ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE
ENGINEERING AND TECHNOLOGY

OPTICAL SOLITONS FOR THE HIGHER-ORDER CUBIC-QUINTIC NONLINEAR SCHRÖDINGER EQUATION WITH A $\mathscr{PT}$-SYMMETRIC POTENTIAL

M.Sc. THESIS

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Mathematical Engineering Department

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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

$\mathscr{PT}$ -SİMETRİK BİR POTANSİYEL İÇEREN DOĞRUSAL OLMAYAN YÜKSEK MERTEBE KÜBİK-KUİNTİK SCHRÖDINGER DENKLEMİNDE OPTİK SOLİTONLAR

YÜKSEK LİSANS TEZİ

Ayşe Şebnem YAR
(509141201)

Matematik Mühendisliği Anabilim Dalı

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Tez Danışmanı: Assoc. Prof. Dr. İlkay BAKIRTAŞ AKAR

AĞUSTOS 2017

Ayşe Şebnem YAR, a M.Sc. student of ITU Graduate School of Science Engineering and Technology 509141201 successfully defended the thesis entitled "OPTICAL SOLITONS FOR THE HIGHER-ORDER CUBIC-QUINTIC NONLINEAR SCHRÖDINGER EQUATION WITH A $\mathscr{PT}$-SYMMETRIC POTENTIAL", which he prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

Thesis Advisor : **Assoc. Prof. Dr. İlkay BAKIRTAŞ AKAR** ....
Istanbul Technical University

Jury Members : **Assoc. Prof. Dr. İlkay BAKIRTAŞ** ....
Istanbul Technical University

Prof. Dr. Nalan ANTAR ....
Istanbul Technical University

Assist. Prof. Dr. Güler KAYGUSUZOĞLU ....
Namık Kemal University

Date of Submission : **25 August 2017**
Date of Defense : **29 May 2017**

To my family and future,

FOREWORD

I would like to express my deep appreciation to my advisor, Assoc. Prof. Dr. Ilkay Bakırtaş for standing behind me throughout this project. With her support, I took the opportunity to work on one of the most modern and excellent subjects of applied mathematics. I am grateful to her and I will always be. I would also like to thank my colleagues Mahmut Bağcı and Izzet Göksel, for their assistance with their infinite wisdom about numerical methods, solitons and Matlab programming. I love and want to thank my family. They always stand by me in this process and prevented me from giving up. Finally, I'm grateful to all my friends. Special thanks to Selda Türkmen for her psychological support and optimistic attitude.

August 2017 Ayşe Şebnem YAR

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ABBREVIATIONS

NLS	**:** Nonlinear Schrödinger
CNLS	**:** Cubic Nonlinear Schrödinger
CQNLS	**:** Cubic Quintic Nonlinear Schrödinger
3OD	**:** Third Order Dispersion
4OD	**:** Fourth Order Dispersion
$\mathscr{PT}$	**:** Parity - Time
SR	**:** Spectral Renormalization
KdV	**:** Korteweg-de Vries

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OPTICAL SOLITONS FOR THE HIGHER-ORDER CUBIC-QUINTIC NONLINEAR SCHRÖDINGER EQUATION WITH A $\mathscr{PT}$-SYMMETRIC POTENTIAL

SUMMARY

In nature, most of the systems are nonlinear and as a result of this fact, those systems are modeled by nonlinear systems of equations. Some of the most remarkable progress in nonlinear science is observed in wave propagation phenomena. Often, research on a given nonlinear system begins by investigating a one-dimensional partial differential equation (PDE) as an approximation to an experimental system in optics, fluid dynamics, plasma physics and biology. Many of these nonlinear equations have known nonlinear wave type solutions and some are commonly referred to as soliton solutions. Solitons are localized waves that collide elastically, suffering only a shift in phase. The history of solitons dates back to 1834, the year in which John Scott Russell observed a wave form propagated for several kilometers in a shallow canal of Scotland without being distorted. Solitons represent the solutions of nonlinear wave-type partial differential equations, including sine-Gordon, Korteweg-de Vries (KdV) and nonlinear Schrödinger (NLS) equations.

