}{\partial \beta_1 \partial \beta_2}\right) & E\left(\frac{\partial^2 l}{\partial \beta_1 \partial \beta_3}\right) \\ E\left(\frac{\partial^2 l}{\partial \beta_2 \partial \beta_1}\right) & E\left(\frac{\partial^2 l}{\partial \beta_2^2}\right) & E\left(\frac{\partial^2 l}{\partial \beta_2 \partial \beta_3}\right) \\ E\left(\frac{\partial^2 l}{\partial \beta_3 \partial \beta_1}\right) & E\left(\frac{\partial^2 l}{\partial \beta_3 \partial \beta_2}\right) & E\left(\frac{\partial^2 l}{\partial \beta_3^2}\right) \end{pmatrix} \\ &= \begin{pmatrix} \frac{n_1}{\beta_1^2} & 0 & 0 \\ 0 & \frac{n_2}{\beta_2^2} & 0 \\ 0 & 0 & \frac{m}{\beta_3^2} \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{33} & I_{32} & I_{33} \end{pmatrix} \end{aligned} \tag{3.7}$$

So, it must be the case that $I^{-1}(\boldsymbol{\theta}) = \begin{pmatrix} \frac{\beta_1^2}{n_1} & 0 & 0 \\ 0 & \frac{\beta_2^2}{n_2} & 0 \\ 0 & 0 & \frac{\beta_3^2}{m} \end{pmatrix}$ The MLE estimator $\hat{R}_{s,k_1,k_2}$ is approximately normally distributed with mean R_{s,k_1,k_2} and variance

$$\sigma^2_{R_{s,k_1,k_2}} = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial R_{s,k_1,k_2}}{\partial \beta_i}\frac{\partial R_{s,k_1,k_2}}{\partial \beta_j} I_{ij}^{-1},$$

where I_{ij}^{-1} is the ij-th term of the matrix $I^{-1}(\theta)$, the inverse matrix of $I(\theta)$. So

$$\begin{aligned}\sigma^2_{R_{s,k_1,k_2}} &= \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial R_{s,k_1,k_2}}{\partial \beta_i}\frac{\partial R_{s,k_1,k_2}}{\partial \beta_j} I_{ij}^{-1} \\ &= \left(\frac{\partial R_{s,k_1,k_2}}{\partial \beta_1}\right)^2 \left(\frac{\beta_1^2}{n_1}\right) + \left(\frac{\partial R_{s,k_1,k_2}}{\partial \beta_2}\right)^2 \left(\frac{\beta_2^2}{n_2}\right) + \left(\frac{\partial R_{s,k_1,k_2}}{\partial \beta_3}\right)^2 \left(\frac{\beta_3^2}{m}\right)\end{aligned} \tag{3.8}$$

where $\frac{\partial R_{s,k_1,k_2}}{\partial \beta_1}$, $\frac{\partial R_{s,k_1,k_2}}{\partial \beta_2}$, and $\frac{\partial R_{s,k_1,k_2}}{\partial \beta_3}$ are as defined in Section 2.2. Therefore, the $(1-\gamma)100\%$ asymptotic confidence interval for R_{s,k_1,k_2} when α is known is given by

$$\left(\hat{R}_{s,k_1,k_2} - z_{1-\frac{\gamma}{2}}\sqrt{\frac{1}{m}}\hat{\sigma}_{R_{s,k_1,k_2}}, \hat{R}_{s,k_1,k_2} + z_{1-\frac{\gamma}{2}}\sqrt{\frac{1}{m}}\hat{\sigma}_{R_{s,k_1,k_2}}\right), \tag{3.9}$$

where z_γ is the 100γ-th percentile of the standard normal distribution $\boldsymbol{N}(0,1)$.

3.2 Uniformly Minimum Variance Unbiased Estimator of R_{s,k_1,k_2}

As [15] point out, despite possessing a useful invariance property, the MLE method may be susceptible to bias, especially if sample sizes are very small. Since records sample sizes are particularly not always large and it is often of intrinsic interest to consider only estimators that are unbiased. This leads to the inevitable task of deriving an unbiased estimator for R_{s,k_1,k_2} which is optimal in the MSE. In this subsection an attempt is made to find an unbiased estimator which performs best among all unbiased estimators, the so called UMVUE, for R_{s,k_1,k_2}. In deriving the UMVUE, it is often necessary determine whether or not a statistic of the parameter under study is complete. Showing that a sufficient statistic is complete is generally quite difficult. However, it is well known that if the parameter vector $\boldsymbol{\theta}$ is viewed as unknown but non-random, with the only available information as the measurements $\mathbf{X}$ and the observation model specified by the density $f_{\mathbf{X}|\boldsymbol{\theta}}(\mathbf{X}|\boldsymbol{\theta})$, that is, the likelihood function. Then if $f_{\mathbf{X}|\boldsymbol{\theta}}(\mathbf{x}|\boldsymbol{\theta})$ belongs to the exponential class of densities of the form

$$f_{\mathbf{X}|\boldsymbol{\theta}}(\mathbf{x}|\boldsymbol{\theta}) = u(\mathbf{x})exp\left(\boldsymbol{\theta}^{\mathbf{T}}\mathbf{T}(\mathbf{x}) - t(\boldsymbol{\theta})\right) \tag{3.10}$$

it must be true that $\mathbf{T}(\mathbf{X})$ is a complete sufficient statistic for $\boldsymbol{\theta}$, [44]. Most of the fundamental definitions and theorems in this section have been taken from [44] unless otherwise stated.

Definition 1. *A statistic* $\mathbf{T}(\mathbf{X})$ *is said to be sufficient for a parameter* $\boldsymbol{\theta}$ *if it contains all the information about the observation vector* $\mathbf{X}$ *necessary to estimate* $\boldsymbol{\theta}$. *Formally,*

$\mathbf{T}(\mathbf{X})$ *is sufficient for* $\boldsymbol{\theta}$ *if the conditional density of* $\mathbf{X}$ *given* $\mathbf{T}(\mathbf{X})$ *is independent of* $\boldsymbol{\theta}$. *This independence property indicates that all the information about* $\boldsymbol{\theta}$ *has been "squeezed" in* $f_{\mathbf{T}}(\mathbf{T}|\boldsymbol{\theta})$ *and there is no leftover information about* $\boldsymbol{\theta}$ *that could be extracted from* $f_{\mathbf{X}|\mathbf{T}}(\mathbf{X}|\mathbf{T})$ *, which means that the desnsity must be independent of* $\boldsymbol{\theta}$ *.*

Theorem 4. ***Neyman-Fisher Factorization Theorem*** *The statistic* $\mathbf{T}(\mathbf{X})$ *is sufficient if and only if the density* $f_{\mathbf{X}}$ *can be written in the form* $\left[f_{\mathbf{X}|\boldsymbol{\theta}}(\mathbf{x}|\boldsymbol{\theta}) = H(\mathbf{T}(\mathbf{x}),\mathbf{x})I(\mathbf{x})\right]$ *A proof of this result can be found in [44].*

Definition 2. *Let* $\mathbf{T}(\mathbf{X})$ *be a sufficient statisic. We say* $\mathbf{T}$ *is complete if any function* $\mathbf{h}(\mathbf{T})$ *that satisfies*

$$E\left[\mathbf{h}(\mathbf{T})\right] = 0 \tag{3.11}$$

for all $\boldsymbol{\theta}$ *must necessarily be identically zero. Equivalently, the sufficient statistic* $\mathbf{T}$ *is complete if there is at most one unbiased estimator* $\hat{\boldsymbol{\theta}}(\mathbf{T})$ *of* $\boldsymbol{\theta}$ *depending on* $\mathbf{T}$ *only .*

Lemma 1. *The statistic*

$$(U,V,W) = (-\ln\left[1-G(r_{n_1})\right], -\ln\left[1-G(p_{n_2})\right], -\ln\left[1-G(s_m)\right])$$

is a complete sufficient statistic for $\theta = (\beta_1, \beta_2, \beta_3)$.

Proof. The joint pdf of the sets of upper records $R_1,...,R_{n_1}$, $P_1,...,P_{n_2}$, and $S_1,...,S_m$ is given by

$$\begin{aligned} f(\theta|data) =& \beta_1^{n_1}\beta_2^{n_2}\beta_3^{m}\{1-G(r_{n_1})\}^{\beta_1}\{1-G(p_{n_2})\}^{\beta_2}\{1-G(s_m)\}^{\beta_3} \\ &\times \prod_{i=1}^{n_1}\frac{g(r_i)G(r_i)}{1-G(r_i)}\prod_{i=1}^{n_2}\frac{g(p_i)G(p_i)}{1-G(p_i)}\prod_{i=1}^{m}\frac{g(s_i)G(s_i)}{1-G(s_i)} \end{aligned} \tag{3.12}$$

, see [25]. The joint pdf can also be written as

$$f(\theta|data) = H(data)I_\theta\left(U(r_{n_1}),V(p_{n_2}),W(s_m)\right) \tag{3.13}$$

where

$$\begin{aligned} H(data) =& \prod_{i=1}^{n_1}\frac{g(r_i)G(r_i)}{1-G(r_i)}\prod_{i=1}^{n_2}\frac{g(p_i)G(p_i)}{1-G(p_i)}\prod_{i=1}^{m}\frac{g(s_i)G(s_i)}{1-G(s_i)}, \\ U(r_{n_1}),V(p_{n_2}),W(s_m) =& \left(-\ln\left[1-G(r_{n_1})\right], -\ln\left[1-G(s_m)\right], -\ln\left[1-G(s_m)\right]\right), \\ I_\theta\left(U(r_{n_1}),V(p_{n_2}),W(s_m)\right) =& \beta_1^{n_1}\beta_2^{n_2}\beta_3^{m}e^{-\beta_1U(r_{n_1})}e^{-\beta_2V(p_{n_2})}e^{-\beta_3W(s_m)}. \end{aligned} \tag{3.14}$$

From (3.13) it is clear that by employing the Neuman-Fisher Factorization Theorem, (U,V,W) is a sufficient statistic for θ. Furthermore, it can be deduced that (U,V,W)

is a complete sufficient statistic for θ since the likelihood function, $f(\theta|data)$, can be written in the canonical exponential class form given in (3.10). $\square$

Theorem 5. ***Rao-Blackwell Theorem*** *Suppose that* $\mathbf{T}(\mathbf{X})$ *is a sufficient statistic and* $\mathbf{W}(\mathbf{X})$ *is an unbiased estimator of* $\boldsymbol{\theta}$*, then if we define the new unbiased estimator* $E(\mathbf{W}(\mathbf{X})|\mathbf{T}(\mathbf{X}))$*, then* $\left[var\left(E\left(\tilde{\mathbf{W}}(\mathbf{X})|\mathbf{T}(\mathbf{X})\right)\right) \le var\left(\tilde{\mathbf{W}}(\mathbf{X})\right)\right]$ *The Rao-Blackwell theorem tells us that estimators with the smallest variance must be a function of the sufficient statistic. This besgs the question is there a unique estimatior with the minimum variance . This is adressed by the following result.*

Theorem 6. ***Lehmann-Scheffe Theorem*** *. If* $\mathbf{T}(\mathbf{X})$ *is a complete sufficient statistic and* $\mathbf{W}(\mathbf{X})$ *is an unbiased estimator of* $\boldsymbol{\theta}$*, then* $\boldsymbol{\phi}(\mathbf{T}) = E(\mathbf{W}|\mathbf{T})$ *is an UMVUE of* $\boldsymbol{\theta}$*. Furthermore,* $\boldsymbol{\phi}(\mathbf{T})$ *is the unique UMVUE in the sense that if* $\boldsymbol{T}^*$ *is any other UMVUE, then* $P_{\boldsymbol{\theta}}(\boldsymbol{\phi}(\mathbf{T}) = \boldsymbol{T}^*) = 1$ *for all* $\boldsymbol{\theta}$*.*

Let

$$R_1^* = -\ln[1-G(R_1)], P_1^* = -\ln[1-G(P_1)], \text{ and } S_1^* = -\ln[1-G(S_1)] \quad (3.15)$$

Then it is easy to show that R_1^*, P_1^*, and S_1^* are exponentially distributed random variables with means β_1^{-1}, β_2^{-1}, and β_3^{-1} respectively. It follows that the joint distribution of the independent random variables R_1^*, P_1^* , and S_1^* can therefore be written as

$$\begin{aligned} f_{R_1^*,P_1^*,S_1^*}(r_1^*,p_1^*,s_1^*) =& f_{R_1^*}(r_1^*)f_{P_1^*}(p_1^*)f_{S_1^*}(s_1^*) \\ =& \beta_1\beta_2\beta_3 e^{-(\beta_1 r_1^*+\beta_2 p_1^*+\beta_3 s_1^*)}, 0<r_1^*<\infty, 0<p_1^*<\infty, 0<s_1^*<\infty. \end{aligned} \quad (3.16)$$

Lemma 2. *If* $R_1^* = -\ln[1-G(R_1)]$ *and* $U = -\ln[1-G(R_{n_1})]$*, the conditional distribution of* R_1^* *given* U *is given by*

$$f_{R_1^*|U}(r_1^*|U) = \frac{f_{R_1^*,U}(r_1^*,u)}{f_U(u)} = \frac{(n_1-1)(u-r_1^*)^{n_1-2}}{u^{n_1-1}}, 0<r_1^*<u \quad (3.17)$$

