ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY

ON ALMOST PSEUDO RICCI SYMMETRIC MANIFOLDS

M.Sc. THESIS

Ecem BEKTAŞ

Department of Mathematical Engineering

Mathematical Engineering Programme

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To my family,

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FOREWORD

I would like to express my sincere gratitude to Prof. Dr. Füsun ÖZEN ZENGİN, her teachings and guidance helped me on my study and research on my graduation project.

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May 2017

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ABBREVIATIONS

M	**:** Manifold
g	**:** Metric tensor
$T_p(M)$	**:** The vector space of all tangent vectors at p
TM	**:** The tangent bundle of M
$C^\infty(M)$	**:** The set of all smooth functions on M
δ_i^j	**:** The Kronecker symbols
S_{ij}	**:** The Ricci curvature tensor
r	**:** The scalar curvature
$(PS)_n$	**:** An n-dimensional pseudo symmetric manifold
$(PRS)_n$	**:** An n-dimensional pseudo Ricci symmetric manifold
$A(PRS)_n$	**:** Almost pseudo Ricci symmetric manifold
κ	**:** The gravitational constant
T	**:** The energy momentum tensor
σ	**:** The energy density
p	**:** The isotropic pressure of the fluid

ON ALMOST PSEUDO RICCI SYMMETRIC MANIFOLDS

SUMMARY

The main concern of this thesis is to investigate an $n-$dimensional almost pseudo Ricci symmetric manifold (M,g) whose Ricci tensor S satisfies the condition

$$(\nabla_Z S)(X,Y) = [A(Z)+B(Z)]S(X,Y)+A(X)S(Y,Z)+A(Y)S(X,Z)$$

where A and B are two non-zero $1-$forms and ∇ denotes the operator of the covariant differentiation with respect to the metric g. Such a manifold is called almost pseudo Ricci symmetric manifold and denoted by $A(PRS)_n$.

In the first Chapter, some significant definitions and notions which will be used in the next chapters are expressed. Also, some definitions and notions are given in order to identify the Riemannian manifold.

In the second Chapter, a historical overview of the important role of symmetric spaces in differential geometry is given. Especially, Cartan dealt with classification of those spaces and established Riemannian symmetric spaces. A Riemanian manifold is called locally symmetric if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M,g). The class of Riemannian symmetric manifolds is a natural generalization of the class of manifolds of constant curvature. If the Ricci tensor S of $(0,2)$ of the manifold is non-zero and satisfies the condition

$$(\nabla_Z S)(X,Y) = 2A(Z)S(X,Y)+A(X)S(Y,Z)+A(Y)S(X,Z)$$

where ∇ denotes the Levi-Civita connection and A is a non-zero $1-$form such that $g(X,\rho) = A(X)$ for all vector fields X,ρ, then this manifold is called pseudo Ricci symmetric manifold and is denoted by $(PRS)_n$. Pseudo Ricci symmetric manifold was introduced by Chaki. Chaki and Kawaguchi generalized pseudo Ricci symmetric manifold as almost pseudo Ricci symmetric manifolds such that

$$(\nabla_Z S)(X,Y) = [A(Z)+B(Z)]S(X,Y)+A(X)S(Y,Z)+A(Y)S(X,Z).$$

If $A = B$, then $A(PRS)_n$ reduces to $(PRS)_n$. So, a pseudo Ricci symmetric manifold is a particular case of $A(PRS)_n$.

In the third Chapter, an almost pseudo Ricci symmetric manifold admitting W_2-Ricci tensor has been analyzed. Our aim is to examine of some properties of these manifolds and find theorems related by these properties. In this Chapter, firstly, W_2-curvature tensor on manifold (M,g) $n > 3$ is given by

$$W_2(X,Y,Z,U) = R(X,Y,Z,U)+\frac{1}{n-1}[g(X,Z)S(Y,U)-g(Y,Z)S(X,U)].$$

After that, W_2-flat A(PRS)$_n$ is investigated. Then, the contracted W_2-curvature tensor type of $(0,2)$ is called W_2-Ricci tensor and denoted by $\overline{W}_2(X,Y)$ as follow

$$\overline{W}_2(X,Y) = \frac{n}{n-1}[S(X,Y) - \frac{r}{n}g(X,Y)].$$

In this Chapter, $A(PRS)_n$ admitting non-zero W_2-Ricci tensor is studied. A necessary and sufficient condition is found for W_2-curvature tensor to be divergence-free. After that, the conditions for which the W_2-Ricci tensor of type $(0,2)$ is recurent, Codazzi type and covariantly constant are examined. The obtained results are written as theorems. Finally, an example of the existence of these manifolds satisfying special conditions is given.

The last Chapter is concerned with an almost pseudo Ricci symmetric spacetime. Under some conditions, we determine the properties of this spacetime. In the first part of this section, it is considered that our spacetime is a perfect fluid. In the second part, using the results obtained in the first part, we prove that our spacetime reduces to an Einstein, quasi-Einstein or $\eta-$Einstein space with some assumptions. In addition, we show that a dust and a radiation fluid with almost pseudo Ricci symmetric tensor are vacuum.

HEMEN HEMEN PSEUDO RICCI SİMETRİK MANİFOLDLAR HAKKINDA

ÖZET

Bu tez çalışmasında, Riemann manifoldunun genelleştirilmişi olan hemen hemen pseudo Ricci simetrik manifoldu incelenmiştir. Bu tez çalışmasının temel amacı

$$(\nabla_Z S)(X,Y) = [A(Z)+B(Z)]S(X,Y)+A(X)S(Y,Z)+A(Y)S(X,Z)$$

bağıntısını sağlayan $(0,2)$ tipindeki S Ricci tensörünün bazı özel koşullar altında durumunu incelemektir. Burada A ve B, $1-$formlar, ρ and Q, $g(X,\rho) = A(X)$ ve $g(X,Q) = B(X)$ ile tanımlanan A ve B şeklindeki $1-$formlar tarafından üretilen vektör alanları olup, ∇ kovaryant türevi ifade etmektedir.

Bu tez çalışması, 4 ayrı bölümden oluşmaktadır. İlk bölümde, Riemann manifoldunun genel bir tanımı verilmiştir. Bir M bir manifoldu üzerinde

$$<,>: \chi(M)x\chi(M) \longrightarrow C^\infty(M,\Re)$$

dönüşümü $2-$lineer, simetrik ve pozitif tanımlı ise, bu dönüşüme M üzerinde Riemann metriği veya metrik tensörü denir. Üzerinde Riemann metriği tanımlanmış manifolda Riemann manifoldu adı verilir. Bu bölümde ayrıca, Riemann manifoldu üzerinde bazı vektör alanlarına ait temel tanımlar ve kavramlar verilmiştir.

Tez çalışmasının ikinci bölümünde, ilk olarak, simetrik manifoldların diferansiyel geometrideki öneminden bahsedilmiştir ve bu manifoldların tarihsel geçmişi hakkında detaylı bilgi verilmiştir. Düz olmayan, pseudo Ricci simetrik Riemann manifoldunun $(0,2)$ tipindeki S Ricci tensörü

$$(\nabla_Z S)(X,Y) = 2A(Z)S(X,Y)+A(X)S(Y,Z)+A(Y)S(X,Z)$$

ifadesini sağlar ve $(PRS)_n$ ile gösterilir. Bu bağıntı Chaki tarafından bulunmuş ve Chaki ve Kawaguchi tarafından hemen hemen pseudo Ricci simetrik manifoldu olarak genelleştirilerek, $A(PRS)_n$ ile gösterilmiştir. Bu bölümde, söz edilen manifoldlarla ilgili çalışmalar detaylı olarak verilmiştir.