In this thesis, we explore the theoretical and numerical analysis of optical solitons of a higher order cubic quintic nonlinear Schrödinger equation (CQNLS) with a fourth-order dispersion term (4OD) in a $\mathscr{PT}$-symmetric potential.

In Chapter 1, the historical background of optical soliton research is briefly given. The application areas and the mechanism of the NLS 4OD equation are discussed, and the general properties of $\mathscr{PT}$-symmetric potentials are argued. In this chapter, the aim of the thesis, literature review and hypothesis of the thesis are given, respectively.

In Chapter 2, spectral renormalization method (SR), the numerical method which is used to obtain localized soliton solutions is explained. The modification of this method is given in order to apply to CQNLS equation with a fourth order dispersion term and an external potential. Then, the Split-step Fourier method is given for nonlinear stability analysis.

Chapter 3 is dedicated to (1+1)D 4OD cubic-quintic NLS equation without an external potential. Exact and numerical solution of the equation are analyzed and the produced results are shown by some graphs. Lastly, the nonlinear stability of the soliton solutions are investigated for various parameters of the considered equation and the results are compared.

Chapter 4 includes studies of exact soliton solution of the (1+1)D 4OD cubic-quintic NLS equation with a $\mathscr{PT}$-symmetric potential. This $\mathscr{PT}$-symmetric potential is introduced and for different values of parameters, soliton solutions are found in this potential. For this various values of parameters of the equation, exact and numerical results are compared, the effect of the eigenvalue of the numerical solutions are figured out and the maximum amplitude of the solitons are discovered. For final, the nonlinear stability of the produced solitons are demonstrated in terms of various parameters.

Result of this dissertation are summarized in Chapter 5. In this thesis, MATLABR2010a computer programme is used and all of the results are produced by the use of this programme.

$\mathscr{PT}$ -SİMETRİK BİR POTANSİYEL İÇEREN DOĞRUSAL OLMAYAN YÜKSEK MERTEBE KÜBİK-KUİNTİK SCHRÖDINGER DENKLEMİNDE OPTİK SOLİTONLAR

ÖZET

Doğada pek çok olgu, nonlineer (doğrusal olmayan) denklem sistemleriyle modellenir. Nonlineer bilimsel araştırmalar konusundaki en önemli gelişmeler nonlineer dalga yayılımı problemleri konusunda öne çıkmaktadır. Sıklıkla, verilen bir nonlineer sistemin araştırılmasına, optik, akışkanlar mekaniği, plazma fiziği ve biyolojideki ilişkili deneysel sistemin bir yaklaşımı olarak, bir boyutlu bir kısmi türevli diferansiyel denklemin çözümünün elde edilmesiyle başlanır. Bu tip denklemlerin çoğunun, bilinen nonlineer dalga tipi çözümleri vardır ve bunların bir kısmı da soliton olarak isimlendirilmiştir. Solitonlar, elastik çarpışmalar yapan ve çarpışma sonrası formunu koruyarak sadece faz kayması görünen lokalize dalga çözümleridir. Solitonun geçmişi 1834 yılına dayanır. Bu tip lokalize ve formunu koruyarak kilometrelerce ilerleyen bir dalga ilk kez John Scott Russell tarafından İskoçya'da dar ve sığ bir kanalda gözlemlenmiştir. Russell daha sonra bu dalgayı "solitary wave-yalnız dalga" olarak isimlendirmiştir. Nonlineer dalgaların modellenmesinde kullanılan sine-Gordon, Korteweg-de Vries (KdV) ve nonlinear Schrödinger (NLS) gibi kısmi türevli diferansiyel denklemlerin soliton tipi çözümleri kabul ettikleri uzun yıllardır bilimsel literatürde gösterilmiştir.

Evrende ölçülebilen bütün fiziksel gözlemlerin sonuçları, reel büyüklüklerle ifade edilebilir.