Likewise, for $P_1^* = -\ln[1-G(P_1)]$ *and* $V = -\ln[1-G(P_{n_2})]$ *, we have*

$$f_{P_1^*|V}(p_1^*|V) = \frac{f_{P_1^*,V}(p_1^*,v)}{f_V(v)} = \frac{(n_2-1)(v-p_1^*)^{n_2-2}}{v^{n_2-1}}, 0<p_1^*<v, \quad (3.18)$$

and finally for $S_1^* = -\ln[1-G(S_1)]$ *and* $W = -\ln[1-G(S_m)]$ *, we have that*

$$f_{S_1^*|W}(s_1^*|W) = \frac{f_{S_1^*,W}(s_1^*,w)}{f_W(w)} = \frac{(m-1)(w-s_1^*)^{m-2}}{w^{m-1}}, 0<s_1^*<w \quad (3.19)$$

Proof. The joint pdf of any pair (R_m, R_n) of upper records as given by [25] is

$$f_{R_n,R_m}(r_n,r_m) = \frac{[-\ln(1-F(r_m)]^m \left[-\ln\left(\frac{1-F(r_n)}{1-F(r_m)}\right)\right]^{n-m-1} f(r_m)f(r_n)}{m!(n-m-1)!(1-F(r_m))}, \text{ for } m<n. \tag{3.20}$$

Using result (2.18) and (3.20) together with appropriate elementary transformation techniques, joint pdf of R_1^* and U is derived and from this, the pdf of U is found to be

$$f_{R_1^*,U}(r_1^*,u) = \frac{1}{(n_1-2)!}\beta_1^{n_1}(u-r_1^*)^{n_1-2}e^{-u\beta_1}$$

and

$$f_U(u) = \frac{1}{(n_1-1)!}\beta_1^{n_1}e^{-u\beta_1}u^{n_1-1}$$

Consequently we get the conditional distribution of R_1^* given U as follows

$$f_{R_1^*|U}(r_1^*|U) = \frac{f_{R_1^*,U}(r_1^*,u)}{f_U(u)} = \frac{(n_1-1)(u-r_1^*)^{n_1-2}}{u^{n_1-1}}, 0<r_1^*<u$$

Following a similar procedure, we obtain the conditional distributions $f_{P_1^*|V}(p_1^*|V)$ and $f_{S_1^*|W}(s_1^*|W)$ as

$$f_{P_1^*|V}(p_1^*|V) = \frac{f_{P_1^*,V}(p_1^*,v)}{f_V(v)} = \frac{(n_2-1)(v-p_1^*)^{n_2-2}}{v^{n_2-1}}, 0<p_1^*<v, \quad \text{and}$$

$$f_{S_1^*|W}(s_1^*|W) = \frac{f_{S_1^*,W}(s_1^*,w)}{f_W(w)} = \frac{(m-1)(w-s_1^*)^{m-2}}{w^{m-1}}, 0<s_1^*<w$$

respectively to conclude the proof . □

Now the main result of the present work is given in the following theorem.

Theorem 7. *For $n_1 \geq 2$, $n_2 \geq 2$, and $m \geq 2$, the UMVUE of*

$$\varphi(\beta_1,\beta_2,\beta_3) = \frac{\beta_3}{\beta_1(j_1+i_1)+\beta_2(j_2+i_2)+\beta_3}$$

, denoted by $\varphi_U(\beta,\beta_2,\beta_3)$, is

$$\varphi_U(\beta,\beta_2,\beta_3) = \begin{cases} Q_1(n_1,n_2,m,u,v,w) & if\ w \leq max\left\{\frac{v}{h}, \frac{u}{k}\right\} \\ Q_2(n_1,n_2,m,u,v,w) & if\ \frac{u}{k} \leq max\left\{w, \frac{v}{h}\right\} \\ Q_3(n_1,n_2,m,u,v,w) & if\ \frac{v}{h} \leq max\left\{w, \frac{u}{k}\right\} \end{cases} \tag{3.21}$$

where

$$Q_1 = \sum_{a=0}^{n_1-1} (-1)^a \left(\frac{kw}{u}\right)^a \sum_{b=0}^{n_2-1} (-1)^b \left(\frac{hw}{v}\right)^b \frac{\binom{n_1-1}{a}\binom{n_2-1}{b}}{\binom{m+a+b-1}{a+b}}$$
$$= F_1\left(1,(1-n_1),(1-n_2),m;\frac{kw}{u},\frac{hw}{v}\right)$$
$$Q_2 = \left(\frac{m-1}{n_1}\right)\left(\frac{u}{kw}\right) \sum_{a=0}^{n_2-1} (-1)^a \left(\frac{hu}{kv}\right)^a \sum_{b=0}^{m-2} (-1)^b \left(\frac{u}{kw}\right)^b \frac{\binom{n_2-1}{a}\binom{m-2}{b}}{\binom{n_1+a+b}{a+b}}$$
$$= \left(\frac{m-1}{n_1}\right)\left(\frac{u}{kw}\right) F_1\left(1,(1-n_2),(2-m),n_1+1;\frac{hu}{kv},\frac{u}{kw}\right)$$
$$Q_3 = \left(\frac{m-1}{n_2}\right)\left(\frac{v}{hw}\right) \sum_{a=0}^{n_1-1} (-1)^a \left(\frac{kv}{hu}\right)^a \sum_{b=0}^{m-2} (-1)^b \left(\frac{v}{hw}\right)^b \frac{\binom{n_1-1}{a}\binom{m-2}{b}}{\binom{n_2+a+b}{a+b}}$$
$$= \left(\frac{m-1}{n_2}\right)\left(\frac{v}{hw}\right) F_1\left(1,(1-n_1),(2-m),n_2+1;\frac{kv}{hu},\frac{v}{hw}\right)$$

and $k = i_1 + j_1$, $h = i_2 + j_2$.

Proof. Using (2.2), the following functions are defined

$$\phi\left(R_1^*,P_1^*,S_1^*\right) = \begin{cases} 1 \;, & R_1^* > kS_1^*, P_1^* > pS_1^* \\ 0 \;, & Otherwise \end{cases}, \tag{3.22}$$

and

$$\varphi(\beta_1,\beta_2,\beta_3) = \frac{\beta_3}{k\beta_1 + h\beta_2 + \beta_3} \tag{3.23}$$

with $R_1^*, P_1^*,$ and S_1^* defined as in (3.15). It can easily be shown that

$$E\left[\phi\left(R_1^*,P_1^*,S_1^*\right)\right] = \int_0^\infty \int_0^\infty \int_0^\infty \phi\left(r_1^*,p_1^*,s_1^*\right) f_{R_1^*,P_1^*,S_1^*}(r_1^*,p_1^*,s_1^*) dr_1^* dp_1^* ds_1^*$$
$$= \varphi(\beta_1,\beta_2,\beta_3)$$

Thus $\phi\left(R_1^*,P_1^*,S_1^*\right)$ is an unbiased estimator of $\varphi(\beta_1,\beta_2,\beta_3)$. The linearity of unbiased estimators allows one to conclude that an unbiased estimator of R_{s,k_1,k_2} is given by

$$\sum \mathbf{\substack{s\\k}} \phi\left(R_1^*,P_1^*,S_1^*\right) \tag{3.24}$$

Furthermore, since (U,V,W) is a complete sufficient statistic for $\boldsymbol{\theta} = (\beta_1,\beta_2,\beta_3)$, employing the Rao-Blackwell and Lehmann-Scheffe's Theorems, the unique UMVUE for $\varphi(\beta_1,\beta_2,\beta_3)$ is obtained as

$$\varphi_U(\beta_1,\beta_2,\beta_3) = E\left[\phi\left(R_1^*,P_1^*,S_1^*\right)\middle| U=u, V=v, W=w\right]$$
$$= \underset{A}{\int\int\int} f_{R_1^*|U}(x) f_{P_1^*|V}(y) f_{S_1^*|W}(z) dxdydz \tag{3.25}$$

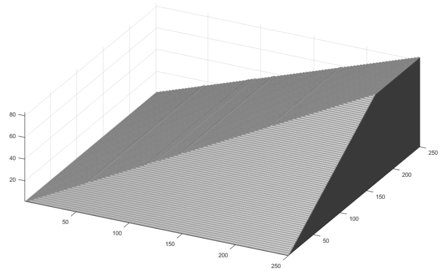

Figure 3.1 : A look of the region of integration, A, for specific values of u, v, and w,.

where

$$A = \{ (r_1^*, p_1^*, s_1^*) : 0 < r_1^* < u, 0 < p_1^* < v, 0 < s_1^* < w, r_1^* > ks_1^*,\ p_1^* > hs_1^* \} \quad (3.26)$$

Let $r_1^* = x$, $p_1^* = y$, $s_1^* = z, \theta = (\beta_1, \beta_2, \beta_3)$ and $f(x,y,z) = (u-x)^{n_1-2}(v-y)^{n_2-2}(w-z)^{m-2}$. Now for the region $w \le max\left\{\frac{v}{h}, \frac{u}{k}\right\}$, it is true that

$$\begin{aligned}
\varphi_U(\boldsymbol{\theta}) &= \frac{(n_1-1)(n_2-1)(m-1)}{u^{n_1-1}v^{n_2-1}w^{m-1}} \int_0^w \int_{hz}^v \int_{kz}^u f(x,y,z)dxdydz \\
&= \frac{(m-1)}{u^{n_1-1}v^{n_2-1}w^{m-1}} \int_0^w (u-kz)^{n_1-1}(v-hz)^{n_2-1}(w-z)^{m-2}dz
\end{aligned}$$

Let $z = wt$ so that

$$\begin{aligned}
\varphi_U(\boldsymbol{\theta}) &= \frac{(m-1)}{u^{n_1-1}v^{n_2-1}w^{m-1}} \int_0^1 (u-kwt)^{n_1-1}(v-hwt)^{n_2-1}(w-wt)^{m-2}wdt \\
&= (m-1)\int_0^1 \left(1-\frac{kwt}{u}\right)^{n_1-1}\left(1-\frac{hwt}{v}\right)^{n_2-1}(1-t)^{m-2}dt \\
&= \sum_{a=0}^{n_1-1}(-1)^a\left(\frac{kw}{u}\right)^a \sum_{b=0}^{n_2-1}(-1)^b\left(\frac{hw}{v}\right)^b \frac{\binom{n_1-1}{a}\binom{n_2-1}{b}}{\binom{m+a+b-1}{a+b}}
\end{aligned}$$

where the following binomial expansions have been used

$$\begin{aligned}
\left(1-\frac{kwt}{u}\right)^{n_1-1} &= \sum_{a=0}^{n_1-1}(-1)^a\binom{n_1-1}{a}\left(\frac{kwt}{u}\right)^a \text{ and} \\
\left(1-\frac{hwt}{v}\right)^{n_2-1} &= \sum_{b=0}^{n_2-1}(-1)^b\binom{n_2-1}{b}\left(\frac{hwt}{v}\right)^b
\end{aligned}$$

together with the assumption that integration and summation are interchangeable. Alternatively, formula 3.211 of [38] is given as

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}(1-ux)^{-\sigma}(1-vx)^{-\theta}dx = B(\mu,\lambda)F_1(\lambda,\sigma,\theta,\lambda+\mu;u,v),$$

with Re $\lambda > 0$ and Re $\mu > 0$. Using this result with $\lambda = 1$, $\mu = m-1$, $\sigma = 1-n_1$, $\theta = 1-n_2$, $u = \frac{kw}{u}$ and $v = \frac{hw}{v}$, one obtains an alternative representation for φ_U as

$$\varphi_U(\boldsymbol{\theta}) = F_1\left(1; (1-n_1), (1-n_2); m; \frac{kw}{u}, \frac{hw}{v}\right) \quad (3.27)$$

where $B(\cdot)$ is the standard beta function and F_1 is the Appell hypergeometric function of the first type which is defined as

$$\begin{aligned} F_1(a;b,c;d;x,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n}{m!n!(d)_{m+n}} x^m y^n \\ &= \frac{\Gamma(d)}{\Gamma(a)\Gamma(d-a)} \int_0^1 t^{a-1}(1-t)^{d-a-1}(1-tx)^{-b}(1-ty)^{-c}dt \end{aligned}$$

and and $(q)_n$ is the rising Pochhammmer symbol, which is defined by :

$$(q)_n = \begin{cases} 1, & n = 0 \\ q(q+1)...(q+n-1), & n > 0, \end{cases}$$

see [45] and [38] for more details about the Appell hypergeometric function. For the region $\frac{u}{k} \leq max\left\{w, \frac{v}{h}\right\}$

$$\begin{aligned} \varphi_U(\beta_1,\beta_2,\beta_3) &= \frac{(n_1-1)(n_2-1)(m-1)}{u^{n_1-1}v^{n_2-1}w^{m-1}} \int_0^{\frac{u}{k}}\int_{hz}^{v}\int_{kz}^{u} f(x,y,z)dxdydz \\ &= \left(\frac{m-1}{n_1}\right)\left(\frac{u}{kw}\right) F_1\left(1,(1-n_2),(2-m),n_1+1;\frac{hu}{kv},\frac{u}{kw}\right) \end{aligned}$$