Çalısmanın üçüncü bölümünde ise, ilk olarak, 1970 yılında Pokhariyal and Mishra tarafından (M^n,g), $(n>3)$ manifoldu üzerinde

$$W_2(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Z)S(Y,U) - g(Y,Z)S(X,U)]$$

bağıntısı ile tanımlanan W_2-eğrilik tensörünün genel özellikleri yer almaktadır. Bu bölümde, hemen hemen pseudo Ricci simetrik manifoldu göz önüne alınmıştır. Daha sonra, daraltılmış W_2-tensörü

$$\overline{W}_2(X,Y) = \frac{n}{n-1}[S(X,Y) - \frac{r}{n}g(X,Y)]$$

şeklinde bulunmuş ve $\overline{W}_2$ ifadesiyle gösterilmiştir. Manifoldumuz W_2-düz kabul edildiğinde, Ricci tensörü

$$S(X,Y) = \frac{r}{n} g(X,Y)$$

bağıntısını gerçeklemektedir. Bu ifadeler kullanılarak

$$(n-1)(n+2)rA(Z) = 0$$

denklemi bulunmuştur. Bu takdirde, $r \neq 0$ olduğundan $A(Z) = 0$ olmalıdır. Dolayısıyla, bu manifold, B formu tarafından üretilen rekürant manifolda indirgenmektedir. Bu bölümde ayrıca, hemen hemen pseudo Ricci simetrik manifoldu için

$$(\nabla_Z r) = [A(Z) + B(Z)]r + 2A(QZ)$$

bağıntısı bulunmuştur. Bu bağıntı kullanılarak, $div\overline{W}_2$ ifadesi

$$div\overline{W}_2(Y) = \frac{n}{n-1}[(\frac{n-1}{n})(2A(QY) + rA(Y)) + B(QY) - \frac{r}{n}B(Y)]$$

şeklinde elde edilmiştir. $\overline{W}_2$ tensörünün diverjansının sıfır olduğu durumda, $-\frac{r}{2}$ ve $\frac{r}{2}$ değerleri S Ricci tensörünün ρ ve Q özvektörlerine göre özdeğerleri olduğu ispatlanmıştır. Ayrıca, $\overline{W}_2$ tensörü rekürant ve Codazzi tipinde tensör alanları olarak kabul edilmesi halinde, A ve B formları için sonuçlar bulunmuştur. Eğer $\overline{W}_2$ tensörü genelleştirilmiş rekürant ise, yani

$$(\nabla_Z \overline{W}_2)(X,Y) = \alpha(Z)\overline{W}_2(X,Y) + \gamma(Z)g(X,Y)$$

bağıntısı mevcutsa, $\gamma(Z) = 0$ zorunda olduğu elde edilmiştir. Bu ise, manifoldun $\overline{W}_2$ Ricci tensörünün genelleştirilmiş rekürant olamayacağını göstermiştir. Daha sonra, r skaler eğriliği sabit ve W_2-Ricci tensörünün de kovaryant türevinin sıfır olduğu kabul edilerek teoremler elde edilmiştir. Bu durumda,

$$(\nabla_Z S)(X,Y) = 0$$

ifadesinin sağlandığı gösterilmiştir. Bu çalışmanın devamında ise, A ve B formları arasında

$$A(QZ) = -\frac{r}{2}[A(Z) + B(Z)]$$

bağıntısının mevcut olduğu bulunmuştur. Eğer, göz önüne alınan manifoldun r skaler eğriliği sabit ve $\overline{W}_2$ tensörü Codazzi tipinde ise, A ve B formları tarafından üretilen vekör alanları arasındaki açının

$$\theta = \arccos(-\frac{1}{3|A|})$$

olduğu elde edilmiştir. Bu bölümün sonunda, skaler eğriliğe sabit olan, eğriliğe sahip hemen hemen pseudo Ricci simetrik manifolda örnek verilmiştir.

Çalışmanın son bölümünde, hemen hemen pseudo Ricci simetrik uzay-zaman için özel bir durum incelenmiştir. Bölümün ilk kısmında, uzay-zaman kavramlarının tarih boyunca araştırılmsına duyulan gereksinim hakkında bilgi verilmiştir. Daha sonra, genel relativistik uzay-zamanı, $(-,+,+,+)$ işaretli g metrik tensörlü (M,g) Lorentz manifoldu olarak alınmıstır. Birinci bölümde, ilk olarak mükemmel akışkan ele alınmıştır. Kozmolojik sabit içermeyen Einstein alan denkleminin tanımı ve mükemmel akışkan için enerji momentum tensörü T' nin tanımı verilmiştir. Bu durumda, κ yerçekimi sabiti, σ enerji yoğunluğu, p izotropik basınç olmak üzere, Ricci tensörü

$$S(X,Y) = \kappa(\sigma + p)A(X)A(Y) + (\kappa p + \frac{r}{2})g(X,Y)$$

ve skaler eğrilik

$$r = \kappa(\sigma - 3p)$$

olarak bulunur. Bu ifadeler kullanılarak, $4-$ boyutlu, hemen hemen pseudo Ricci simetrik uzay-zamanımızda kozmolojik sabit içermeyen mükemmel akışkana ait Einstein alan denklemini sağlaması için gerek ve yeter koşul olarak, enerji yoğunluğunun türevinin, $1-$form A tarafından üretilen vektör alanına dik olması, bunun sonucu olarak, $\sigma + \rho = 0$ olması gerektiği gösterilmiştir. Daha sonra, $4-$boyutlu hemen hemen pseudo Ricci simetrik uzay-zamanımızda mükemmel akıskan için, eğer enerji yoğunluğunun türevi $1-$form A tarafından üretilen vektör alanına dik ise, uzay-zamanımızın Einstein uzayına indirgendiği bulunmuştur. Ayrıca, kozmolojik sabit içermeyen Einstein alan denkleminin izotropik basınç ve A, $1-$formu tarafından üretilen vektör alanı arasındaki ilişki incelenmiştir. Daha sonra, A, $1-$formu tarafından üretilen vektör alanının diverjansı

$$divA = \frac{3(\sigma - p)}{2(\sigma + p)}$$

şeklinde bulunmuştur. Basınç içermeyen, uzay-zaman modeli göz önüne alınırsa, enerji momentum tensörü

$$T(X,Y) = \sigma A(X)A(Y)$$

formundadır. Basınç içermeyen, hemen hemen pseudo Ricci symmetric uzay-zamanda $\sigma = 0$ olmak zorunda olduğu ispatlanmıştır. Bu sonuç, $T(X,Y) = 0$ olmak zorunda olduğunu da göstermiştir. Böylece, basınç içermeyen hemen hemen pseudo Ricci simetrik uzay-zamanın boş olduğunu göstermiştir. Bu bölümün sonunda, uzay-zamanımız radyoaktif akışkan olarak seçilmiş ve enerji momentum tensörü

$$T(X,Y) = p[4A(X)A(Y) + g(X,Y)]$$

ifadesi olarak göz önüne alınmıştır. Böylece,

$$-6\kappa pA(Z) = 0$$

olduğu elde edilmiştir. Bu sonuç, $p = 0$ olması gerektiğini göstermiştir. Yani, $T(X,Y) = 0$ olmak zorundadır. Bu durumda, hemen hemen pseudo Ricci simetrik ve kozmolojik sabit içermeyen Einstein alan denklemine sahip bir uzay-zamanda eğer radyoaktif akışkan göz önüne alanırsa, uzayın boş uzaydan ibaret olacağı ispatlanmıstır.

1. INTRODUCTION

1.1 Basic Definitions

1.1.1 Riemannian manifold

Let M be a manifold if at each point $p \in M$, there is a bilinear, symmetric and positive definite tensor, $g = <, >$ defined on T_pM such that

$$g : T_pMxT_pM \to \Re$$

then g is called the Riemannian metric or Riemannian tensor. Thus, M is called a Riemannian manifold [1].