Kuantum mekaniğinde bütün operatörlerin (örneğin Hamiltonyen) özdeğerlerinin reel olmaları gerekir ve reel spektrumu garanti edebilmek için kullanılan operatörler Hermityen (kendine eş) olamlıdır. Fakat son yıllarda yapılan bazı çalışmalarda bu gerekliliğin zayıflatılabileceği gözlendiği gibi ve operatörlerin uzay-zaman simetrisini ($\mathscr{PT}$-simetri) sağlaması durumunda, Hermityen olmayan operatörlerinde bütünüyle reel spektrum yaratabileceği gösterilmiştir.

Bu çalışmada kullandığımız potansiyel de $\mathscr{PT}$-simetrik olma özelliği taşımaktadır, yani $V(x) = V^*(-x)$ ilişkisini sağlar. Kullanılan $\mathscr{PT}$-simetrik potansiyel aşağıdaki bir kompleks yapıda tanımlanmıştır:

$$V_{PT} = V(x) + iW(x) \tag{1}$$

Burada $V(x)$ ve $W(x)$, sırasıyla, $\mathscr{PT}$-simetrik kompleks potansiyelin reel ve imajiner kısımlarıdır. Potansiyelin reel kısmı çift fonksiyon özelliğine sahipken, imajiner kısmı tek fonksiyondur.

Bu çalışmada, aşağıda ifade edilen, dördüncü mertebeden dispersiyon terimi ve bir dış potansiyel içeren doğrusal olmayan kübik-kuintik nonlineer Schrödinger denkleminin soliton çözümlerinin sayısal olarak varlığı ve kararlılık (stabilite) analizleri incelenmiş

ve sonuçlar kesin çözümlerle karşılaştırılmıştır.

$$iu_z + u_{xx} + \alpha|u|^2u + \gamma u_{xxxx} + \beta|u|^4u + V_{PT}u = 0 \tag{2}$$

Verilen denklemde u kompleks değerli türevlenebilir fonksiyonu, u_{xx} kırılımı modelleyen terimi, α üçüncü mertebeden doğrusal olmayan terimin katsayısını, γ terimi dördüncü mertebeden dispersiyon teriminin katsayısını ,β beşinci mertebeden doğrusal olmayan terimin katsayısını ve V_{PT} $\mathscr{PT}$-simetrisi özelliği sağlayan potansiyeli temsil eder. Bu tezin amacı, $\mathscr{PT}$-simetrisi özelliği sağlayan potansiyelin ve dördüncü mertebe dispersiyon teriminin (u_{xxxx}), soliton çözümünde ve bu çözümlerin kararlılığında yarattığı etkiyi gözlemlemektir.

Bölüm 1'de, optik solitonlarla ilgili çalışmaların tarihsel gelişimlerinden kısaca söz edilmiş ve dördüncü mertebeden dispersiyon terimi ve $\mathscr{PT}$-simetrik potansiyel içeren, doğrusal olmayan kübik ve kübik-kuintik Schrödinger denklemlerinin yapısı ve uygulama alanları anlatılmıştır. Bunların yanı sıra, bu bölümde, denklemin çözümünde kullanılmış olan sayısal analiz yöntemleri de açıklanmıştır. Ayrıca analitik ve sayısal olarak dördüncü mertebe dispersiyon terimi içeren NLS denkleminin soliton çözümlerini inceleyen çalışmalardan da bahsedilmiştir. Çalışmada kullanılan $\mathscr{PT}$-simetrik potansiyelin tanımı verilmiş, fiziksel anlamı ve sağlaması gereken özellikler açıklanmıştır. Bu bölümde, CQNLS denkleminin soliton tipi çözümlerinin elde edilmesinde kullanılan spektral renormalizasyon (SR) metodunun literatürde kullanıldığı problemlerden bahsedilmiş, metodun temel yaklaşımı anlatılmıştır. Tezin amacı, gerekli literatür taraması ve tezin hipotezi sırasıyla verilmiştir.