Finally, for the region $\frac{v}{h} \leq max\left\{w, \frac{u}{k}\right\}$

$$\begin{aligned} \varphi_U(\beta,\beta_2,\beta_3) &= \frac{(n_1-1)(n_2-1)(m-1)}{u^{n_1-1}v^{n_2-1}w^{m-1}} \int_0^{\frac{v}{h}}\int_{hz}^{v}\int_{kz}^{u} f(x,y,z)dxdydz \\ &= \left(\frac{m-1}{n_2}\right)\left(\frac{v}{hw}\right) F_1\left(1,(1-n_1),(2-m),n_2+1;\frac{kv}{hu},\frac{v}{hw}\right) \end{aligned}$$

□

One can therefore conclude from Theorem 3 that the UMVUE of R_{s,k_1,k_2}, denoted by R^U_{s,k_1,k_2}, must be

$$R^U_{s,k_1,k_2} = \sum{}_{\mathbf{k}}^{\mathbf{s}} \varphi_U(\theta) \tag{3.28}$$

3.3 Bayesian Estimation of R_{s,k_1,k_2} For Known α

3.3.1 Conjugate Prior Distributions

As was the case in Section 2.2, in this subsection an attempt is made to derive the Bayesian estimate of R_{s,k_1,k_2} under the assumptions that the independent parameters

β_1, β_2, and β_3 are random variables with independent gamma prior distributions and that α is known. The prior distributions of β_1, β_2, and β_3 are given respectively by

$$\begin{aligned}\pi(\beta_1) &= \frac{\gamma_1^{\delta_1}\beta_1^{\delta_1-1}e^{-\gamma_1\beta_1}}{\Gamma(\delta_1)},\\ \pi(\beta_2) &= \frac{\gamma_2^{\delta_2}\beta_2^{\delta_2-1}e^{-\gamma_2\beta_2}}{\Gamma(\delta_2)}, \text{ and}\\ \pi(\beta_3) &= \frac{\gamma_3^{\delta_3}\beta_3^{\delta_3-1}e^{-\gamma 3\beta_3}}{\Gamma(\delta_3)}\end{aligned}$$

The joint prior distribution function of $\theta = (\beta_1, \beta_2, \beta_3)$ is given by $f(\theta) = \pi(\beta_1)\pi(\beta_2)\pi(\beta_3)$ and the its joint posterior distribution function is given by

$$\begin{aligned}\pi(\theta\,|data) =& \frac{\mathscr{L}(\theta\,|\underline{r},\underline{p},\underline{s})f(\theta)}{\int_0^\infty\int_0^\infty\int_0^\infty \mathscr{L}(\theta\,|data)f(\theta)d\beta_1 d\beta_2 d\beta_3}\\ =& \frac{\lambda_1^{\nu_1}\lambda_2^{\nu_2}\lambda_3^{\nu_3}}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)}\beta_1^{\nu_1-1}\beta_2^{\nu_2-1}\beta_2^{\nu_3-1}e^{-\lambda_1\beta_1}e^{-\lambda_2\beta_2}e^{-\lambda_3\beta_3}\end{aligned} \tag{3.29}$$

where

$$\lambda_1 = \gamma_1 + \ln\{1-G(r_{n_1})\},\ \lambda_2 = \gamma_2 + \ln\{1-G(p_{n_2})\},\ \lambda_3 = \gamma_3 + \ln\{1-G(s_m)\},$$
$$\nu_1 = \delta_1 + n_1,\ \nu_2 = \delta_2 + n_2, \text{ and } \nu_3 = \delta_3 + m$$

An explicit formula for the posterior distribution of R_{s,k_1,k_2} in this case is clearly very complex and is not pursued further. In order to find an estimate of reliability, we will once again use Lindley's approximation and MCMC method with α replaced by 1 in the formulae from Section (2.2). Before proceeding to the approximate methods, a closed form of the Bayes estimator is proposed.

3.3.2 Closed Form of Bayes Estimator

The Bayes estimator of R_{s,k_1,k_2}, denoted by R^B_{s,k_1,k_2}, is as follows

$$\begin{aligned}R^B_{s,k_1,k_2} =& \int_0^\infty\int_0^\infty\int_0^\infty R_{s,k_1,k_2}(\boldsymbol{\theta})\,\pi\left(\boldsymbol{\theta}\,|\alpha,\underline{x},\underline{y},\underline{z}\right)d\boldsymbol{\theta}\\ =& \mathscr{M}\sum_{\mathbf{k}}^{\mathbf{s}}\int_0^\infty\int_0^\infty\int_0^\infty\left(\frac{\beta_1^{\nu_1-1}\beta_2^{\nu_2-1}\beta_3^{\nu_3}\exp\left(-\mu_1\beta_1-\mu_2\beta_2-\mu_3\beta_3\right)}{p\beta_1+q\beta_2+\beta_3}\right)d\boldsymbol{\theta}\end{aligned}$$

where

$$\mathscr{M} = \frac{\mu_1^{\nu_1}\mu_2^{\nu_2}\mu_3^{\nu_3}}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)}$$

Define a one-to-one transformation $\mathscr{T}$ by

$$\left.\begin{array}{l} x=\dfrac{p\beta_1}{p\beta_1+q\beta_2+\beta_3} \\ \\ y=\dfrac{q\beta_2}{p\beta_1+q\beta_2+\beta_3} \\ \\ z=p\beta_1+q\beta_2+\beta_3 \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \beta_1=xz/p \\ \\ \beta_2=yz/q \\ \\ \beta_3=z(1-x-y) \end{array}\right. ,$$

where $p\neq 0$ and $q\neq 0$. It is evident that $R^B_{s,k_1,k_2}=1$ for the case when $p=0$ and $q=0$. We note that

$$0<x+y=\frac{p\beta_1+q\beta_2}{p\beta_1+q\beta_2+\beta_3}<1,\ 0<z<\infty \text{ and } \frac{\beta_3}{p\beta_1+q\beta_2+\beta_3}=1-x-y.$$

Furthermore, the Jacobian of $\mathscr{T}$ is given by

$$|J(x,y,z)|=\begin{vmatrix} z/p & 0 & x/p \\ 0 & z/q & y/q \\ -z & -z & 1-x-y \end{vmatrix}=\frac{z^2}{pq}.$$

Therefore, we have that

$$\begin{aligned} R^B_{s,k_1,k_2}=&\frac{\mu_1^{v_1}\mu_2^{v_2}\mu_3^{v_3}}{p^{v_1}q^{v_2}\Gamma(v_1)\Gamma(v_2)\Gamma(v_3)}\sum\mathbf{\tfrac{s}{k}}\int_0^1\int_0^{1-y}\int_0^{\infty}x^{v_1-1}y^{v_2-1}(1-x-y)^{v_3} \\ &\times z^{v_1+v_2+v_3-1}\exp\{-z[\mu_1x/p+\mu_2y/q+\mu_3(1-x-y)]\}\,dzdxdy \\ =&\mathscr{K}\sum\mathbf{\tfrac{s}{k}}\int_0^1\int_0^{1-y}x^{v_1-1}y^{v_2-1}(1-x-y)^{v_3}(1-\sigma_1x-\sigma_2y)^{-(v_1+v_2+v_3)}\,dxdy \end{aligned}$$

where

$$\mathscr{K}=\frac{(1-\sigma_1)^{v_1}(1-\sigma_2)^{v_2}\Gamma(v_1+v_2+v_3)}{\Gamma(v_1)\Gamma(v_2)\Gamma(v_3)},\quad \sigma_1=1-\frac{\mu_1}{p\mu_3},\quad \sigma_2=1-\frac{\mu_2}{q\mu_3}$$

A Euler type integral representation of the Appell hypergeometric function of the first kind, denoted by F_1, as shown in [38] is as follows

$$\frac{\Gamma(d)F_1(a,b,c,d;x,y)}{\Gamma(b)\Gamma(c)\Gamma(d-b-c)}=\int_0^1\int_0^{1-v}u^{b-1}v^{c-1}(1-u-v)^{d-b-c-1}(1-ux-vy)^{-a}\,dudv$$
$$b>0,\ c>0,\ d-b-c>0,\ |x|<1,\ |y|<1. \tag{3.30}$$

Using the result (4.28), together with functional relations of F_1 (see [38] formulae 9.183(1)), we can conclude that

$$R^B_{s,k_1,k_2}=\begin{cases} \varpi_1(\sigma_1,\sigma_2),\ \text{if } |\sigma_1|<1,\ |\sigma_2|<1 \\ \\ \varpi_2(\sigma_1,\sigma_2),\ \text{if } \sigma_1<-1,\sigma_2<-1 \\ \\ \varpi_3(\sigma_1,\sigma_2),\ \text{if } |\sigma_1|<1,\sigma_2<-1 \\ \\ \varpi_4(\sigma_1,\sigma_2),\ \text{if },\sigma_1<-1,|\sigma_2|<1 \end{cases} \tag{3.31}$$

where

$$\varpi_1 = \sum \mathbf{s}_{\mathbf{k}} \frac{(1-\sigma_1)^{v_1}(1-\sigma_2)^{v_2} v_3}{(v_1+v_2+v_3)} F_1\left(\sum_{i=1}^{3} v_i, v_1, v_2, 1+\sum_{i=1}^{3} v_i; \sigma_1, \sigma_2\right),$$

$$\varpi_2 = \sum \mathbf{s}_{\mathbf{k}} \left(\frac{v_3}{v_1+v_2+v_3}\right) F_1\left(1, v_1, v_2, 1+\sum_{i=1}^{3} v_i; \frac{\sigma_1}{\sigma_1-1}, \frac{\sigma_2}{\sigma_2-1}\right),$$

$$\varpi_3 = \sum \mathbf{s}_{\mathbf{k}} \left(\frac{v_3(1-\sigma_1)}{v_1+v_2+v_3}\right) F_1\left(1, v_3+1, v_2, 1+\sum_{i=1}^{3} v_i; \sigma_1, \frac{\sigma_1-\sigma_2}{1-\sigma_2}\right),$$

and

$$\varpi_4 = \sum \mathbf{s}_{\mathbf{k}} \left(\frac{v_3(1-\sigma_2)}{v_1+v_2+v_3}\right) F_1\left(1, v_1, v_3+1, 1+\sum_{i=1}^{3} v_i; \frac{\sigma_2-\sigma_1}{1-\sigma_1}, \sigma_2,\right).$$

Using Jeffery's non informative priors, it can easily be shown that the Bayes estimator assumes the same form as above with $a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 0$.

3.3.3 Lindley Approximation

$$\begin{aligned}
\hat{R}^{L}_{s,k_1,k_2} &= w + (w_1 v_1 + w_2 v_2 + w_3 v_3 + v_5 + v_6) + \frac{1}{2}[\mathscr{A}(w_1\sigma_{11} + w_2\sigma_{12} + w_3\sigma_{13}) \\
&\quad + \mathscr{B}(w_1\sigma_{21} + w_2\sigma_{22} + w_3\sigma_{23}) + \mathscr{C}(w_1\sigma_{31} + w_2\sigma_{32})]
\end{aligned}$$

$$v_i = \rho_1\sigma_{i1} + \rho_2\sigma_{i2} + \rho_3\sigma_{i3},\ i = 1,2,3.$$

$$v_5 = u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{23}\sigma_{23}$$

$$v_6 = \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33})$$

$$\mathscr{A} = \tau_{111}\sigma_{11} + 2\tau_{121}\sigma_{12} + 2\tau_{131}\sigma_{13} + 2\tau_{231}\sigma_{23} + \tau_{221}\sigma_{22} + \tau_{331}\sigma_{33}$$

$$\mathscr{B} = \tau_{112}\sigma_{11} + 2\tau_{122}\sigma_{12} + 2\tau_{132}\sigma_{13} + 2\tau_{232}\sigma_{23} + \tau_{222}\sigma_{22} + \tau_{332}\sigma_{33}$$

$$\mathscr{C} = \tau_{113}\sigma_{11} + 2\tau_{123}\sigma_{12} + 2\tau_{133}\sigma_{13} + 2\tau_{233}\sigma_{23} + \tau_{223}\sigma_{22} + \tau_{333}\sigma_{33}$$

Since $w(\beta) = R_{s,k_1,k_2}$, we have:

$$\rho_1 = \frac{b_1-1}{\theta_1} - a_1, \quad \rho_2 = \frac{b_2-1}{\theta_2} - a_2, \quad \rho_3 = \frac{b_3-1}{\theta_3} - a_3,$$

$$\tau_{11} = -\frac{n_1}{\theta_1^2}, \quad \tau_{22} = -\frac{n_2}{\theta_2^2}, \quad \tau_{33} = -\frac{m}{\theta_3^2},$$

$$\tau_{111} = \frac{2n_1}{\theta_1^3}, \quad \tau_{222} = \frac{2n_2}{\theta_2^3} \quad \tau_{333} = \frac{2n_1}{\theta_1^3},$$

All other terms are zero and the $w_{ij}, i, j = 1,2,3$ are as defined in Section 2.2.1. So, $\mathscr{A} = \tau_{111}\sigma_{11}$, $\mathscr{B} = \tau_{112}\sigma_{11}$, and $\mathscr{C} = \tau_{113}\sigma_{11}$

3.3.4 MCMC Method

Since α is known, the posterior distributions of $\beta_1, \beta_2, \beta_3$, and α are simply:

$$\begin{aligned}
&\beta_1|\beta_2, \beta_3, data \sim Gamma\left(n_1+\delta_1, \gamma_1 - \ln\{1-G(r_{n_1})^{\alpha}\}\right), \\
&\beta_2|\beta_1, \beta_3, data \sim Gamma\left(n_2+\delta_2, \gamma_2 - \ln\{1-G(p_{n_2})^{\alpha}\}\right), \\
&\beta_3|\beta_1, \beta_2, data \sim Gamma\left(m+\delta_3, \gamma_3 - \ln\{1-G(s_m)^{\alpha}\}\right)
\end{aligned} \tag{3.32}$$

Once again random samples are generated from these distributions using Gibbs sampling. The algorithm is as follows:

1. Start with an initial guess $\left(\beta_1^{(0)}, \beta_2^{(0)}, \beta_3^{(0)}\right)$.

2. Set $t = 1$.

3. Generate $\beta_1^{(t)}$ from $Gamma\left(n_1+\delta_1, \gamma_1 - \ln\{1-G(r_{n_1})^{\alpha}\}\right)$.

4. Generate $\beta_2^{(t)}$ from $Gamma\left(n_2+\delta_2, \gamma_2 - \ln\{1-G(p_{n_2})^{\alpha}\}\right)$.

5. Generate $\beta_3^{(t)}$ from $Gamma\left(m+\delta_3, \gamma_3 - \ln\{1-G(s_m)^{\alpha}\}\right)$.

6. Compute $R_{s,k_1,k_2}^{(t)} = \sum_{\mathbf{k}}^{\mathbf{s}} \dfrac{\beta_3^{(t)}}{(i_1+j_1)\beta_1^{(t)} + (i_2+j_2)\beta_2^{(t)} + \beta_3^{(t)}}$.