1.1.2 Connections on manifolds

Let M be a smooth manifold. A covariant differentiation or a connection on M is an operator ∇ that assigns to each pair of C^∞ vector fields X and Y and a scalar function f on domain U that satisfies the following properties for $X, Y, Z \in \chi(U)$, [2]

- $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$
- $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$
- $\nabla_{fX} Z = f \nabla_X Z$
- $\nabla_X(fZ) = (Xf)ZY + f \nabla_X Z.$

1.1.3 Torsion tensor

Let X, Y be vector fields on a Riemannian manifold M. Then the vector field

$$\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

defines a tensor field T of type $(0,2)$ [3]. Thus, this tensor is called torsion tensor and for any $1-$form ϕ, we have

$$T(\phi, X, Y) = \phi(\tau(X,Y))$$

.

1.1.4 Affine connection

An affine connection on ∇ on a smooth manifold M is a mapping

$$\nabla : \chi(M) x \chi(M) \to \chi(M)$$

$$(X,Y) \mapsto \nabla_X Y$$

subject to the properties :

- $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$
- $\nabla_X(\alpha Y + \beta Z) = \alpha\nabla_X Y + \beta\nabla_X Z$
- $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$ (Leibniz Rule)

where $X,Y,Z \in \chi(M)$ and f,g be linear functions, [4].

If the torsion vanishes, then the affine connection is called torsion-free, in a word, for all X,Y

$$\nabla_X Y - \nabla_Y X = [X,Y]$$

holds.

1.1.5 Levi-Civita connection

Let M be a Riemannian manifold. There exists an affine connection ∇ on M that is compatible with g and symmetric. Then, it satisfies the condition

$$Z(g(X,Y)) = g(\nabla_Z X,Y) + g(X,\nabla_Z Y)$$

for vector fields X,Y,Z. An affine connection which is torsion-free and compatible with the metric g is called the $Levi-Civita$ connection or Riemannian connection, [2].

1.1.6 Riemannian curvature tensor

Let M be a Riemannian manifold and $\chi(M)$ denote the space of C^∞ vector fields on M. The Riemannian curvature tensor is the map

$$R : \chi(M) X \chi(M) X \chi(M) \to \chi(M)$$

defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for all $X,Y,Z \in \chi(M)$ [5]. Clearly, we have the antisymmetry in X and Y :

$$R(Y,X)Z = -R(X,Y)Z.$$

Using metric g, we can change a $(0,3)-$tensor to a $(0,4)-$tensor such that

$$R(X,Y,Z,W) = g(R(X,Y,Z),W).$$

Also, the Riemannian curvature tensor satisfies the following rules:

- R(X,Y,Z,W)=-R(Y,X,Z,W)=R(Y,X,W,Z)
- R(X,Y,Z,W)=R(Z,W,X,Y)
- R(X,Y)Z+R(Z,X)Y+R(Y,Z)X=0 (I. Bianchi Identity).

1.1.7 Covariant derivative of tensor fields

Let $\varphi \in T^r(M)$. The covariant derivative of φ is a tensor of order $(r+1)$ given by

$$\begin{aligned}(\nabla_Y \varphi)(X_1,X_2,...,X_r) =& Y(\varphi(X_1,X_2,...,X_r)) - \varphi(\nabla_Y X_1,X_2,...,X_r) - ...\\ & \varphi(X_1,X_2,...,\nabla_Y X_{r-1},X_r) - \varphi(X_1,X_2,...,X_{r-1},\nabla_Y X_r)\end{aligned}$$

where X,Y be vector fields on M [6].

1.2 Some Special Tensor Fields and Vector Fields

1.2.1 Ricci tensor

Let M be a manifold with an affine connection ∇ and R be the curvature tensor of ∇. The Ricci tensor Ric of the connection is a tensor field of type $(0,2)$ assigning to the vector fields X and Y. Then the function $Ric(X,Y)$ the value of which at $p \in M$ is the trace of the linear mapping

$$T_pM \to T_pM$$

$$Z_p \to R(Z_p,X(p);Y(p))$$

where $Z_p \in T_pM$ [7].

1.2.2 Torse-forming vector field

A vector field ψ in a Riemannian manifold M is called torse-forming if it satisfies

$$\nabla_X \psi = \rho X + \alpha(X)\psi$$

where $X \in TM$, α is a linear form and ρ is a scalar function [8]. In local transcription, this reads

$$\psi^h{}_{,i} = \rho \delta^h{}_i + \psi^h \alpha_i \qquad \textbf{(1.1)}$$

where ψ^h and α_i are the components of the vector fields generated by ψ and α and $\delta^h{}_i$ is Kronecker symbol.

Torse-forming vector field ψ is called

- recurrent, if $\rho = 0$
- concircular, if α_i is a gradient covector $(i.e., \alpha_i = \alpha_{,i})$
- convergent, if it is concircular, and $\rho = const.exp(\alpha)$

1.2.3 Codazzi type tensor field

Let$M$ be a smooth Riemannian manifold. A tensor field T of type $(0,2)$ is called Codazzi type if it satisfies Codazzi equation

$$(\nabla_X T)(Y,Z) = (\nabla_Y T)(X,Z) \qquad \textbf{(1.2)}$$

for every vector fields X,Y,Z, [9]

1.3 Some Special Riemannian Manifolds

1.3.1 Recurrent manifold

A non-flat Riemannian manifold M is called recurrent manifold if its curvature tensor satisfies the condition

$$(\nabla_W R)(X,Y,Z,U) = \alpha(W)R(X,Y,Z,U) \qquad \textbf{(1.3)}$$

for a non-zero $1-$form α, [10].

1.3.2 Generalized recurrent manifold

A non-flat, $n-$dimensional Riemannian manifold M, $(n > 2)$ is called a generalized recurrent manifold if its curvature tensor R of type $(0,4)$ satisfies the condition

$$(\nabla_X R)(Y,Z,W,U) = \alpha(X)R(Y,Z,W,U) + \beta(X)G(Y,Z,W,U) \quad \textbf{(1.4)}$$

where $G(Y,Z,W,U) = g(Y,U)g(Z,W) - g(Y,W)g(Z,U)$ and α,β are non-zero $1-$forms, [11].

1.3.3 Ricci recurrent manifold

A non-flat Riemannian manifold M is called a Ricci-recurrent manifold if its Ricci tensor S satisfies the condition

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) \quad \textbf{(1.5)}$$

where ∇ is the Levi-Civita connection of the Riemannian metric g and A is a $1-$form on M, [12].

2. RIEMANNIAN MANIFOLD WITH W_2-CURVATURE TENSOR

2.1 W_2-Curvature Tensor

In 1970, Pokhariyal and Mishra [13] introduced a new tensor, called W_2 , in a Riemannian manifold and studied their properties. According to them [13], a W_2-curvature tensor on a manifold (M,g), $(n>3)$ is defined by

$$W_2(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Z)S(Y,U) - g(Y,Z)S(X,U)]. \qquad \textbf{(2.1)}$$

After that, W_2-curvature tensor on some special manifolds has been examined by many authors such as Taleshian and Hosseinzadeh [14], Özen Zengin [15], Hui [16], Mallick and De [17], etc.

2.2 W_2-flat Riemannian Manifold

In this section, we denote the contracted W_2-curvature tensor which is type of $(0,2)$ as $\overline{W}_2$ and call it W_2-Ricci tensor.

Now, contracting (2.1) over X and U, we obtain the contracted W_2 tensor, i.e., W_2-Ricci tensor

$$\overline{W}_2(X,Y) = \frac{n}{n-1}[S(X,Y) - \frac{r}{n}g(X,Y)]. \qquad \textbf{(2.2)}$$

If we assume that our manifold is W_2-flat, then from (2.2),

$$S(X,Y) = \frac{r}{n}g(X,Y). \qquad \textbf{(2.3)}$$

Thus, this manifold reduces Einstein manifold.