Bölüm 2'de Ablowitz ve Musslimani'nin ortaya koyduğu Spektral Renormalizasyon (SR) yönteminden ve yöntemin temel prensiplerinden bahsedilmiştir. Bu sayısal yöntemin bir modifikasyonu ile dış potansiyel içeren ve dördüncü mertebeden dispersiyon terimi bulunan CQNLS denkleminin sayısal çözümleri elde edilmiştir. Bu yöntemde, denkleme $u(x,z) = f(x)e^{i\mu z}$ formunda bir çözüm aranmış olup $f(x)$ kompleks değerli fonksiyonu Fourier uzayında iteratif olarak çözülmüştür. Daha sonra, elde edilen solitonların stabilite analizi için Ayrık adımlı Fourier metodu (Split-step Fourier Method) kullanılmıştır.

Bölüm 3, aşağıdaki gibi verilen potansiyelsiz halde, (1+1) boyutlu dördüncü mertebeden bir dispersiyon terimi içeren, kübik-kuintik NLS denklemine ayrılmıştır:

$$iu_z + \alpha u_{xx} + |u|^2u + \gamma u_{xxxx} + \beta|u|^4u = 0 \tag{3}$$

Literatürde, kuintik terimin ihmal edildiği halde, (β=0) bu denklemin analitik çözümleri belli parametreler için

$$u(x,z) = \sqrt{\frac{3\alpha^2}{10\gamma}} \sec h^2\left(\frac{x}{\sqrt{20\gamma/\alpha}}\right) \exp\left(i\frac{4\alpha^2}{25\gamma}z\right) \tag{4}$$

formunda elde edilmiş ve soliton tipi çözümler incelenmiştir. Bu bölümde, sayısal algoritmamızın doğruluğunu test etmek amacıyla, elde edilen bu kesin çözüm ve SR algoritmasından elde edilen sayısal çözümlerin üst üste düştüğü gösterilmiştir. Bu çözümlerde dördüncü mertebeden dispersiyon teriminin soliton yapıları üzerindeki etkisi araştırılmış ve son olarak bu solitonların stabiliteleri incelenmiştir.

Bölüm 4, (1+1) boyutlu dördüncü mertebeden dispersiyon terimi ve $\mathscr{PT}$-simetrik potansiyel içeren kübik-kuintik NLS denkleminin sayısal çözümlerini ve kararlılık analizlerini içermektedir. Bu bölümde, çalışmada kullanılan $\mathscr{PT}$-simetrik potansiyel tanımlanmış ve farklı parametreler altında bu potansiyel altında analitik çözümleri üretebilmek için $u(x,z) = f(x)e^{i(\mu z+g(x))}$ çözüm önerisi yapılmıştır. Burada $f(x)$ ve $g(x)$ henüz yapısı belli olmayan reel değerli fonksiyonlar olarak kabul edilmiştir. Bu çözüm önerisi denklemde yerine konarak elde edilen çözümler kesin çözümle karşılaştırılmış ve kullanılan $\mathscr{PT}$-simetrik potansiyelin yapısı aşağıda verilen şekilde elde edilmiştir:

$$V_{PT} = [V_1 \mathrm{sech}^2(x) + V_2 \mathrm{sech}^4(x)] + i[W_3 \mathrm{sech}^3(x) \tanh(x)]. \quad (5)$$

Potansiyelde bulunan katsayıların soliton çözümüne nasıl etki ettiği sayısal olarak incelenmiş ve sonuçlar tartışılmıştır. Ayrıca dispersiyon teriminin katsayısı olan (γ) ve özdeğer (μ) ile maksimum genlik arasındaki ilişkiler yine sayısal olarak incelenmiş ve sonuçlar grafiklerle gösterilmiştir. Sayısal çözümleri elde etmek için kullanılan spektral renormalizasyon metodu ile elde edilen soliton çözümleri, çeşitli parametreler için analitik çözüm ile karşılaştırılmıştır. Daha sonra, bu solitonların stabilite analizi için Ayrık adımlı Fourier metodu (Split-step Fourier Method) kullanılmış ve elde edilen sonuçlar grafikler üzerinde gösterilmiştir. Son olarak, dispersiyon teriminin (γ) ve özdeğerlerin (μ) solitonun kararlılığı üzerindeki etkisi incelenip bulunan sonuçlar grafiklerle gösterilmiş ve sonuçlar yorumlanmıştır.