7. Set $t = t+1$.

8. Repeat steps $2-7$ T times.

The sample obtained in the above algorithm is then used to obtain the Bayes estimate of R_{s,k_1,k_2} as well as the HPD credible intervals for R_{s,k_1,k_2}. The Bayes estimates of R_{s,k_1,k_2} under the SE and LINEX loss functions are given respectively by

$$\hat{R}^B_{s,k_1,k_2} = \frac{1}{T}\sum_{t=1}^{T} R^{(t)}_{s,k_1,k_2} \tag{3.33}$$

$$\hat{R}^B_{s,k_1,k_2} = -\frac{1}{v}\ln E\left(e^{-vR_{s,k_1,k_2}}\right) = -\frac{1}{v}\ln\left(\frac{1}{T}\sum_{t=1}^{T} e^{-vR^{(t)}_{s,k_1,k_2}}\right) \tag{3.34}$$

The $100(1-\eta)\%$ HPD credible intervals for R_{s,k_1,k_2} can be obtained by the method of [43]. For illustration purposes, a much simplified system made up $k_1 = 1$ component of type 1 and $k_2 = 1$ component of type 2 which functions as long as both the components are functioning is considered. In this case, the reliability expression (2.2) simplifies to

$$R_{2,1,1} = R = \frac{\beta_3}{\beta_1+\beta_2+\beta_3} \tag{3.35}$$

From (3.29), we can deduce that β_1, β_2, and β_3 have gamma marginal posterior distributions. Suppose $\beta_1 \sim Gamma(\delta_1, \gamma_1)$, $\beta_2 \sim Gamma(\delta_2, \gamma_2)$, $\beta_3 \sim Gamma(\delta_3, \gamma_3)$. Let $X = \beta_3$ and $Y = \beta_1 + \beta_2$ so that $R = \dfrac{X}{Y+X}$. The pdf of R needs to be derived. [46] derived the distribution for a sum of n independent gamma random variables with different parameters expressed as a single gamma-series. This representation is computationally friendly as coefficients are calculated using a simple iteration and a truncation error is easily attainable. Using this result, one can obtain the distribution of $Y = \beta_1 + \beta_2$ and it is given by

$$f_Y(y) = C\sum_{i=0}^{\infty} \frac{\tau_i y^{\rho+i-1} e^{-\frac{y}{\gamma_1}}}{\gamma_1^{i+\rho}\Gamma(i+\rho)} \tag{3.36}$$

where

$$\begin{aligned}
C &= \prod_{j=1}^{2}\left(\frac{\gamma_1}{\gamma_j}\right)^{\delta_j} = \left(\frac{\gamma_1}{\gamma_2}\right)^{\delta_2} \\
\rho &= \sum_{j=1}^{2} \delta_j = \delta_1 + \delta_2 \\
\xi_j &= \sum_{i=1}^{2}\left(\delta_i \frac{\left(1-\frac{\gamma_1}{\gamma_i}\right)^j}{j}\right) = \frac{\delta_2\left(1-\frac{\gamma_1}{\gamma_2}\right)^j}{j}, \qquad j = 1,2,\ldots \\
\tau_{j+1} &= \frac{1}{(j+1)}\sum_{i=1}^{j+1} i\xi_i\tau_{j+1-i}, \qquad j = 0,1,2,\ldots \text{with } \tau_0 = 1.
\end{aligned} \tag{3.37}$$

In order to derive the pdf of R its first noted that R must take values between 0 and 1 and so the derivation proceeds as follows :

$$\text{For } r \in (0,1), \qquad \begin{aligned} F_R(r) &= \Pr\left(\frac{X}{Y+X} \le r\right) \\ &= \Pr\left(Y \ge X(\frac{1}{r}-1)\right) \\ &= \int_0^{\infty} f_X(x)\Pr\left(Y \ge x(\frac{1}{r}-1)\right)dx \end{aligned}$$

, see Chaitanya [47]. Differentiating the resulting expression for $F_R(r)$ with respect to r and simplifying yields

$$f_R(r) = C\sum_{i=0}^{\infty} \tau_i \frac{(\gamma_1\gamma_3)^{\delta_3}(z+1)^2 z^{k+\rho-1}(\gamma_1\gamma_3+z)^{-(\delta_3+k+\rho)}}{B(\delta_3, k+\rho)}, \quad 0 < r < 1 \tag{3.38}$$

where $z = \dfrac{1-r}{r}$ and $B(x,y)$ is the standard beta function. The Bayes estimator of $R_{2,1,1}$, denoted by $\hat{R}_{2,1,1}^{BE}$, under the SE loss function is the mean of the posterior distribution in (3.38) , and is therefore given by

$$\begin{aligned}\hat{R}_{2,1,1}^{BE} = E(R) = &\int_0^1 r f_R(r) dr \\ = &C \sum_{i=0}^{\infty} \tau_i \frac{\delta_3 (1-d)^{\delta_3} {}_2F_1\left(a, b; b+1; d\right)}{b}\end{aligned} \tag{3.39}$$

where $a = 1 + \delta_3$, $b = \delta_3 + \rho + i$, and $d = 1 - \gamma_1 \gamma_3$ and ${}_2F_1$ is the Gauss-hypergeometric function with C, ρ, and τ_i as defined in (3.37). It is also clear that in the case of a system with n components of n different types, the reliability expression in (3.35) becomes

$$R_{n,1,1,\ldots,1} = R = \frac{\beta_n}{\sum\limits_{i=1}^{n-1} \beta_i + \beta_n} \tag{3.40}$$

Therefore the technique used above may be generalized to get the Bayes estimate of the reliability expression in (3.40). For the reliability expression (3.40), if the β_i, $i = 1, 2, 3, \ldots, n$ have a common scale parameter γ, using the well known facts that

$$\sum_{i=1}^{n-1} \beta_i \sim Gamma\left(\sum_{i=1}^{n-1} \delta_i, \gamma\right),$$

and therefore

$$\frac{\beta_n}{\beta_n + \sum\limits_{i=1}^{n-1} \beta_i} \sim Beta\left(\delta_n, \sum_{i=1}^{n-1} \delta_i\right)$$

Using properties of the beta distribution, the Bayesian estimate of reliability (3.40) is given by

$$E\left(\frac{\beta_n}{\beta_n + \sum\limits_{i=1}^{n-1} \beta_i}\right) = \frac{\delta_n}{\delta_n + \sum\limits_{i=1}^{n-1} \delta_i}$$

4. A BRIEF OVERLOOK AT THE MLE, UMVUE and BAYES ESTIMATION FOR R_{s,k_1,k_2} USING COMPLETE SAMPLES

4.1 MLE Estimation of R_{s,k_1,k_2} For Unknown α II

Let $(X_1, X_2, ..., X_{n_1})$, $(Y_1, Y_2, ..., Y_{n_2})$ and $(Z_1, Z_2, ..., Z_m)$ be random samples of sizes n_1, n_2 and m from Kw-$G(\alpha, \beta_1)$, Kw-$G(\alpha, \beta_2)$, and Kw-$G(\alpha, \beta_3)$ respectively. Then the respective likelihood functions of the observed samples are given by

$$\mathscr{L}_1(\alpha, \beta_1|\underline{x}) = (\alpha\beta_1)^{n_1} \prod_{i=1}^{n_1} \left(\frac{g(x_i)G(x_i)^{\alpha-1}}{1-G(x_i)^{\alpha}} \right) \prod_{i=1}^{n_1} (1-G(x_i)^{\alpha})^{\beta_1} \tag{4.1}$$

$$\mathscr{L}_2(\alpha, \beta_2|\underline{y}) = (\alpha\beta_2)^{n_2} \prod_{i=1}^{n_2} \left(\frac{g(y_i)G(y_i)^{\alpha-1}}{1-G(y_i)^{\alpha}} \right) \prod_{i=1}^{n_2} (1-G(y_i)^{\alpha})^{\beta_2} \tag{4.2}$$

$$\mathscr{L}_3(\alpha, \beta_3|\underline{z}) = (\alpha\beta_3)^{m} \prod_{i=1}^{m} \left(\frac{g(z_i)G(z_i)^{\alpha-1}}{1-G(z_i)^{\alpha}} \right) \prod_{i=1}^{m} (1-G(z_i)^{\alpha})^{\beta_3} \tag{4.3}$$

Thus the overall likelihood function of $\theta = (\beta_1, \beta_2, \beta_3, \alpha)$ based on the observed samples $\underline{x}$,$\underline{y}$ and $\underline{z}$ can be written as $\mathscr{L}\left(\beta_1, \beta_2, \beta_3, \alpha \,|\underline{x}, \underline{y}, \underline{z}\right) = \prod_{i=1}^{3} \mathscr{L}_i$ and the corresponding log-likelihood is given by

$$\begin{aligned}\ell\left(\theta|\underline{x}, \underline{y}, \underline{z}\right) =& (n_1+n_2+m)\ln\alpha + (\alpha-1)\left(\sum_{i=1}^{n_1} \ln G(x_i) + \sum_{i=1}^{n_2} \ln G(y_i) + \sum_{i=1}^{m} \ln G(z_i) \right) \\ & + (\beta_1-1)\sum_{i=1}^{n_1} \ln\left[g(x_i)\left(1-G(x_i)^{\alpha}\right)\right] + (\beta_2-1)\sum_{i=1}^{n_2} \ln\left[g(y_i)\left(1-G(y_i)^{\alpha}\right)\right] \\ & + (\beta_3-1)\sum_{i=1}^{m} \ln\left[g(z_i)\left(1-G(z_i)^{\alpha}\right)\right] + n_1\ln\beta_1 + n_2\ln\beta_2 + m\ln\beta_3\end{aligned} \tag{4.4}$$

The MLEs of α, and β_i , denoted by $\hat{\alpha}$ and $\hat{\beta}_i$, $i = 1, 2, 3$ respectively are the solutions to the following system of equations :

$$\frac{\partial\ell}{\partial\beta_1} = \frac{n_1}{\beta_1} + \sum_{i=1}^{n_1} \ln(1-G(x_i)^{\alpha}) = 0 \tag{4.5}$$

$$\frac{\partial\ell}{\partial\beta_2} = \frac{n_2}{\beta_2} + \sum_{i=1}^{n_2} \ln(1-G(y_i)^{\alpha}) = 0 \tag{4.6}$$

$$\frac{\partial\ell}{\partial\beta_3} = \frac{m}{\beta_3} + \sum_{i=1}^{m} \ln(1-G(z_i)^{\alpha}) = 0 \tag{4.7}$$

$$\frac{\partial \ell}{\partial \alpha} = \frac{(n_1+n_2+m)}{\alpha} + (\beta_1-1)\sum_{i=1}^{n_1}\frac{G(x_i)^{\alpha}\ln G(x_i)}{1-G(x_i)^{\alpha}} + (\beta_2-1)\sum_{i=1}^{n_2}\frac{G(y_i)^{\alpha}\ln G(y_i)}{1-G(y_i)^{\alpha}}$$
$$+ (\beta_3-1)\sum_{i=1}^{m}\frac{G(z_i)^{\alpha}\ln G(z_i)}{1-G(z_i)^{\alpha}} + \sum_{i=1}^{n_1}\ln G(x_i) + \sum_{i=1}^{n_2}\ln G(y_i) + \sum_{i=1}^{m}\ln G(z_i) = 0 \tag{4.8}$$

Therefore,

$$\hat{\beta}_1 = -\frac{n_1}{\sum_{i=1}^{n_1}\ln\left(1-G(x_i)^{\hat{\alpha}}\right)}, \quad \hat{\beta}_2 = -\frac{n_2}{\sum_{i=1}^{n_2}\ln\left(1-G(y_i)^{\hat{\alpha}}\right)} \tag{4.9}$$

$$\hat{\beta}_3 = -\frac{m}{\sum_{i=1}^{m}\ln\left(1-G(z_i)^{\hat{\alpha}}\right)} \tag{4.10}$$