3. ALMOST PSEUDO RICCI SYMMETRIC MANIFOLDS

3.1 The Importance of Symmetric Manifolds

In the late twenties, because of the important role of symmetric spaces in differential geometry, Cartan [18], obtaining a classification of those spaces, established Riemannian symmetric spaces.

Let (M,g) be an n-dimensional Riemannian manifold with the metric g and let ∇ be the Levi-Civita connection of (M,g). A Riemanian manifold is called locally symmetric [18] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M,g). This condition of locally symmetric is equivalent to the fact that every point $p \in M$, the locally geodesic symmetry $F(p)$ is an isometry [19]. The class of Riemannian symmetric manifolds is very natural generalization of the class of manifolds of constant curvature.

In the last five decades, the notion of locally symmetric manifolds have been weakended by many authors in several ways by extending to special manifolds such as conformally symmetric manifolds by Chaki and Gupta [20], recurrent manifolds introduced by Walker [21], conformally recurrent manifolds by Adati and Miyazawa [22], conformally symmetric Ricci-recurrent spaces by Roter [23], pseudo-Riemannian manifolds with recurrent concircular curvature tensor by Olszak and Olszak [24], semi-symmetric manifolds by Szabo [25] ,pesudo symmetric manifolds by Chaki [26], weakly symmetric manifolds by Tamassy and Binh [27], projective symmetric manifolds by Soos [28], etc.

The Einstein equations [19], imply that the energy-momentum tensor is of vanishing divergence. This requirement is satisfied [28] if the energy- momentum tensor is covariant-constant. In the paper [28], Chaki and Ray had shown that a general relativistic spacetime with covariant-constant energy-momentum tensor is Ricci symmetric, that is, $\nabla S = 0$ where S is the Ricci tensor of the spacetime. If however $\nabla S \neq 0$, then such a spacetime may be called pseudo Ricci symmetric. It can be said that the Ricci symmetric condition is only a special case of the pseudo Ricci symmetric condition. It is, therefore, meaningful to study the properties of pseudo Ricci symmetric spacetimes in general relativity.

3.2 Almost Pseudo Ricci Symmetric Manifolds

Let Q be the symmetric endomorphism corresponding to the Ricci tensor as indicated below

$$S(X,Y) = g(QX,Y)$$

for all vector fields X and Y.

A non-flat Riemannian manifold is called pseudo Ricci symmetric and denoted by $(PRS)_n$ if the Ricci tensor S of type $(0,2)$ of the manifold is non-zero and satisfies the condition, [29]

$$(\nabla_Z S)(X,Y) = 2A(Z)S(X,Y) + A(X)S(Y,Z) + A(Y)S(X,Z) \tag{3.1}$$

where ∇ denotes the Levi-Civita connection and A is a non-zero $1-$form such that

$$g(X,\rho) = A(X) \tag{3.2}$$

for all vector fields X, ρ being the vector field corresponding to the associated $1-$form A. If in (3.1), the $1-$form $A = 0$, then the manifold reduces to Ricci symmetric manifold or covariantly constant

$$(\nabla_Z S)(X,Y) = 0. \tag{3.3}$$

The notion of pseudo Ricci symmetry is different from that of R. Deszcz [30].

So, pseudo Ricci symmetric manifolds have some importance in general theory of relativity. By this motivation, Chaki and Kawaguchi [31] generalized pseudo Ricci symmetric manifold and introduced the notion of almost pseudo Ricci symmetric manifold as

$$(\nabla_Z S)(X,Y) = [A(Z) + B(Z)]S(X,Y) + A(X)S(Y,Z) + A(Y)S(X,Z) \tag{3.4}$$

where A and B are two non-zero $1-$forms and ∇ denotes the operator of the covariant differentiation with respect to the metric g. In such a case, A and B are called the associated $1-$forms and an $n-$dimensional manifold of this kind is denoted by $A(PRS)_n$.

If $B = A$, then the equation (3.3) reduces to (3.1), that is, $A(PRS)_n$ reduces to a pseudo Ricci symmetric manifold [29]. Thus, pseudo Ricci symmetric manifold is a particular case of $A(PRS)_n$. In 1993, Tamassy and Binh [32] introduced the notion of weakly Ricci symmetric manifold which is the generalization of pseudo Ricci symmetric manifold in the sense of Chaki. It may be mentioned that an $A(PRS)_n$ is not a particular case of a weakly Ricci symmetric manifold introduced by Tamassy and Binh [32].

Let $g(X,\rho) = A(X)$ and $g(X,Q) = B(X)$ for all X. Then ρ, Q are called basic vector fields of the manifold corresponding to the associated $1-$forms A and B, respectively.

Almost pseudo Ricci symmetric manifolds on some structures have been studied by many authors such as De and Gazi [33], Shaikh, Hui and Bagewadi [34], De, Özgür and De [35], Hui and Özen Zengin [36], De and Mallick [37], De and Pal [38], Kırık and Özen Zengin [39], etc.

3.3 W_2-flat $A(PRS)_n$

Assuming that our manifold (M,g) is $A(PRS)_n$ admitting W_2 curvature tensor, from (2.3), we get

$$(\nabla_Z S)(X,Y) = \frac{1}{n}(\nabla_Z r)g(X,Y). \tag{3.5}$$

Putting (3.4) in (3.5), we obtain

$$(\nabla_Z r)g(X,Y) = r[(A(Z)+B(Z))g(X,Y)+A(X)g(Y,Z)+A(Y)g(X,Z)]. \tag{3.6}$$

Contracting (3.6) over X and Y, then we have

$$n(\nabla_Z r) = r[(n+2)A(Z)+nB(Z)]. \tag{3.7}$$

Again, contracting (3.6) over X and Z, then we have

$$(\nabla_Z r) = r[(n+2)A(Z)+B(Z)]. \tag{3.8}$$

Comparing (3.7) and (3.8), we get

$$r[(n+2)A(Z)+nB(Z)] = nr[(n+2)A(Z)+B(Z)].$$

Thus, we have

$$(n+2)(n-1)rA(Z) = 0.$$

Since $r \neq 0$, we have from the above equation it must be $A(Z) = 0$. Thus, we have the following theorem:

Theorem 3.3.1 *W_2-flat $A(PRS)_n$ reduces to a Ricci recurrent manifold with the recurrence vector field generated by the $1-$form B.*

3.4 A(PRS)$_n$ Admitting Non-zero W_2-Ricci Tensor

Now, we assume that our manifold $A(PRS)_n$ is of non-zero W_2-curvature tensor. By taking the covariant derivative of (2.2), we get

$$(\nabla_Z \overline{W}_2)(X,Y) = \frac{n}{n-1}[(\nabla_Z S)(X,Y) - \frac{1}{n}(\nabla_Z r)g(X,Y)]. \qquad \textbf{(3.9)}$$

If we contract (3.4) over X and Y, then we obtain

$$(\nabla_Z r) = [A(Z) + B(Z)]r + 2A(QZ). \qquad \textbf{(3.10)}$$

By putting (3.4) and (3.10) in (3.9), we find

$$(\nabla_Z \overline{W}_2)(X,Y) = \frac{n}{n-1}\{[A(Z)+B(Z)]S(X,Y) + A(X)S(Y,Z) + A(Y)S(X,Z)$$
$$- \frac{r}{n}[A(Z)+B(Z)]g(X,Y) - \frac{2}{n}A(QZ)g(X,Y)\}. \qquad \textbf{(3.11)}$$

Now, contracting (3.11) over X and Z,

$$div\overline{W}_2(Y) = \frac{n}{n-1}[(\frac{n-1}{n})(2A(QY) + rA(Y)) + B(QY) - \frac{r}{n}B(Y)]. \qquad \textbf{(3.12)}$$

By considering the tensor $\overline{W}_2$ as divergence free, from (3.12), it can be obtained that

$$\frac{n}{n-1}[(\frac{n-1}{n})(2A(QY) + rA(Y)) + B(QY) - \frac{r}{n}B(Y)] = 0. \qquad \textbf{(3.13)}$$

If $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector Q where $g(X,Q) = B(X)$, then $-\frac{r}{2}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ where $g(X,\rho) = A(X)$. Conversely, if the equation (3.13) holds then from (3.12), $\overline{W}_2$ must be divergence-free.