Bölüm 5'de tezde elde edilen tüm sonuçlar ayrıntılı olarak açıklanmıştır. Potansiyelsiz denklemde ve bir dış potansiyel içeren denklemdeki sonuçlar özetlenip, sisteme eklenen dış potansiyelin etkisi tartışılmıştır. Ayrıca önceki bölümlerden elde edilen sonuçlar ışığında dispersiyon terimi γ ve özdeğer μ'nün de çözümler üzerindeki etkileri de bu bölümde tartışılmıştır.

Bu tezde MATLABR2010a bilgisayar programı kullanılmış ve bütün çözümler bu program ile elde edilmiştir.

1. INTRODUCTION

In the field of optics, a soliton denote to any optical field that does not change its shape during propagation because of the sensitive balance between linear and nonlinear effects in the medium [1]. Interest in optical solitons has grown steadily in recent years. The field has considerable potential for technological applications, and it presents many exciting research problems both from a fundamental and an applied point of view. Over the past thirty-five years, soliton research has been conducted in fields as diverse as particle physics, molecular biology, quantum mechanics, geology, meteorology, oceanography, astrophysics and cosmology [2]. NLS equation is usually defined by the nonlinear dynamics of pulses on a picosecond time-scale. This equation is developed by Erwin Schrödinger in 1927. A considerable amount of research work has been devoted to the study of nonlinear Schrödinger equations with a variety of nonlinearities. Several methods, numerical and analytical, have been effectively used to handle these problems [3]. The inverse scattering method, Lax pair, Backlund transformation are some of these methods. These works and results have important scientific values and application prospects such as transmitting digital signals over long distances. Mathematical and numerical analysis of the considered equation with application areas can be found in the reference [2]. Nonlinear Schrödinger equation is usually define as the propagation of an optical pulse in optical materials is given as

$$iu_z + u_{xx} + \alpha|u|^2u = 0. \tag{1.1}$$

In optics, u corresponds to the differentiable complex valued, slowly varying amplitude of the electric field; u_{xx} corresponds to diffraction; z is a scaled propagation distance; the coefficients α represents the cubic nonlinearities of the medium.

The dynamics of pulses with widths smaller than 1 picosecond can not be governed by the cubic NLS equation. For example, in a solid state laser, pulses are generated shorter than 10 femtoseconds and the approximation of the standard NLS equation breaks down. In order to describe the dynamics in such systems, we need higher order dispersion terms. One needs to consider the third order (3OD) dispersion for performance enhancement along trans-oceanic and trans-continental distances. Also, for short pulse widths where the group velocity dispersion changes, within the spectral bandwidth of the signal, can no longer be neglected, one needs to take into account the presence of fourth order (4OD) dispersion [4].

In this thesis, we investigate higher order (we refer to fourth-order dispersion (4OD) term), cubic-quintic nonlinear Schrödinger equation with a $\mathscr{PT}$-symmetric optical potential given below:

$$iu_z + u_{xx} + \alpha|u|^2u + \gamma u_{xxxx} + \beta|u|^4u + V_{PT}u = 0. \tag{1.2}$$

Here γ is a fourth-order diffraction coupling constant taken as a negative constant value and V_{PT} is a $\mathscr{PT}$-symmetric external potential (lattice).

The aim of this thesis is to find the exact and the numerical solutions of the equation 1.2 and discover the effect of the fourth order dispersion term γu_{xxxx} on the soliton solutions and their stabilities.