The MLE of α on the hand can be obtained as a solution to the following nonlinear-equation $\xi(\alpha) = \alpha$, where

$$\xi(\alpha) = -(n_1+n_2+m)\left[\left(\hat{\beta}_1-1\right)\sum_{i=1}^{n_1}\frac{G(x_i)^{\alpha}\ln G(x_i)}{1-G(x_i)^{\alpha}} + \left(\hat{\beta}_2-1\right)\sum_{i=1}^{n_2}\frac{G(y_i)^{\alpha}\ln G(y_i)}{1-G(y_i)^{\alpha}}\right.$$
$$\left. + \left(\hat{\beta}_3-1\right)\sum_{i=1}^{m}\frac{G(z_i)^{\alpha}\ln G(z_i)}{1-G(z_i)^{\alpha}} + \sum_{i=1}^{n_1}\ln G(x_i) + \sum_{i=1}^{n_2}\ln G(y_i) + \sum_{i=1}^{m}\ln G(z_i)\right]^{-1} \tag{4.11}$$

It is clear that $\hat{\alpha}$ is a fixed point of the equation $\xi(\alpha) = \alpha$ and can therefore be obtained via an iterative scheme as follows

$$\xi(\alpha_i) = \alpha_i$$

Where α_i is the i-th iterate of $\hat{\alpha}$. The iterative procedure will be halted when the quantity $|\alpha_{i+1} - \alpha_i|$ is sufficiently small. Therefore, the maximum likelihood estimator of R_{s,k_1,k_2} is given by

$$R_{s,k_1,k_2} = \sum \mathbf{\frac{s}{k}} \left(\frac{\hat{\beta}_3}{k\hat{\beta}_1 + h\hat{\beta}_2 + \hat{\beta}_3}\right) \tag{4.12}$$

4.2 Asymptotic Confidence Interval II

In this subsection we derive the asymptotic distribution of $\hat{\boldsymbol{\theta}} = \left(\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3, \hat{\alpha}\right)$ and from this, the asymptotic distribution of R_{s,k_1,k_2} is derived. We later construct an asymptotic confidence interval based on the asymptotic distribution of R_{s,k_1,k_2} . The expected Fisher information matrix of $\boldsymbol{\theta} = (\beta_1, \beta_2, \beta_3, \alpha)$ is given by $\P(\boldsymbol{\beta}) = E(\mathbf{I}(\boldsymbol{\theta}))$, where $\mathbf{I}(\boldsymbol{\theta}) = [I_{ij}(\boldsymbol{\beta})] = \left[-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j}\right]$ for $i, j = 1,2,3,4$ is the observed information matrix. Thus we have

$$\frac{\partial^2 \ell}{\partial \beta_1 \partial \alpha} = \sum_{i=1}^{n_1} \frac{G(x_i)^\alpha \ln G(x_i)}{1 - G(x_i)^\alpha}, \qquad \frac{\partial^2 \ell}{\partial \beta_2 \partial \alpha} = \sum_{i=1}^{n_2} \frac{G(y_i)^\alpha \ln G(y_i)}{1 - G(y_i)^\alpha},$$

$$\frac{\partial^2 \ell}{\partial \beta_1 \partial \alpha} = \sum_{i=1}^{m} \frac{G(z_i)^\alpha \ln G(z_i)}{1 - G(z_i)^\alpha}$$

and

$$\begin{aligned}\frac{\partial^2 \ell}{\partial \alpha^2} =& \frac{-(n_1+n_2+m)}{\alpha^2} + (\beta_1 - 1) \sum_{i=1}^{n_1} \frac{G(x_i)^\alpha (\ln G(x_i))^2}{(1 - G(x_i)^\alpha)^2} \\ &+ (\beta_2 - 1) \sum_{i=1}^{n_1} \frac{G(y_i)^\alpha (\ln G(y_i))^2}{(1 - G(y_i)^\alpha)^2} + (\beta_3 - 1) \sum_{i=1}^{n_1} \frac{G(z_i)^\alpha (\ln G(z_i))^2}{(1 - G(z_i)^\alpha)^2}.\end{aligned}$$

Lemma 3. *Let $(X_1, X_2, ..., X_{n_1})$ be a random sample of size n_1 from the Kw-$G(\alpha, \beta_1)$. Then the following hold*

(i) $E\left(\frac{G(X_i)^\alpha \ln G(X_i)}{1 - G(X_i)^\alpha}\right) = \frac{1}{\alpha}\left([\psi(\beta_1+1) - \psi(1)] - \frac{\beta_1}{(\beta_1 - 1)}[\psi(\beta_1) - \psi(1)]\right),$

(ii) $E\left(\frac{G(X_i)^\alpha (\ln G(X_i))^2}{(1 - G(X_i)^\alpha)^2}\right) = \frac{2\beta_1}{\alpha}\left[\sum_{k=0}^{\infty} \frac{1}{(k+1)}\left(\frac{1}{(\beta_1+k-1)} - \frac{1}{(\beta_1+k)}\right)\sum_{j=1}^{k}\frac{1}{j}\right]$

Where $\psi(t) = \frac{d}{dt}\Gamma(t)$ is the Psi (polygamma) function.

Proof. If $U(X_i) = G(X_i)^\alpha$, then it is easy to show that $f_U(u) = \theta_1(1-u)^{\theta_1 - 1}$, with $0 < u < 1$ is the pdf of U. Now define functions

$$\zeta_1(G(X_i)^\alpha) = \frac{G(X_i)^\alpha \ln G(X_i)}{1 - G(X_i)^\alpha} = \frac{G(X_i)^\alpha \left(\frac{1}{\alpha}\ln G(X_i)^\alpha\right)}{1 - G(X_i)^\alpha}$$

and

$$\xi_1(G(X_i)^\alpha) = \frac{G(X_i)^\alpha (\ln G(X_i))^2}{(1 - G(X_i)^\alpha)^2} = \frac{G(X_i)^\alpha \left(\frac{1}{\alpha}\ln G(X_i)^\alpha\right)^2}{(1 - G(X_i)^\alpha)^2}$$

Employing formulae 4.293(8) and 1.516(1)) of [38] for (i) and (ii) respectively , we proceed as follows

$$\begin{aligned}E(\zeta_1(U)) =& \int_0^1 \left(\frac{\beta_1}{\alpha}\right) u(1-u)^{\beta_1 - 2} \ln u du, \quad (\text{set } t = 1-u) \\ =& \left(\frac{\beta_1}{\alpha}\right)\left(\int_0^1 t^{\beta_1 - 2}\ln(1-t)dt - \int_0^1 t^{\beta_1 - 1}\ln(1-t)dt\right) \\ =& \frac{1}{\alpha}\left([\psi(\beta_1+1) - \psi(1)] - \frac{\beta_1}{(\beta_1 - 1)}[\psi(\beta_1) - \psi(1)]\right), \beta_1 > 0\end{aligned}$$

$$
\begin{aligned}
E\left(\xi_1(U)\right) &= \left(\frac{\beta_1}{\alpha^2}\right)\int_0^1 u(1-u)^{\beta_1-3}\ln^2 u du, \quad (\text{set } t=1-u) \\
&= \left(\frac{\beta_1}{\alpha^2}\right)\left(\int_0^1 t^{\beta_1-3}\ln^2(1-t)dt - \int_0^1 t^{\beta_1-2}\ln^2(1-t)dt\right) \\
&= \left(\frac{2\beta_1}{\alpha^2}\right)\left[\sum_{k=0}^{\infty}\frac{1}{(k+1)}\left(\frac{1}{(\beta_1+k-1)}-\frac{1}{(\beta_1+k)}\right)\sum_{j=1}^{k}\frac{1}{j}\right], \beta_1>0
\end{aligned}
$$

□

Similarly, we can derive $E\left(\zeta_2(G(Y_i)^{\alpha})\right)$, $E\left(\xi_2(G(Y_i)^{\alpha})\right)$ and $E\left(\zeta_3(G(Z_i)^{\alpha})\right)$, $E\left(\xi_3(G(Z_i)^{\alpha})\right)$ for the random samples $(Y_1,Y_2,...,Y_{n_2})$ and $(Z_1,Z_2,...,Z_m)$ of sizes n_2, m and from *Kw-G*(α,β_2), and *Kw-G*(α,β_3) respectively. From Lemma 3, we can conclude that

$$
\begin{aligned}
I_{14} &= \frac{1}{\alpha}\left([\psi(\beta_1+1)-\psi(1)] - \frac{\beta_1}{(\beta_1-1)}[\psi(\beta_1)-\psi(1)]\right), \\
I_{24} &= \frac{1}{\alpha}\left([\psi(\beta_2+1)-\psi(1)] - \frac{\beta_2}{(\beta_2-1)}[\psi(\beta_2)-\psi(1)]\right), \\
I_{34} &= \frac{1}{\alpha}\left([\psi(\beta_3+1)-\psi(1)] - \frac{\beta_3}{(\beta_3-1)}[\psi(\beta_3)-\psi(1)]\right),
\end{aligned} \tag{4.13}
$$

and

$$
\begin{aligned}
I_{44} =& \frac{(n_1+n_2+m)}{\alpha^2} + (1-\beta_1)\sum_{i=1}^{n_1}E\left(\xi_1(G(X_i)^{\alpha})\right) + (1-\beta_2)\sum_{i=1}^{n_2}E\left(\xi_2(G(Y_i)^{\alpha})\right) \\
&+ (1-\beta_3)\sum_{i=1}^{m}E\left(\xi_3(G(Z_i)^{\alpha})\right)
\end{aligned} \tag{4.14}
$$

Furthermore, it can be shown that

$$
I_{11} = \frac{n_1}{\beta_1^2}, I_{22} = \frac{n_2}{\beta_2^2}, I_{33} = \frac{m}{\beta_3^2}, \text{ and } I_{12}=I_{13}=I_{21}=I_{23}=I_{31}=I_{32}=0. \tag{4.15}
$$

Theorem 8. *If* $\hat{\boldsymbol{\theta}} = \left(\hat{\beta}_1,\hat{\beta}_2,\hat{\beta}_3,\hat{\alpha}\right)$ *is the maximum likelihood estimator of* $\boldsymbol{\theta} = (\beta_1,\beta_2,\beta_3,\alpha)$, *then*

$$
\left[\hat{\beta}_1-\beta_1,\hat{\beta}_2-\beta_2,\hat{\beta}_3-\beta_3,\hat{\alpha}-\alpha\right]^T \to N_4\left(0,\mathbf{B}^{-1}(\boldsymbol{\theta})\right)
$$

Where $\mathbf{B}(\boldsymbol{\theta})$ *and* $\mathbf{B}^{-1}(\boldsymbol{\theta})$ *are symmetric matrices such that*

$$
\mathbf{B}(\boldsymbol{\theta}) = \begin{pmatrix} b_{11} & 0 & 0 & b_{14} \\ & b_{22} & 0 & b_{24} \\ & & b_{33} & b_{34} \\ & & & b_{44} \end{pmatrix}, \quad \mathbf{B}^{-1}(\boldsymbol{\theta}) = \frac{1}{|\mathbf{B}(\boldsymbol{\theta})|}\begin{pmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ & d_{22} & d_{23} & d_{24} \\ & & d_{33} & d_{34} \\ & & & d_{44} \end{pmatrix}
$$

with

$$|\mathbf{B}(\boldsymbol{\theta})| = b_{11}b_{22}\left(b_{33}b_{44}-b_{34}^2\right) - b_{11}b_{33}\left(b_{23}^2-b_{14}^2\right)$$

and *the entries of each of the matrices being*

$$d_{11} = b_{22}\left(b_{33}b_{44}-b_{34}^2\right) - b_{33}b_{24}^2, \quad d_{12} = b_{14}b_{24}b_{33}, \quad d_{13} = b_{14}b_{22}b_{34},$$

$$d_{22} = u_{11}\left(u_{22}u_{44}-u_{34}^2\right) - u_{33}u_{14}^2, \quad d_{23} = b_{11}b_{24}b_{34}, \quad d_{24} = -b_{11}b_{24}b_{33},$$

$$d_{33} = u_{11}\left(u_{22}u_{44}-u_{24}^2\right) - u_{22}u_{14}^2, \quad d_{34} = -b_{11}b_{22}b_{34},$$

$$d_{44} = b_{11}b_{22}b_{33}, \quad d_{14} = b_{14}b_{22}b_{33}$$

Proof. The proof of the theorem follows fromt the asymptotic normality of MLE, see [39]. □

Theorem 9. *If $\hat{R}_{s,k_1,k_2}$ is the MLE of R_{s,k_1,k_2}, then*

$$\left(\hat{R}_{s,k_1,k_2} - R_{s,k_1,k_2}\right) \to \mathbf{N}(0,\sigma^2)$$

where

$$\sigma^2 = \frac{1}{|\mathbf{B}(\boldsymbol{\theta})|}\left[\left(\frac{\partial R_{s,k_1,k_2}}{\partial \beta_1}\right)^2 d_{11} + 2\frac{\partial R_{s,k_1,k_2}}{\partial \beta_1}\frac{\partial R_{s,k_1,k_2}}{\partial \beta_2}d_{12} + \left(\frac{\partial R_{s,k_1,k_2}}{\partial \beta_2}\right)^2 d_{22}\right.$$
$$\left. + 2\frac{\partial R_{s,k_1,k_2}}{\partial \beta_1}\frac{\partial R_{s,k_1,k_2}}{\partial \beta_3}d_{13} + 2\frac{\partial R_{s,k_1,k_2}}{\partial \theta_2}\frac{\partial R_{s,k_1,k_2}}{\partial \beta_3}d_{23} + \left(\frac{\partial R_{s,k_1,k_2}}{\partial \beta_3}\right)^2 d_{33}\right]$$