Thus, we have the following theorem:

Theorem 3.4.1 *For an $A(PRS)_n$, a necessary and sufficient condition the contracted W_2 curvature tensor $\overline{W}_2$ be divergence free is that $-\frac{r}{2}$ and $\frac{r}{n}$ be eigenvalues of the Ricci tensor S corresponding to the eigenvectors ρ and Q where $g(X,\rho) = A(X)$ and $g(X,Q) = B(X)$, respectively.*

Let $\overline{W}_2$ be recurrent, i.e., from (1.3)

$$(\nabla_Z \overline{W}_2)(X,Y) = \alpha(Z)\overline{W}_2(X,Y) \quad \textbf{(3.14)}$$

where α is a $1-$form.

Using (2.2) and (3.11) in (3.14), it can be found that

$$\begin{aligned}\alpha(Z)[S(X,Y) - \frac{r}{n}g(X,Y)] &= [A(Z)+B(Z)]S(X,Y) + A(X)S(Y,Z) + A(Y)S(X,Z)\\ &\quad - \frac{r}{n}[A(Z)+B(Z)]g(X,Y) - \frac{2}{n}A(QZ)g(X,Y) \quad \textbf{(3.15)}\end{aligned}$$

If we contract (3.15) over X and Z, then we get

$$\alpha(QZ) - \frac{r}{n}\alpha(Z) = \frac{(n-1)}{n}[2A(QZ) + rA(Z)] + B(QZ) - \frac{r}{n}B(Z). \quad \textbf{(3.16)}$$

This leads to the following theorem.

Theorem 3.4.2 *Let us assume that $A(PRS)_n$ be of recurrent W_2-Ricci tensor with the recurrence vector field generated by the $1-$ form α. If $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvectors both Q and μ where $g(X,Q) = B(X)$, $g(X,\mu) = \alpha(X)$ then $-\frac{r}{2}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ where $g(X,\rho) = A(X)$.*

If we take $\alpha(Z) = A(Z)$ in (3.16), we find

$$(\frac{2-n}{n})A(QZ) - A(Z)r = B(QZ) - \frac{r}{n}B(Z).$$

If $\frac{r}{n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector Q then $\frac{nr}{2-n}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ. Thus we have the following theorem:

Theorem 3.4.3 *In an $A(PRS)_n$, let us consider that W_2-Ricci tensor is recurrent with the recurrence vector field generated by the $1-$form A. A necessary and sufficient condition $\frac{r}{n}$ be an eigenvalue of the Ricci tensor S corresponding to the eigenvector Q where $g(X,Q) = B(X)$ is that $\frac{nr}{2-n}$ $(n > 2)$ be an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ where $g(X,\rho) = A(X)$.*

Now, by taking $\alpha(Z) = B(Z)$ in (3.16) then we obtain

$$A(QZ) = -\frac{r}{2}A(Z). \tag{3.17}$$

If we differentiate the equation (3.17), then we can find

$$(\nabla_X A)(QZ) = -\frac{1}{2}[(\nabla_X r)A(Z) + r(\nabla_X A)(Z)]. \tag{3.18}$$

By using (3.4), (3.10) and (3.17) in (3.18), we get

$$|A|A(Z)r = 0.$$

Since $r \neq 0$ then from the above equation $A(Z)$ must be 0. Thus, we have the following theorem:

Theorem 3.4.4 *An $A(PRS)_n$ admits recurrent W_2-Ricci tensor which is recurrent with the recurrence vector field generated by the $1-$ form B does not exist.*

Let us assume that $\overline{W}_2$ is generalized recurrent. Thus, from (1.4)

$$(\nabla_Z \overline{W}_2)(X,Y) = \alpha(Z)\overline{W}_2(X,Y) + \gamma(Z)g(X,Y).$$

Using (2.2) and (3.11) in the above equation, we have

$$\begin{aligned} \frac{n}{n-1}[S(X,Y) - \frac{r}{n}g(X,Y)] + \gamma(Z)g(X,Y) = \frac{n}{n-1}[&(A(Z)+B(Z))S(X,Y) \\ &+ A(X)S(Y,Z) + A(Y)S(X,Z) \\ &- \frac{r}{n}(A(Z)+B(Z))g(X,Y) \\ &- \frac{2}{n}A(QY)]. \end{aligned} \tag{3.19}$$

Contracting (3.19) over X and Y,

$$n\gamma(Z) = 0.$$

Hence, $\gamma(Z) = 0$. Thus, we have the following theorem :

Theorem 3.4.5 *An $A(PRS)_n$ admits generalized W_2-Ricci tensor does not exist.*

If r is constant, then (3.9) reduces to

$$(\nabla_Z \overline{W}_2)(X,Y) = \frac{n}{n-1}(\nabla_Z S)(X,Y). \quad \textbf{(3.20)}$$

Using (3.4) and (3.20), we get

$$(\nabla_Z \overline{W}_2)(X,Y) = \frac{n}{n-1}[(A(Z)+B(Z))S(X,Y)+A(X)S(Y,Z)+A(Z)S(X,Y)]. \quad \textbf{(3.21)}$$

If $\overline{W}_2$ is Codazzi type, from (1.2) and (3.21), it is easy to see that

$$B(Z)S(X,Y) - B(Y)S(X,Z) = 0.$$

Contracting the above equation over X and Y , we find

$$B(Z)r = B(QZ). \quad \textbf{(3.22)}$$

In this case, we have the following theorem :

Theorem 3.4.6 *In an $A(PRS)_n$ admitting constant scalar curvature, if W_2-Ricci tensor is Codazzi type then r is an eigenvalue of the Ricci tensor S corresponding to the eigenvector Q where $g(X,Q) = B(X)$.*

Now, if we assume that r is constant and $\overline{W}_2$ is covariantly constant. Then from (2.2)

$$(\nabla_Z S)(X,Y) = 0. \quad \textbf{(3.23)}$$

Using (3.23) in (3.4), we have

$$(\nabla_Z S)(X,Y) = [A(Z)+B(Z)]S(X,Y)+A(X)S(Y,Z)+A(Y)S(X,Z).$$

Contracting the above equation over X and Y, we get

$$[A(Z)+B(Z)]r + 2A(QZ) = 0. \quad \textbf{(3.24)}$$

Finally, we obtain from (3.24)

$$A(QZ) = -\frac{r}{2}[A(Z)+B(Z)]. \quad \textbf{(3.25)}$$

Hence, we have the following theorem:

Theorem 3.4.7 *If an $A(PRS)_n$ admitting constant scalar curvature and covariantly constant W_2-Ricci tensor then the Ricci tensor of this manifold is covariantly constant and the $1-$forms A and B are related by*

$$A(QZ) = -\frac{r}{2}[A(Z)+B(Z)].$$

Let us assume that r is constant and $\overline{W}_2$ is Codazzi type. If we take the covariant derivative of (3.22), then we get

$$(\nabla_X B)(QZ) = r(\nabla_X B)(Z). \tag{3.26}$$

By using (3.4) and contracting over X and Z in (3.26), we find

$$r|B|(3 < A,B > +|B|) = 0 \tag{3.27}$$

where $<,>$ is the inner product. Since $r \neq 0$ and $|B| \neq 0$ in (3.27), we obtain

$$< A,B >= -\frac{|B|}{3}. \tag{3.28}$$

We know that $< A,B >= |A||B|cos\theta$, from (3.28), we get

$$|A|cos\theta = -\frac{1}{3} \qquad where \qquad \frac{\pi}{2} < \theta \leq \pi.$$

Therefore, we can state the following theorem:

Theorem 3.4.8 *If an $A(PRS)_n$ with the constant scalar curvature tensor is of Codazzi type W_2-Ricci tensor then the angle between the vector fields generated by the 1-forms A and B is*

$$\theta = \arccos(-\frac{1}{3|A|}).$$

Thus, $\theta \in (\frac{\pi}{2}, \pi]$ where $|A|$ is the length of the vector field generated by the $1-$form A.