Proof. Using Theorem 1 and the delta method (see [39]) ,the asymptotic distribution of the $R_{s,k_1,k_2} = g^*(\theta)$ may be written as follows

$$\left(\hat{R}_{s,k_1,k_2} - R_{s,k_1,k_2}\right) \to \mathbf{N}(0,\sigma^2)$$

where $\sigma^2 = \mathbf{c^T B^{-1} c}$ with $\mathbf{c} = \left[\frac{\partial R_{s,k_1,k_2}}{\partial \boldsymbol{\theta}}\right]^T = \left[\frac{\partial R_{s,k_1,k_2}}{\partial \beta_1}, \frac{\partial R_{s,k_1,k_2}}{\partial \beta_2}, \frac{\partial R_{s,k_1,k_2}}{\partial \beta_3}, 0\right]^T$,

$$\frac{\partial R_{s,k_1,k_2}}{\partial \beta_1} = \sum_{\mathbf{k}}^{\mathbf{s}}\left(\frac{-p\beta_3}{(p\theta_1+q\beta_2+\beta_3)^2}\right),$$

$$\frac{\partial R_{s,k_1,k_2}}{\partial \beta_2} = \sum_{\mathbf{k}}^{\mathbf{s}}\left(\frac{-q\beta_3}{(p\beta_1+q\beta_2+\beta_3)^2}\right),$$

$$\frac{\partial R_{s,k_1,k_2}}{\partial \beta_3} = \sum_{\mathbf{k}}^{\mathbf{s}}\left(\frac{p\beta_1+q\beta_2}{(p\beta_1+h\beta_2+\beta_3)^2}\right),$$

and $\mathbf{B^{-1}}(\theta)$ is as defined in Theorem 1. Thus, it must be true that

$$\sigma^2 = \frac{1}{|\mathbf{B}(\boldsymbol{\theta})|}\left[\left(\frac{\partial R_{s,k_1,k_2}}{\partial \beta_1}\right)^2 d_{11} + 2\frac{\partial R_{s,k_1,k_2}}{\partial \beta_1}\frac{\partial R_{s,k_1,k_2}}{\partial \beta_2} d_{12} + \left(\frac{\partial R_{s,k_1,k_2}}{\partial \beta_2}\right)^2 d_{22}\right.$$
$$\left. + 2\frac{\partial R_{s,k_1,k_2}}{\partial \beta_1}\frac{\partial R_{s,k_1,k_2}}{\partial \beta_3} d_{13} + 2\frac{\partial R_{s,k_1,k_2}}{\partial \beta_2}\frac{\partial R_{s,k_1,k_2}}{\partial \beta_3} d_{23} + \left(\frac{\partial R_{s,k_1,k_2}}{\partial \beta_3}\right)^2 d_{33}\right]$$

This concludes the proof. □

Therefore, a $100(1-\gamma)\%$ asymptotic confidence interval of R_{s,k_1,k_2} is given by

$$\left(\hat{R}_{s,k_1,k_2} - z_{1-\frac{\gamma}{2}}\hat{\sigma},\, \hat{R}_{s,k_1,k_2} + z_{1-\frac{\gamma}{2}}\hat{\sigma}\right) \tag{4.16}$$

Where z_γ is the $100\gamma-$th percentile of $\mathbf{N}(0,1)$.

4.3 Uniformly Minimum Variance Unbiased Estimator of R_{s,k_1,k_2} II

In order to derive the UMVUE of reliability, the thesis starts with the following lemma.

Lemma 4. *Let $(X_1,X_2,...,X_{n_1})$, $(Y_1,Y_2,...,Y_{n_2})$, and $(Z_1,Z_2,...,Z_m)$ be independent random samples of sizes n_1, n_2 and m from the distributions Kw-G(α,β_1), Kw-G(α,β_2) and Kw-G(α,β_3) respectively. The statistic*

$$(R,S,T) = \left(-\sum_{i=1}^{n_1}\ln\left[1-G(X_i)^\alpha\right], -\sum_{i=1}^{n_2}\ln\left[1-G(Y_i)^\alpha\right], -\sum_{i=1}^{m}\ln\left[1-G(Z_i)^\alpha\right]\right),$$

is a complete sufficient statistic for $\boldsymbol{\theta} = (\beta_1,\beta_2,\beta_3)$.

Proof. The joint pdf of the random sample $\boldsymbol{X} = (X_1,...,X_{n_1})$, is given by

$$\begin{aligned} f_{\boldsymbol{X}}(\boldsymbol{x}) &= (\alpha\beta_1)^{n_1}\prod_{i=1}^{n_1}\left(\frac{g(x_i)G(x_i)^{\alpha-1}}{1-G(x_i)^\alpha}\right)\prod_{i=1}^{n_1}(1-G(x_i)^\alpha)^{\beta_1} \\ &= \left[\alpha^{n_1}\prod_{i=1}^{n_1}\left(\frac{g(x_i)G(x_i)^{\alpha-1}}{1-G(x_i)^\alpha}\right)\right]\left[\beta_1^{n_1}\prod_{i=1}^{n_1}(1-G(x_i)^\alpha)^{\beta_1}\right] \\ &= \left[\alpha^{n_1}\prod_{i=1}^{n_1}\left(\frac{g(x_i)G(x_i)^{\alpha-1}}{1-G(x_i)^\alpha}\right)\right]\left[\beta_1^{n_1}\exp\left(-\beta_1\left(-\sum_{i=1}^{m}\ln(1-G(z_i)^\alpha)\right)\right)\right] \end{aligned} \tag{4.17}$$

It follows that $R = -\sum_{i=1}^{n_1}\ln(1-G(X_i)^\alpha)$ is a sufficient statistic for β_1 by the Neyman-Fisher factorization criterion. R is also a complete sufficient statistic since the pdf of β_1 can be written in the canonical exponential form 3.10. A similar approach can be followed for S and T. □

Let

$$X_1^* = -\ln\left[1 - G(X_1)^\alpha\right],\ Y_1^* = -\ln\left[1 - G(Y_1)^\alpha\right],$$

and

$$Z_1^* = -\ln\left[1 - G(Z_1)^\alpha\right]$$

Then it is easy to show that X_1^*, Y_1^*, and Z_1^* are exponentially distributed random variables with means β_1^{-1}, β_2^{-1}, and β_3^{-1} respectively.

Lemma 5. *If* $X_1^* = -\ln\left[1 - G(X_1)^\alpha\right]$ *and* $U = -\sum_{i=1}^{n_1} \ln\left(1 - G(X_i)^\alpha\right)$*, the conditional distribution of* X_1^* *given* U *is given by*

$$f_{X_1^*|U}(x_1^*|U) = \frac{f_{X_1^*,U}(x_1^*,u)}{f_U(u)} = \frac{(n_1-1)(u-x_1^*)^{n_1-2}}{u^{n_1-1}}, 0 < x_1^* < u \tag{4.18}$$

Likewise, for $Y_1^* = -\ln\left[1 - G(Y_1)^\alpha\right]$ *and* $V = -\sum_{i=1}^{n_2} \ln\left[1 - G(Y_i)\right]$*, we have*

$$f_{Y_1^*|V}(y_1^*|V) = \frac{f_{Y_1^*,V}(y_1^*,v)}{f_V(v)} = \frac{(n_2-1)(v-y_1^*)^{n_2-2}}{v^{n_2-1}}, 0 < y_1^* < v, \tag{4.19}$$

and finally for $Z_1^* = -\ln\left[1 - G(Z_1)^\alpha\right]$ *and* $W = -\sum_{i=1}^{m} \ln\left[1 - G(Z_i)^\alpha\right]$*, we have that*

$$f_{Z_1^*|W}(z_1^*|W) = \frac{f_{Z_1^*,W}(z_1^*,w)}{f_W(w)} = \frac{(m-1)(w-z_1^*)^{m-2}}{w^{m-1}}, 0 < z_1^* < w \tag{4.20}$$

Proof. The proof runs parallel to a similar proof used by [48] in deriving the UMVUE of $P(X > Y)$ under progressive type-II sampling scheme. The proof is as follows: Let $Q = \sum_{i=2}^{n_1} P_i$ where $P_i = -\ln\left(1 - G(X_i)^\alpha\right)$. Since the $-\ln\left(1 - G(X_i)^\alpha\right)'s$ are independent exponential random variables, each with mean β_1^{-1}, it must be true that $Q \sim Gamma(n_1 - 1, \beta_1)$. Moreover Q and P_1 are independent and their joint distribution must be given by

$$f_{Q,P_1}(q,p) = f_Q(q) f_{P_1}(p) = \frac{\beta_1^{n_1}}{\Gamma(n_1-1)} q^{n_1-2} e^{-\beta_1(q+p)} \tag{4.21}$$

Defining $X_1^* = P_1$ and $U = Q + P_1$ and applying elementary transformation techniques yields the joint distribution of X_1^* and U as

$$f_{X_1^*,U}(x_1^*,u) = \frac{1}{(n_1-2)!} \beta_1^{n_1} (u - x_1^*)^{n_1-2} e^{-u\beta_1} \tag{4.22}$$

and similarly the marginal distribution of U is given by

$$f_U(u) = \frac{1}{(n_1-1)!} \beta_1^{n_1} e^{-u\beta_1} u^{n_1-1}$$

Consequently we get the conditional distribution of X_1^* given U as follows

$$f_{X_1^*|U}(x_1^*|U) = \frac{f_{X_1^*,U}(x_1^*,u)}{f_U(u)} = \frac{(n_1-1)(u-x_1^*)^{n_1-2}}{u^{n_1-1}}, 0 < x_1^* < u \tag{4.23}$$

Thus, the expressions are still similar to the ones derived in the records case and certainly for censored samples as well. The derivation of the remaining conditional distributions as well as the proof of the following theorem are trivial. □

Theorem 10. *For $n_1 \geq 2$, $n_2 \geq 2$, and $m \geq 2$ and the UMVUE of*

$$\varphi(\beta_1,\beta_2,\beta_3) = \frac{\beta_3}{k\beta_1 + h\beta_2 + \beta_3},$$

denoted by $\varphi_U(\beta,\beta_2,\beta_3)$, is given by

$$\varphi_U(\beta,\beta_2,\beta_3) = \begin{cases} Q_1(n_1,n_2,m,u,v,w) & if\ w \leq max\left\{\frac{v}{h}, \frac{u}{k}\right\} \\ Q_2(n_1,n_2,m,u,v,w) & if\ \frac{u}{k} \leq max\left\{w, \frac{v}{h}\right\} \\ Q_3(n_1,n_2,m,u,v,w) & if\ \frac{v}{h} \leq max\left\{w, \frac{u}{k}\right\} \end{cases} \tag{4.24}$$

where

$$\begin{aligned} Q_1 &= \sum_{a=0}^{n_1-1} (-1)^a \left(\frac{kw}{u}\right)^a \sum_{b=0}^{n_2-1} (-1)^b \left(\frac{hw}{v}\right)^b \frac{\binom{n_1-1}{a}\binom{n_2-1}{b}}{\binom{m+a+b-1}{a+b}} \\ &= F_1\left(1,(1-n_1),(1-n_2),m;\frac{kw}{u},\frac{hw}{v}\right) \\ Q_2 &= \left(\frac{m-1}{n_1}\right)\left(\frac{u}{kw}\right) \sum_{a=0}^{n_2-1} (-1)^a \left(\frac{hu}{kv}\right)^a \sum_{b=0}^{m-2} (-1)^b \left(\frac{u}{kw}\right)^b \frac{\binom{n_2-1}{a}\binom{m-2}{b}}{\binom{n_1+a+b}{a+b}} \\ &= \left(\frac{m-1}{n_1}\right)\left(\frac{u}{kw}\right) F_1\left(1,(1-n_2),(2-m),n_1+1;\frac{hu}{kv},\frac{u}{kw}\right) \\ Q_3 &= \left(\frac{m-1}{n_2}\right)\left(\frac{v}{hw}\right) \sum_{a=0}^{n_1-1} (-1)^a \left(\frac{kv}{hu}\right)^a \sum_{b=0}^{m-2} (-1)^b \left(\frac{v}{hw}\right)^b \frac{\binom{n_1-1}{a}\binom{m-2}{b}}{\binom{n_2+a+b}{a+b}} \\ &= \left(\frac{m-1}{n_2}\right)\left(\frac{v}{hw}\right) F_1\left(1,(1-n_1),(2-m),n_2+1;\frac{kv}{hu},\frac{v}{hw}\right) \end{aligned}$$

and $k = i_1 + j_1 \neq 0$, $h = i_2 + j_2 \neq 0$.

As was the case with upper record values, the UMVUE of R_{s,k_1,k_2}, denoted by R^U_{s,k_1,k_2}, must be

$$R^U_{s,k_1,k_2} = \sum{}_{\mathbf{k}}^{\mathbf{s}} \varphi_U(\theta) \tag{4.25}$$

4.4 Closed Form of The Bayes Estimator Using Conjugate And Non-Informative Priors II

Assuming gamma priors for β_1, β_2, and β_3 given by

$$\pi(\beta_i) = \frac{a_i^{b_i} \beta_i^{b_i - 1} e^{-a_i \beta_i}}{\Gamma(b_i)}, \quad \beta_i, b_i, a_i > 0,\, i = 1,2,3, \tag{4.26}$$

the joint posterior function of β_1, β_2, and β_3 (see Section 2.2) can be written as

$$\pi(\boldsymbol{\theta}\,|\underline{r}, \underline{p}, \underline{s}) = \frac{\mu_1^{v_1} \mu_2^{v_2} \mu_3^{v_3}}{\Gamma(v_1)\Gamma(v_2)\Gamma(v_3)} \beta_1^{v_1 - 1} \beta_2^{v_2 - 1} \beta_2^{v_3 - 1} \exp\left(-\mu_1 \beta_1 - \mu_2 \beta_2 - \mu_3 \beta_3\right) \tag{4.27}$$

Where

$$\mu_1 = a_1 - \sum_{i=1}^{n_1} \ln\left[1 - G(x_i)^{\alpha}\right],\ \mu_2 = a_2 - \sum_{i=1}^{n_2} \ln\left[1 - G(y_i)^{\alpha}\right]$$

$$\mu_3 = a_3 - \sum_{i=1}^{m} \ln\left[1 - G(z_i)^{\alpha}\right]$$

The Bayes estimator of R_{s,k_1,k_2}, denoted by R^B_{s,k_1,k_2}, is as follows

$$\begin{aligned} R^B_{s,k_1,k_2} &= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} R_{s,k_1,k_2}(\beta_1, \beta_2, \beta_3)\, \pi\left(\beta_1, \beta_2, \beta_3\,|\alpha, \underline{x}, \underline{y}, \underline{z}\right) d\beta_1 d\beta_2 d\beta_3 \\ &= \mathscr{M} \sum_{\mathbf{k}}^{\mathbf{s}} \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \left(\frac{\beta_1^{v_1 - 1} \beta_2^{v_2 - 1} \beta_3^{v_3} \exp\left(-\mu_1 \beta_1 - \mu_2 \beta_2 - \mu_3 \beta_3\right)}{p\beta_1 + q\beta_2 + \beta_3} \right) d\boldsymbol{\theta} \end{aligned}$$

where

$$\mathscr{M} = \frac{\mu_1^{v_1} \mu_2^{v_2} \mu_3^{v_3}}{\Gamma(v_1)\Gamma(v_2)\Gamma(v_3)}$$

Define a one-to-one transformation $\mathscr{T}$ by

$$\left.\begin{aligned} x &= \frac{p\beta_1}{p\beta_1 + q\beta_2 + \beta_3} \\ y &= \frac{q\beta_2}{p\beta_1 + q\beta_2 + \beta_3} \\ z &= p\beta_1 + q\beta_2 + \beta_3 \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} \beta_1 &= xz/p \\ \beta_2 &= yz/q \\ \beta_3 &= z(1 - x - y) \end{aligned}\right. ,$$

where $p \neq 0$ and $q \neq 0$. It is evident that $R^B_{s,k_1,k_2} = 1$ for the case when $p = 0$ and $q = 0$. We note that

$$0 < x + y = \frac{p\beta_1 + q\beta_2}{p\beta_1 + q\beta_2 + \beta_3} < 1,\ 0 < z < \infty \text{ and } \frac{\beta_3}{p\beta_1 + q\beta_2 + \beta_3} = 1 - x - y.$$

Furthermore, the Jacobian of the transformation is given by

$$|J(x,y,z)| = \begin{vmatrix} z/p & 0 & x/p \\ 0 & z/q & y/q \\ -z & -z & 1 - x - y \end{vmatrix} = \frac{z^2}{pq}.$$

Therefore, we have that

$$\begin{aligned} R^B_{s,k_1,k_2} =& \frac{\mu_1^{v_1}\mu_2^{v_2}\mu_3^{v_3}}{p^{v_1}q^{v_2}\Gamma(v_1)\Gamma(v_2)\Gamma(v_3)}\sum_{\mathbf{k}}^{\mathbf{s}}\int_0^1\int_0^{1-y}\int_0^{\infty} x^{v_1-1}y^{v_2-1}\left(1-x-y\right)^{v_3} \\ &\times z^{v_1+v_2+v_3-1}\exp\left\{-z\left[\mu_1 x/p+\mu_2 y/q+\mu_3(1-x-y)\right]\right\}dzdxdy \\ =& \mathscr{K}\sum_{\mathbf{k}}^{\mathbf{s}}\int_0^1\int_0^{1-y} x^{v_1-1}y^{v_2-1}\left(1-x-y\right)^{v_3}\left(1-\sigma_1 x-\sigma_2 y\right)^{-(v_1+v_2+v_3)}dxdy \end{aligned}$$

where

$$\mathscr{K} = \frac{\left(1-\sigma_1\right)^{v_1}\left(1-\sigma_2\right)^{v_2}\Gamma\left(v_1+v_2+v_3\right)}{\Gamma\left(v_1\right)\Gamma\left(v_2\right)\Gamma\left(v_3\right)}, \quad \sigma_1 = 1-\frac{\mu_1}{p\mu_3}, \quad \sigma_2 = 1-\frac{\mu_2}{q\mu_3}$$

A Euler type integral representation of the Appell hypergeometric function of the first kind, denoted by F_1, as shown in [38] is as follows

$$\frac{\Gamma(d)F_1\left(a,b,c,d;x,y\right)}{\Gamma(b)\Gamma(c)\Gamma(d-b-c)} = \int_0^1\int_0^{1-v} u^{b-1}v^{c-1}(1-u-v)^{d-b-c-1}\left(1-ux-vy\right)^{-a}dudv$$
$$b>0,\ c>0,\ d-b-c>0,\ |x|<1,\ |y|<1. \tag{4.28}$$

Using the result (4.28), together with functional relations of F_1 (see [38] formulae 9.183(1)), we can conclude that

$$R^B_{s,k_1,k_2} = \begin{cases} \varpi_1\left(\sigma_1,\sigma_2\right), \text{ if } |\sigma_1|<1,\ |\sigma_2|<1 \\ \varpi_2\left(\sigma_1,\sigma_2\right), \text{ if } \sigma_1<-1, \sigma_2<-1 \\ \varpi_3\left(\sigma_1,\sigma_2\right), \text{ if } |\sigma_1|<1, \sigma_2<-1 \\ \varpi_4\left(\sigma_1,\sigma_2\right), \text{ if }, \sigma_1<-1, |\sigma_2|<1 \end{cases} \tag{4.29}$$

where

$$\varpi_1 = \sum_{\mathbf{k}}^{\mathbf{s}}\frac{\left(1-\sigma_1\right)^{v_1}\left(1-\sigma_2\right)^{v_2}v_3}{\left(v_1+v_2+v_3\right)}F_1\left(\sum_{i=1}^3 v_i, v_1, v_2, 1+\sum_{i=1}^3 v_i; \sigma_1,\sigma_2\right),$$

$$\varpi_2 = \sum_{\mathbf{k}}^{\mathbf{s}}\left(\frac{v_3}{v_1+v_2+v_3}\right)F_1\left(1, v_1, v_2, 1+\sum_{i=1}^3 v_i; \frac{\sigma_1}{\sigma_1-1}, \frac{\sigma_2}{\sigma_2-1}\right),$$

$$\varpi_3 = \sum_{\mathbf{k}}^{\mathbf{s}}\left(\frac{v_3\left(1-\sigma_1\right)}{v_1+v_2+v_3}\right)F_1\left(1, v_3+1, v_2, 1+\sum_{i=1}^3 v_i; \sigma_1, \frac{\sigma_1-\sigma_2}{1-\sigma_2}\right),$$

and

$$\varpi_4 = \sum_{\mathbf{k}}^{\mathbf{s}}\left(\frac{v_3\left(1-\sigma_2\right)}{v_1+v_2+v_3}\right)F_1\left(1, v_1, v_3+1, 1+\sum_{i=1}^3 v_i; \frac{\sigma_2-\sigma_1}{1-\sigma_1}, \sigma_2,\right).$$

Using Jeffery's non informative priors, it can easily be shown that the Bayes estimator assumes the same form as above with $a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 0$.

5. NUMERICAL EXPERIMENTS AND SIMULATIONS

In this section, Monte Carlo simulations are conducted to compare the performance of the Bayes estimator, MLE and UMVUE of R_{s,k_1,k_2} using upper record values from the family of Kumaraswamy generalized distributions, *Kw-G*. **Case 1:** The exponential distribution with rate parameter equal to 2 is used as the baseline, G, when α is unknown. The performances of the estimators are compared in terms of mean squared error (MSE). The HDP credible intervals are compared in terms of average confidence lengths, and coverage probabilities (cp). All results are based on 1000 replications and computations are performed in MATLAB2010. When α is unknown, the performance of the MLE and Bayes estimators under Lindley approximation and MCMC method using four the parameter values $\theta = (\alpha, \beta_1, \beta_2, \beta_3) = (1,1,2,2)$ are compared. The confidence levels are held at 5% level of significance. The true value of R_{s,k_1,k_2} is evaluated for $(s_1, s_2, k_1, k_2) = (2,1,2,2), (2,3,4,6)$. Bayes estimators and HDP credible intervals are computed using following choices of prior distributions : Prior 1: $(\delta_1, \delta_2, \delta_3, \delta_4) = (2,1,1,3), (\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (2,2,1,1)$, Prior 2: $(\delta_1, \delta_2, \delta_3, \delta_4) = (1,1,2,3), (\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (1,2,3,1)$. Records samples are generated using an algorithm from [49]. The results for Case 1 are reported in Table A.5 with $\bar{L}_{CR}$ and $\bar{L}_{CI}$ denoting the average asymptotic and credible interval lengths respectively. cp_{CI} and cp_{CR} denote the respective coverage probabilities. It is observed that the MSE decreases with increase in sample sizes as expected. This confirms the consistency of the estimates. The average lengths of the asymptotic and Bayesian credible intervals also decrease with increase in sample sizes and the coverage probabilities are at least 0.80 and 0.90 respectively. For low sample sizes, Lindley's approximation has the smallest MSE followed by the MCMC and MLE methods but the performance is almost the same with increase in sample size. So, for samples large enough, any of the estimates may be be employed. **Case 2** : In the case that α is known the Weibull distribution with scale and shape parameters given by 2 and 3 respectively, is used as the baseline. The performance of the UMVUE, MLE and Bayes estimators under Lindley approximation and MCMC method are compared with R_{s,k_1,k_2} evaluated at $\theta =$

$(\alpha, \beta_1, \beta_2, \beta_3) = (1,1,2,2)$ with $(s_1, s_2, k_1, k_2) = (2,1,2,2)$. The asymptotic and HDP credible intervals are compared in terms of average confidence lengths, and coverage probabilities (cp). The prior used is Prior 3: $(\delta_1, \delta_2, \delta_3) = (1,1,2), (\gamma_1, \gamma_2, \gamma_3) = (2,3,1)$. The results of the simulations for Case 2 are reported in Table A.7 and Table A.6. It is noted that the MSE in all the estimates decreases as sample sizes increases. For low sample sizes, Lindley's approximation has the least MSE followed by the MLE and the MCMC method. Here we note that the UMVUE is not comparable under the MSE criterion, see [20]. However, even for small sample sizes, the UMVUE performs best among all the estimates in terms of biases and is therefore preferred in practice. The asymptotic and Bayesian credibles intervals lengths together with their corresponding coverage probabilities increase with increase in sample size.

5.1 Real Data Application

Recall the proposed model description from the introduction. Fatigue strength is a factor of paramount importance in structural materials in order to ensure long-term reliability of structures. It is vital to ensure that material used can sustain huge loads without failure. High-strength low alloy steels are much stronger and tougher than ordinary carbon steels and are highly resistant to corrosion, see [50] and [51]. Their increased strength means that structures can be built to contain less steel and therefore be lighter than they otherwise would be. They are often used in cars and trucks because it leads to fuel economy and less damage to road surfaces. In this section, we demonstrate how the model may be applied in real life by considering three sets of data which were produced by [52] for evaluating specimen size effect in gigacycle fatigue of high-strength JIS-SCM440 (AISI-4140) low-alloy steel under ultrasonic fatigue testing. This paper extends an idea previously used by [53], [54] and recently by Sales et al. [55]. In the present work, we propose a way of studying a well known phenomenon in materials science and engineering that fatigue strength of a material decreases with increasing specimen size (size effect), see [56] and [57]. The fatigue test results on the three specimens were extracted with WebPlotDigitizer [58] due to lack of raw data. The results are reported in Tables A.1, A.2, and A.3. As mentioned in the introduction, the material-testing experiments produced results which are naturally of records type, confirming [22]'s claim. So, there are a total of 18, 16, and 9 upper