3.5 An Example For The Existence $A(PRS)_n$ Admitting W$_2-$ Curvature Tensor

In this section, we want to construct an example of an four-dimensional almost pseudo Ricci symmetric manifold with constant scalar curvature and W_2-Ricci tensor.

In local coordinates, let us consider a Riemannian metric g on $\Re^4$ with coordinates (x^1, x^2, x^3, x^4) by

$$ds^2 = g_{ij}dx^i dx^j = e^{x^1}(dx^1)^2 + e^{x^2}(dx^2)^2 + (dx^3)^2 + (sinx^3)^2(dx^4)^2. \tag{3.29}$$

Then the only non vanishing components of the Christoffel symbols and curvature tensors are, respectively,

$$\Gamma^1_{11} = \Gamma^2_{22} = \frac{1}{2}, \quad \Gamma^3_{44} = -sinx^3cosx^3, \quad \Gamma^4_{43} = cotx^3, \tag{3.30}$$

and

$$R^3_{443} = -(sinx^3)^2, \quad R^3_{443} = -1 \tag{3.31}$$

and the components obtained by the symmetry properties. The non-vanishing components of the Ricci tensors are

$$S_{33} = -1, \quad S_{44} = -(sinx^3)^2. \tag{3.32}$$

It can be shown that the scalar curvature r of $(\Re^4, g)$ is -2. By using (2.2), (3.29) and (3.32), we get the only non-vanishing components of $\overline{W}_2$ are

$$\overline{W}_{11} = \frac{2}{3}e^{x^1}, \quad \overline{W}_{22} = \frac{2}{3}e^{x^2}, \quad \overline{W}_{33} = -\frac{2}{3}, \quad \overline{W}_{44} = -\frac{2}{3}(sinx^3)^2. \tag{3.33}$$

By taking the covariant derivatives of each of $\overline{W}_{ij}$ in (3.33), we find that $\overline{W}_{ij,k} = 0$ for all i, j, k. This shows that $\overline{W}_{ij}$ are covariantly constant.

In this case, by taking the covariant derivatives of S_{33} and S_{44} and by using (3.32), we obtain that $S_{ij,k} = 0$ for all i, j, k.

Let us choose the $1-$forms A and B as

$$A = e^{x^1+x^2} + e^{x^3+x^4} \tag{3.34}$$

and

$$B = -e^{x^1+x^2} - 2e^{x^3+x^4}. \tag{3.35}$$

Now, by taking the derivatives of (3.34) and (3.35), we get

$$A_i = \begin{cases} e^{x^1+x^2}, & i = 1,2 \\ e^{x^3+x^4}, & i = 3,4 \end{cases} \tag{3.36}$$

and

$$B_i = \begin{cases} -e^{x^1+x^2}, & i = 1,2 \\ -2e^{x^3+x^4}, & i = 3,4\ . \end{cases} \tag{3.37}$$

With the help of (3.29), (3.32), (3.36) and (3.37), the equations

$$\begin{aligned} A^1 R_{11} &= -\frac{R}{2}(A_1 + B_1) \\ A^2 R_{22} &= -\frac{R}{2}(A_2 + B_2) \\ A^3 R_{33} &= -\frac{R}{2}(A_3 + B_3) \\ A^4 R_{44} &= -\frac{R}{2}(A_4 + B_4) \end{aligned}$$

are satisfied.

From these results, it is clear that $(\Re^4, g)$ given by (3.29) is an $A(PRS)_4$ satisfying Theorem 3.4.7. Thus, we can state the following theorem :

Theorem 3.5.1 *Let us consider a Riemannian metric g on* $\Re^4$ *given by*

$$ds^2 = g_{ij} dx^i dx^j = e^{x^1}(dx^1)^2 + e^{x^2}(dx^2)^2 + (dx^3)^2 + (sinx^3)^2 (dx^4)^2.$$

Assume that this manifold is an $A(PRS)_4$ *with the constant scalar curvature and covariantly constant* $\overline{W}_{ij}$ *tensor. If we choose the* $1-$ *forms A and B related to this manifold as*

$$A = e^{x^1+x^2} + e^{x^3+x^4} \quad and \quad B = -e^{x^1+x^2} + 2e^{x^3+x^4},$$

then Theorem 3.4.7 holds.

4. ON ALMOST PSEUDO RICCI SYMMETRIC SPACETIMES

Astronomers have struggled with basic questions about the size and age of the universe for thousand of years. At the beginning of the twentieth century, the astronomer Edwin Hubble made a critical discovery that soon led to reasonable answers to these equations. Those measurements marked the first evidence that our universe is expanding. This discovery caused a profound revolution in our in view of the universe, and understanding the source of its expansion is arguably the most dominant question in cosmology today.

In recent years, the standard cosmological model has dramatically improved our understanding of the universe by emerging as a particular solution of the Einstein field equations. It is built on several fundamental assumptions and principles; such as cosmological principle is particularly important. It states that our current universe is homogeneous and isotropic, that is, there is neither a preffered place nor direction, at least approximately on large scales. Although this mode has been successful in explaining the majority of the current observations, such as the current expansion of the universe and the spectrum of the cosmic microwave background [40] it lack of the fundamental justification. In fact, the theory of quantum mechanics [41] states that the early universe (just after the universe was born in the big bang) should have been extremely inhomogeneous and anisotropic due to quantum fluctuations that yield the creation of the fundamental particles and eventually to the formation of different kinds of matter distributions such as galaxies, stars and planets. Hence, fundamental questions a rise about how much primordial homogeneities evolved and why they are essential absent in the present universe on large scales that any consistent cosmological model must be able to answer.

This can be understood by the assumption of an extremely short, but particularly violent, phase of expansion just after the big bang, called inflation. This basic idea is that during this phase initial inhomogeneities are effectively smoothed out, and from the point of view of any local observer, the universe rapidly becomes essentially homogeneous and isotropic. However, there is theoretical evidence, [42], [43] that some assumptions of this model lead to significant drawbacks as important phenomena are ignored. In fact, it is conceivable that some inhomogeneities caused by quantum fluctuations would trigger the formation of primordial black holes [44] during the evolution of the universe. Therefore, if such phenomenon was better understood and taken into account, then it could be obtained more general and realistic cosmological models.

The gravitational evolution of celestial bodies may be modeled by the Einstein field equations. There is a system of ten highly coupled partial differential equations expressing an equivalence between matter and geometry. These equations are extremely difficult to solve in general and so simpler cases have to be treated to gain an understanding into how certain types of matter behave under the influence of the gravitational field. For example, the most studied configuration of a matter distribution is that of a static spherically symmetric perfect fluid.