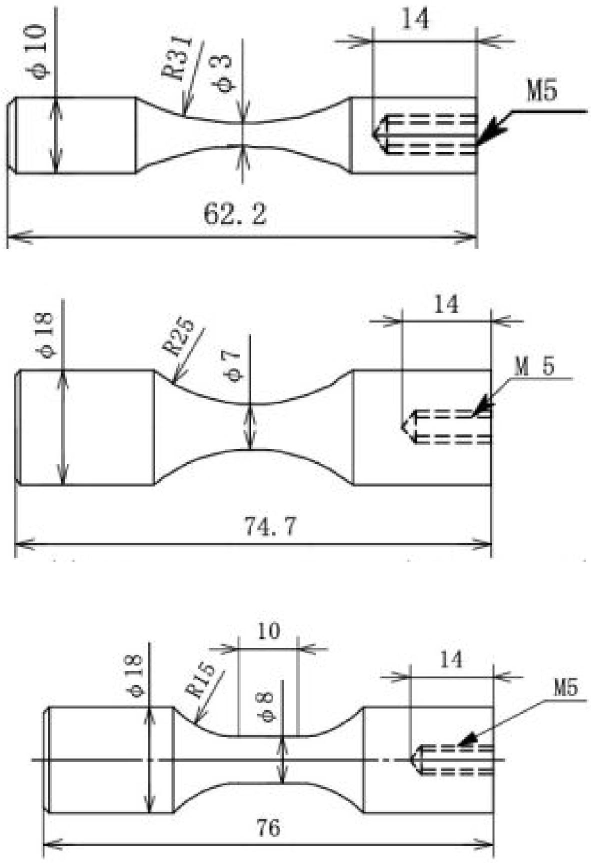

Figure 5.1 : ϕ3mm, $\phi 7mm$ and ϕ8x10mm specimens drawings.

record values form experiments 1, 2, and 3 respectively. In order to compare the specimen fatigue lives, the samples for system component strengths of type 1 and type 2 will be represented by the fatigue life samples from any two specimens with the third specimen sample used as stress sample. The Kumaraswamy-Lomax distribution with a CDF given by

$$F(x) = 1 - \left\{ 1 - \left(1 - \frac{\delta}{(x+\delta)^{\gamma}} \right)^{\alpha} \right\}^{\beta}, \; x > 0, \alpha > 0, \delta > 0, \gamma > 0, \beta > 0, \qquad (5.1)$$

was found to fit the data well with parameter values $(\alpha, \delta, \gamma, \beta_1, \beta_2, \beta_3) = (1, 6234000, 0.32958, 0.9112, 1.0831, 1.5113)$, see Table A.4 for the corresponding K-S distances and p-values. The goodness-of-fit tests were performed on the three data sets with the help of EasyFit [59], MATLAB2010 [60], and R [61]. The lifetimes of the ϕ3mm (stress), ϕ7mm (type 1 strength) and $\phi 8\times$10mm (type 2 strength) specimens are compared with the combination $(s_1, s_2, k_1, k_2) = (1, 1, 1, 1)$ which yields UMVUE of R_{s,k_1,k_2} as 0.4151. That is, the fatigue lifetime of the ϕ3mm specimen exceeds that of both the $\phi 8\times$10mm and ϕ7mm specimens. These result is in agreement with [52]'s findings that under ultrasonic fatigue testing, the fatigue strength of specimens of various sizes fell with increase in specimen sizes.

5.2 Conclusions And Possible Future Considerations

In this thesis, the estimation of stress-strength reliability in a multicomponent system with non-identical component strengths based on upper record values was considered under Bayesian and frequentist methods. The upper record value samples used were generated using the algorithm of [49]. When the common shape parameter is unknown, the MLE and Lindley estimator have similar performances for samples sizes large enough. When the common parameter is known the paper also proposes for the first time, the UMVUE of the reliability parameter using upper record values. The UMVUE performs best in terms of biases and a preferred choice in practice as it is more accurate even for small sample sizes. The asymptotic intervals perform better than Bayesian credible intervals in terms of average lengths and vise versa in terms of coverage probabilities. Despite having considered a system with non-identical components in the present work, the family of distributions used was still the same. The assumption of completely different probability distributions would certainly yield more realistic models as they would account reasonably for the differences in system components' structures. Additionally a lot has not been done in multicomponent stress-strength models for the case of components' strengths degradation over time as alluded by [15]. Investigations are underway on these issues. The mesh of the present model with that of [12] and [11] could also lead to interesting findings.

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APPENDICES

APPENDIX A.1

Table A.1 : ϕ3mm specimen fatigue test data

Experiment 1	
Specimen Number	Fatigue Life(number of cycles to failure)
1	1017286
2	2989152
3	4059346
4	4256299
5	8376572
6	9560400
7	13007977
8	25303118
9	33621704
10	55951560
11	101155984
12	144322192
13	376711232
14	731957760
15	9444513800
16	9912163300
17	9918688300
18	9921105900

Table A.2 : ϕ7mm specimen fatigue test data

Experiment 2	
Specimen Number	Fatigue Life(number of cycles to failure)
1	611670
2	890099
3	974460
4	3461990
5	13640537
6	26045358
7	28147395
8	31216343
9	39400852
10	134652209
11	217309470
12	277856285
13	350706504
14	6441526000
15	6783606914
16	8452132412

Table A.3 : $\phi 8\times 10$mm specimen fatigue test data

Experiment 3	
Specimen Number	Fatigue Life(number of cycles to failure)
1	289867
2	1291756
3	6404257
4	7848468
5	9374890
6	31500474
7	211678768
8	5575744500
9	5926607400

Table A.4 : Kolmogorov-Smirnov Goodness of Fit Test

Specimen size	Test Statistic	p-value
ϕ3mm specimen	0.1113	0.9606
ϕ7mm specimen	0.1369	0.8860
$\phi 8\times 10$mm specimen	0.1886	0.8501

APPENDIX A.2

Table A.5 : Estimating R_{s,k_1,k_2} using Prior 1 and Prior 2 with $(s_1,s_2,k_1,k_2) = (2,1,2,2)$ and $(s_1,s_2,k_1,k_2) = (2,3,4,6)$.

(n_1,n_2,m)	R_{s,k_1,k_2}	$\hat{R}_{s,k_1,k_2}$	$\hat{R}^{L}_{s,k_1,k_2}$	$\hat{R}^{MC}_{s,k_1,k_2}$	CI/CR	$\bar{L}_{CI}/\bar{L}_{CR}$	cp_{CI}/cp_{CR}
(5,5,6)	0.4667	0.4641	0.4005	0.3954	$(0.2660, 0.6621)$	0.3961/0.4236	0.8020/0.9280
		0.0204	0.0234	0.0123	(0.1885,0.6121)		
(10,10,11)		0.4672	0.4360	0.4204	$(0.3111, 0.6232)$	0.3121/0.3377	0.8360/0.9190
		0.0106	0.0084	0.0078	(0.2538,0.5915)		
(15,15,16)		0.4657	0.4487	0.4312	$(0.3336, 0.5979)$	0.2644/0.2901	0.8680/0.9280
		0.0075	0.0068	0.0061	(0.2873,0.5774)		
(20,20,21)		0.4640	0.4541	0.4408	$(0.3479, 0.5801)$	0.2322/0.2582	0.8750/0.9100
		0.0055	0.0055	0.0064	(0.3125,0.5707)		
(25,25,26)		0.4636	0.4577	0.4692	$(0.3588, 0.5685)$	0.2097/0.2386	0.8820/0.8370
		0.0042	0.0043	0.0125	(0.3507,0.5893)		
(5,5,6)	0.7009	0.6549	0.4575	0.5558	$(0.4385, 0.8661)$	0.4276/0.5115	0.7510/0.9040
		0.0300	0.0796	0.0281	(0.3010,0.8125)		
(15,15,16)		0.6844	0.6156	0.6173	$(0.5382, 0.8306)$	0.2924/0.3443	0.8490/0.8920
		0.0097	0.0136	0.0122	(0.4434,0.7878)		
(20,20,21)		0.6928	0.6438	0.6379	$(0.5641, 0.8215)$	0.2574/0.3044	0.8570/0.9040
		0.0070	0.0087	0.0085	(0.4835,0.7879)		
(30,30,31)		0.7006	0.6687	0.6961	$(0.5954, 0.8058)$	0.2104/0.2666	0.9260/0.9390
		0.0032	0.0039	0.0063			
(35,35,36)		0.7000	0.6697	0.7362	$(0.6046, 0.7954)$	0.1908/0.2629	0.9350/0.9060
		0.0022	0.0029	0.0094			

* The second row represents the MSE of the estimates.

Table A.6 : Estimating R_{s,k_1,k_2} using Prior 3 with $(s_1,s_2,k_1,k_2) = (2,1,2,2)$.

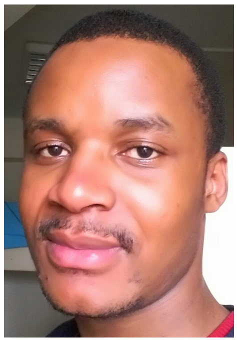
(n_1,n_2,m)	R_{s,k_1,k_2}	$\hat{R}_{s,k_1,k_2}$	$\hat{R}^L_{s,k_1,k_2}$	$\hat{R}^{MC}_{s,k_1,k_2}$	UMVUE	CI/CR	$\bar{L}_{CI}/\bar{L}_{CR}$	cp_{CI}/cp_{CR}
(5,5,6)	0.5000	0.4810	0.5562	0.3634	0.4972	$(0.3504, 0.6117)$	0.2613/0.4341	0.6330/0.8400
		0.0198	0.0169	0.0238	0.0233	(0.1564,0.5906)		
		-0.0190	0.0562	-0.1366	-0.0028			
(10,10,11)		0.4927	0.5271	0.4058	0.5000	$(0.3907, 0.5946)$	0.2039/0.3522	0.6850/0.8800
		0.0098	0.0086	0.0132	0.0107	(0.2340,0.5862)		
		-0.0073	0.0271	-0.0942	0.0000269			
(15,15,16)		0.4922	0.5145	0.4333	0.4968	$(0.4061, 0.5783)$	0.1721/0.3041	0.6830/0.9010
		0.0072	0.0064	0.0084	0.0077	(0.2831,0.5872)		
		-0.0078	0.0145	-0.0667	-0.0032			
(20,20,21)		0.4907	0.5075	0.4501	0.4968	$(0.4146, 0.5668)$	0.1523/2711	0.7070/0.9130
		0.0052	0.0047	0.0057	0.0054	(0.3154,0.5865)		
		-0.0093	0.0075	-0.0499	-0.0061			
(25,25,26)		0.4980	0.5112	0.4677	0.5007	$(0.4294, 0.5666)$	0.1372/0.2471	0.7150/0.9520
		0.0041	0.0039	0.0037	0.0042	(0.3445,0.5916)		
		-0.0020	0.0112	-0.0323	0.00066			
(30,30,35)		0.5147	0.5251	0.4973	0.5176	$(0.4563, 0.5731)$	0.1168/0.2230	0.7930/0.9840
		0.0023	0.0026	0.00181	0.0025	(0.3860,0.6091)		
		0.0147	0.0251	-0.0027	0.0176			
(35,35,36)		0.5012	0.5031	0.4841	0.5031	$(0.4447, 0.5577)$	0.1130/0.2139	0.8630/0.9840
		0.0014	0.0015	0.00141	0.0015	(0.3773,0.5912)		
		0.0012	0.0114	-0.0159	0.0031			

* The second and third rows represent the MSE and Bias of the estimates respectively.

Table A.7 : Estimating R_{s,k_1,k_2} using Prior 3 with $(s_1,s_2,k_1,k_2) = (2,1,2,2)$.

(n_1,n_2,m)	R_{s,k_1,k_2}	$\hat{R}_{s,k_1,k_2}$	$\hat{R}^{L}_{s,k_1,k_2}$	$\hat{R}^{MC}_{s,k_1,k_2}$	UMVUE	CI/CR	$\bar{L}_{CI}/\bar{L}_{CR}$	cp_{CI}/cp_{CR}
(5,5,6)	0.5000	0.4841	0.6479	0.5731	0.5010	$(0.3536, 0.6146)$	0.2610/0.4592	0.6420/0.9720
		0.0191	0.0385	0.0106	0.0010	(0.3536,0.6146)		
		-0.0159	0.1479	0.0731	-0.0010			
(15,15,16)		0.4953	0.5431	0.5580	0.5000	$(0.4088, 0.5818)$	0.1730/0.3046	0.6800/0.9080
		0.0075	0.0071	0.0077	0.0078	(0.4045,0.7046)		
		0.0047	0.0431	0.0580	-0.00004			
(20,20,21)		0.4943	0.5307	0.5548	0.4977	$(0.4185, 0.5701)$	0.1516/0.2705	0.7050/0.9060
		0.0053	0.0051	0.0052	0.0066	(0.3154,0.5865)		
		0.0057	0.0307	0.0548	-0.0023			
(25,25,26)		0.4941	0.5234	0.5527	0.4967	$(0.4257, 0.5625)$	0.1368/0.2451	0.7130/0.8700
		0.0043	0.0042	0.0062	0.0044	(0.4297,0.6747)		
		-0.0059	0.0234	0.0527	-0.00033			
(30,30,31)		0.4850	0.5115	0.5348	0.4869	$(0.4209, 0.5491)$	0.1282/0.2234	1.00/1.00
		0.0023	0.0026	0.0018	0.0025	(0.4176,0.6410)		
		-0.0150	0.0115	0.0348	-0.0131			
(35,35,36)		0.5018	0.5239	0.5577	0.5037	$(0.4453, 0.5582)$	0.1129/0.2106	0.8450/0.9130
		0.0015	0.0015	0.0019	0.0045	(0.4517,0.6623)		
		0.0018	0.0239	-0.0577	0.0037			
(40,40,41)		0.5030	0.5241	0.5574	0.5047	$(0.4519, 0.5541)$	0.1022/0.1986	0.9060/0.9250
		0.00084	0.0013	0.0040	0.00088	(0.4576,0.6562)		
		0.0030	0.0241	0.0574	0.0047			

* The second and third rows represent the MSE and Bias of the estimates respectively.



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