In this section, we concern is concerned with certain investigations in general relativity by the coordinate free method of differential geometry. In this method of study, spacetime of general relativity is regarded as a connected four dimensional semi-Riemannian manifold (M,g) with Lorentzian metric g with signature $(-,+,+,+)$. The geometry of the Lorentzian manifold begins with the study of the casual character of vectors of the manifold. It is due to this causality that the Lorentzian manifold becomes a convenient choice for the study of general relativity. By these equations, it is implied that the energy-momentum tensor is of vanishing divergence [19]. This requirement is satisfied if the energy-momentum tensor is covariant-constant [28]. In Ref [28], M.C. Chaki and S. Roy showed that a general relativistic spacetime with covariant-constant energy-momentum tensor is Ricci symmetric, that is, $\nabla S = 0$ where S is the Ricci tensor of the spacetime. Many authors have been studied spacetimes with special properties such as spacetimes with semisymmetric energy momentum tensor by De and Velimirovic [45], M-projectively flat spacetime by Zengin [46], pseudo Z symmetric spacetime by Mantica and Suh [47], [48], generalized quasi-Einstein spacetimes by Güler and Demirbag [49], a spacetime with pseudo-projective curvature tensor by Mallick, Suh and De, [50], on generalized

Ricci recurrent manifolds with applications to relativity by Mallick, De and De [51], generalized Robertson-Walker spacetimes by Arslan et al, [52] and many others.

4.1 Perfect Fluid $A(PRS)_4$ Spacetimes

It is known that Einstein field equations without cosmological constant can be written as

$$S(X,Y) - \frac{1}{2} r g(X,Y) = \kappa T(X,Y), \tag{4.1}$$

where κ is the gravitational constant, T is the energy momentum tensor for a perfect fluid given by [53]

$$T(X,Y) = (\sigma + p)A(X)A(Y) + pg(X,Y) \tag{4.2}$$

where σ is the energy density and p is the isotropic pressure of the fluid, respectively and $g(X,\rho)$=$A(X)$ for all X, ρ is the flow vector field of the fluid.

Using (4.2), we can express (4.1) by

$$S(X,Y) = \kappa(\sigma + p)A(X)A(Y) + (\kappa p + \frac{r}{2})g(X,Y). \tag{4.3}$$

Contracting (4.3) over X and Y, then we have

$$r = \kappa(\sigma - 3p). \tag{4.4}$$

From (4.3) and (4.4), we obtain

$$S(X,Y) = \kappa(\sigma + p)A(X)A(Y) + \frac{\kappa}{2}(\sigma - p)g(X,Y). \tag{4.5}$$

If we differentiate the equation (4.5), we can find

$$\begin{aligned}(\nabla_Z S)(X,Y) =& \kappa(d\sigma + dp)A(X)A(Y) + \kappa(\sigma + p)[(\nabla_Z A)(X)A(Y) \\ &+ A(X)(\nabla_Z A)(Y)] + \frac{\kappa}{2}(d\sigma - dp)g(X,Y).\end{aligned} \tag{4.6}$$

Comparing (3.4) with (4.6) and using (4.5), then since $\kappa \neq 0$, we get

$$\begin{aligned}
&[A(Z)+B(Z)][(\sigma+p)A(X)A(Y)+\frac{1}{2}(\sigma-p)g(X,Y)] \\
&\quad +2(\sigma+p)A(X)A(Y)A(Z)+\frac{1}{2}(\sigma-p)[A(X)g(Y,Z) \\
&\quad +A(Y)g(X,Z)]=(d\sigma+dp)A(X)A(Y) \\
&\quad +(\sigma+p)[(\nabla_Z A)(X)A(Y)+A(X)(\nabla_Z A)(Y)] \\
&\quad +\frac{1}{2}(d\sigma-dp)g(X,Y). \qquad \textbf{(4.7)}
\end{aligned}$$

Contracting (4.7) over X and Y, we find

$$-6pA(Z)+(\sigma-3p)B(Z)=d\sigma-3dp. \qquad \textbf{(4.8)}$$

If we contract (4.8) over Z, we obtain

$$6p=d\sigma(\rho)-3dp(\rho). \qquad \textbf{(4.9)}$$

On the other hand, contracting (4.7) over X and Y,

$$\frac{3}{2}A(Z)(\sigma+3p)+\frac{1}{2}B(Z)(\sigma+3p)=\frac{1}{2}(d\sigma+3dp). \qquad \textbf{(4.10)}$$

And, contracting (4.10) over Z,

$$-3(\sigma+3p)=d\sigma(\rho)+3dp(\rho) \qquad \textbf{(4.11)}$$

Hence, from (4.8) and (4.11), it can be seen that

$$d\sigma(\rho)=-\frac{3}{2}(\sigma+p). \qquad \textbf{(4.12)}$$

If the derivative of σ is orthogonal to the vector field generated by the $1-$form A then we have the following theorem:

Theorem 4.1.1 *In a perfect fluid $A(PRS)_4$ spacetime satisfying Einstein field equations without cosmological constant, a necessary and sufficient condition the derivative of the energy density be orthogonal to the vector field generated by the 1-form A is that it must be $\sigma+p=0$.*

Putting (4.12) in (4.5), we get

$$S(X,Y) = \kappa\sigma g(X,Y). \tag{4.13}$$

Comparing (4.4) and (4.13), we find $S(X,Y) = \frac{r}{4}g(X,Y)$.

Hence, we can state the following theorem:

Theorem 4.1.2 *In a perfect fluid $A(PRS)_4$ spacetime, if the derivative of the energy density is orthogonal to the vector field generated by the $1-$form A then this spacetime reduces to an Einstein space.*

In this case, comparing the equations (4.9) and (4.12), we get

$$dp(\rho) = -\frac{1}{2}(\sigma + 5p). \tag{4.14}$$

Thus, we have the following theorem:

Theorem 4.1.3 *In a perfect fluid $A(PRS)_4$ spacetime satisfying Einstein field equations without cosmological constant, a necessary and sufficient condition the derivative of the isotropic pressure be orthogonal to the vector field generated by the $1-$form A is that it must be $\sigma = -5p$.*

With the help of (4.5) and (4.14), we find

$$S(X,Y) = \frac{4\kappa\sigma}{5}A(X)A(Y) + \frac{3\kappa\sigma}{5}g(X,Y). \tag{4.15}$$

This shows us that our spacetime is a quasi-Einstein.

Hence, we have the following theorem:

Theorem 4.1.4 *In a perfect fluid $A(PRS)_4$ spacetime satisfying Einstein field equations without cosmological constant, if the derivative of the isotropic pressure is orthogonal to the vector field generated by the $1-$form A then this spacetime reduces to a quasi-Einstein with the form in $S(X,Y) = \frac{4\kappa\sigma}{5}A(X)A(Y) + \frac{3\kappa\sigma}{5}g(X,Y)$.*

Now, contracting (4.5) over X, then we obtain

$$A(QY) = -\frac{\kappa}{2}(\sigma + 3p)A(Y). \tag{4.16}$$

If we take the covariant derivative of (4.16), we find

$$(\nabla_Z A)(QY) = -\frac{\kappa}{2}(d\sigma + 3dp)A(Y) - -\frac{\kappa}{2}(\sigma + 3p)(\nabla_Z A). \tag{4.17}$$

Using (3.4), (4.5) and (4.16) in (4.17), we get

$$-2(\sigma + 2p)A(Y)A(Z) - \frac{1}{2}(\sigma + 3p)A(Y)B(Z) - \frac{1}{2}(\sigma - p)g(Y,Z) + (\sigma + p)(\nabla_Z A)(Y) + \frac{1}{2}(\sigma + 3p)A(Y) = 0. \tag{4.18}$$

Contracting over Y and Z in (4.5), we find

$$6p + (\sigma + p)divA + \frac{1}{2}[d\sigma(\rho) + 3d(\rho)] = 0. \tag{4.19}$$

From (4.11) and (4.19),

$$divA = \frac{3(\sigma - p)}{2(\sigma + p)}. \tag{4.20}$$

Thus, we have the following theorem :

Theorem 4.1.5 *In a perfect fluid $A(PRS)_4$ spacetime satisfying Einstein field equations without cosmological constant, the divergence of the vector field generated by the $1-$form A is $divA = \frac{3(\sigma-p)}{2(\sigma+p)}$.*

If the vector field generated by the $1-$form A is divergence$-$free then by the aid of (4.20), it must be $\sigma = p$. In this case, we have from (4.5),

$$S(X,Y) = 2\kappa\sigma A(X)A(Y).$$

We can say that our spacetime is $\eta-$Einstein spacetime where $\eta = 2\kappa\sigma$.

Let us assume that the vector field generated by the $1-$form A is torse-forming vector field,

$$(\nabla_Z A)(X) = \lambda(Z)A(X) + \beta g(X,Z). \tag{4.21}$$

If we contract (4.21) over X, we get

$$\lambda(Z) = \beta A(Z). \tag{4.22}$$

Comparing (4.6) and (4.22), it is concluded that

$$(\nabla_Z S)(X,Y) = \kappa(d\sigma + dp)A(X)A(Y) + 2\kappa(\sigma + p)A(X)A(Y)A(Z) + \frac{\kappa}{2}(d\sigma - dp)g(X,Y). \quad \textbf{(4.23)}$$

By putting (4.23) in (3.4), we find

$$\kappa(d\sigma + dp)A(X)A(Y) + 2\kappa(\sigma + p)A(X)A(Y)A(Z) + \frac{\kappa}{2}(d\sigma - dp)g(X,Y) = [A(Z) + B(Z)]S(X,Y) + A(X)S(Y,Z) + A(Y)S(X,Z). \quad \textbf{(4.24)}$$

Since $\kappa \neq 0$, from (4.5) and (4.24),

$$(d\sigma + dp)A(X)A(Y) + \frac{1}{2}(d\sigma - dp)g(X,Y) = (\sigma + p)[A(Z) + B(Z)]A(X)A(Y) + \frac{1}{2}(\sigma - p)[A(Z) + B(Z)]g(X,Y) + \frac{1}{2}(\sigma - p)[A(Y)g(X,Z) + A(Z)g(X,Y)]. \quad \textbf{(4.25)}$$

Now, contracting (4.25) over X and Y,

$$d\sigma - 3dp = 2(\sigma - 2p)A(Z) + (\sigma - 2p)B(Z). \quad \textbf{(4.26)}$$

Contracting (4.26) over Z,

$$d\sigma(\rho) - 3dp = -2\sigma + 4p. \quad \textbf{(4.27)}$$

From (4.9) and (4.27),

$$\sigma + p = 0.$$

Thus, we have the following theorem:

Theorem 4.1.6 *If the vector field generated by the* $1-$*form* A *of a perfect fluid* $A(PRS)_4$ *spacetime satisfying Einstein field equations without cosmological constant, then a necessary and sufficient condition the derivative of the energy density be orthogonal to the vector field generated by the 1-form A is that it must be* $\sigma + p = 0$.

4.2 A Pressureless Fluid $A(PRS)_4$ Spacetimes

Assuming that our spacetime is a pressureless fluid spacetime (a dust), the energy momentum tensor is the form

$$T(X,Y) = \sigma A(X)A(Y). \quad \textbf{(4.28)}$$

In this case, from (4.3) and (4.28), we find that

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa\sigma A(X)A(Y). \quad \textbf{(4.29)}$$

Contracting (4.29) over X and Y,

$$r = \kappa\sigma. \quad \textbf{(4.30)}$$

In this case, from (4.29) and (4.30), we get

$$S(X,Y) = \kappa\sigma[A(X)A(Y) + \frac{1}{2}g(X,Y)]. \quad \textbf{(4.31)}$$

By taking the covariant derivative of (4.31), it can be found that

$$(\nabla_Z S)(X,Y) = \kappa d\sigma[A(X)A(Y) + \frac{1}{2}g(X,Y)] + \kappa\sigma[(\nabla_Z A)(X)A(Y) \\ + A(X)(\nabla_Z A)(Y)]. \quad \textbf{(4.32)}$$

By putting (3.4) in (4.32), we obtain

$$[A(Z) + B(Z)]S(X,Y) + A(X)S(Y,Z) + A(Y)S(X,Z) = \\ \kappa d\sigma[A(X)A(Y) + \frac{1}{2}g(X,Y)] + \kappa\sigma[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)]. \quad \textbf{(4.33)}$$

From (4.31) and (4.33), we find

$$\kappa\sigma[A(X)A(Y) + \frac{1}{2}g(X,Y)] + \kappa\sigma A(X)[A(Y)A(Z) + \frac{1}{2}g(Y,Z)] \\ + \kappa\sigma A(Y)[A(X)A(Z) + \frac{1}{2}g(X,Z)] = \kappa(d\sigma)[A(X)A(Y) + \frac{1}{2}g(X,Y)] \\ + \kappa\sigma[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)]. \quad \textbf{(4.34)}$$

Contracting (4.34) over X and Y, we get

$$\sigma B(Z) = d\sigma. \quad \textbf{(4.35)}$$

Then from (4.35), we have

$$d\sigma(\rho) = 0. \tag{4.36}$$

If the derivative of σ is orthogonal to the vector field associated by the $1-$form A then from (4.11) and (4.36), we find

$$\sigma = 0. \tag{4.37}$$

Thus with the help of(4.28) and (4.37), we conclude that

$$T(X,Y) = 0.$$

In this case, the spacetime is devoid of the matter. Thus, we can state the following theorem:

Theorem 4.2.1 *An $A(PRS)_4$ dust fluid spacetime satisfying Einstein field equations without cosmological constant is vacuum.*

4.3 A Radiation Fluid $A(PRS)_4$ Spacetimes

Now, we assume that our spacetime is a radiation fluid. Thus, we have

$$T(X,Y) = p[4A(X)A(Y) + g(X,Y)]. \tag{4.38}$$

In this case, from (4.1) and (4.38), we find

$$S(X,Y) - \frac{r}{2} = \kappa p[4A(X)A(Y) + g(X,Y)]. \tag{4.39}$$

Now, contracting (4.39) over X and Y, we get

$$r = 0. \tag{4.40}$$

Thus, by the aid of (4.39) and (4.40), it can be found that

$$S(X,Y) = \kappa p[4A(X)A(Y) + g(X,Y)]. \tag{4.41}$$

If we take the covariant derivative of (4.41), we obtain

$$(\nabla_Z S)(X,Y) = \kappa dp[4A(X)A(Y) + g(X,Y)] + 4\kappa p[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)]. \tag{4.42}$$

By putting (3.4) in (4.42), we get

$$[A(Z)+B(Z)]S(X,Y)+A(X)S(Y,Z)+A(Y)S(X,Z)= \kappa[dp(4A(X)A(Y)+g(X,Y)) +4p((\nabla_Z A)(X)A(Y)+A(X)(\nabla_Z A)(Y))]. \quad \textbf{(4.43)}$$

Contracting (4.43) over X and Y,

$$[A(Z)+B(Z)]r+2A(QZ)=0. \quad \textbf{(4.44)}$$

Using (4.40) and (4.41) in (4.44),

$$-6\kappa pA(Z)=0. \quad \textbf{(4.45)}$$

Thus, it can be obtained from (4.45) that $p=0$.

In this case, we get from (4.38), $T(X,Y)=0$. Hence, we have the following theorem:

Theorem 4.3.1 *An $A(PRS)_4$ radiation fluid spacetime satisfying Einstein field equations without cosmological constant is vacuum.*

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