In order to investigate the evolution of the ultrashort optical pulses for NLS equation with fourth order dispersion term 4OD without any potential and compare with our numerical method, we used the results obtained in [5]. In this study by Karlsson and Höök, both second and fourth order dispersion terms are taken into account and an exact soliton type solution is given. In this work, the effect of the fourth order dispersion on the shape and stability of the soliton is investigated. Also, in [6] and [7], the dynamics and interactions of bright solitons in an optical fiber with fourth order dispersion are investigated. In [8], 4OD cubic-quintic nonlinear Schrödinger equation with potential is solved through the extended elliptic sub-equation method. As a consequence, many types of exact traveling wave solutions are obtained which including bell and kink profile solitary wave solutions, triangular periodic wave solutions and singular solutions.

Any measurement of a physical observable in our universe obviously yields a real quantity. Eigenvalues of operators are observable in quantum mechanics. Therefore, all the eigenvalues of operators are need to be real for the reality. All observables corresponded to eigenvalues of Hermitian (i.e. self adjoint) operators was postulated to guarantee a real spectrum. In fact, a Hermitian Hamiltonian ensures a real energy spectrum. Instead only space time reflection symmetry or $\mathscr{PT}$-symmetry, weaker version of Hermicity axiom which requires that the Hamiltonian has seen considerable attention in the past decade [9–12]. In addition to this, threshold value above which the spectrum becomes complex are determined in many cases. $\mathscr{PT}$-symmetric is defined by means parity operator $\hat{P}$ and the time operator $\hat{T}$ whose actions are given by $\hat{P}: \hat{p} \to -\hat{p}, \hat{x} \to \hat{x}, i \to -i$, where $\hat{p}$ is the momentum operator, $\hat{x}$ is the position operator and i is the imaginary unit [13].The $\hat{P}\hat{T}$ operator and satisfies the commutativity $\hat{P}\hat{T}\hat{H} = \hat{H}\hat{P}\hat{T}$, namely $V(x) = V^*(-x)$ and a Hamiltonian $\hat{H} = \hat{p}^2 + V(x)$ has the same eigenfunctions then it is said to be $\mathscr{PT}$-symmetric [14]. If the same eigenfunctions are not shared then we can speak of broken $\mathscr{PT}$-symmetry. $\mathscr{PT}$ symmetric structures have been realized in optical models governed by NLS type equations in which the propagation distance z replaces time in quantum mechanics [10].

We will consider the case of 4OD cubic-quintic nonlinear Schrödinger equation with a $\mathscr{PT}$ - symmetric potential defined as

$$V_{PT} = V(x) + iW(x) \tag{1.3}$$

where $V(x)$ and $W(x)$ are the real and imaginary components of the complex $\mathscr{PT}$-symmetric potential, respectively. Here, the real part of a $\mathscr{PT}$ potential is a even, symmetric function whereas the imaginary component should be odd, anti-symmetric. In Figure 1.1, the real and the imaginary parts of the $\mathscr{PT}$-symmetric potential that is derived in Chapter 4 is plotted.

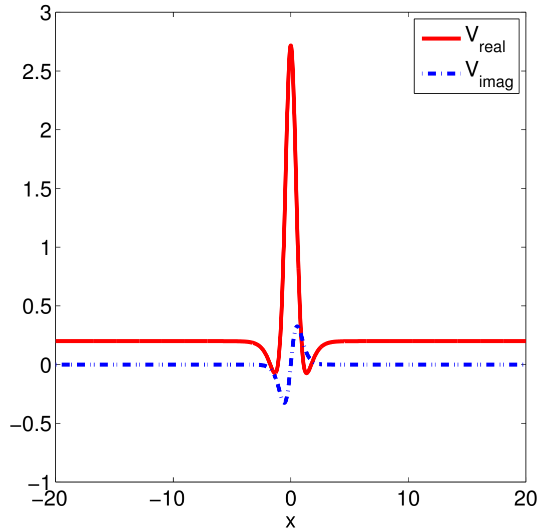

Figure 1.1 : Real and imaginary parts of $\mathscr{PT}$-symmetric potential plotted on top of each other.

In this thesis, to obtain numerical solutions for 4OD cubic-quintic nonlinear Schrödinger equation with and without external potential, we will use spectral renormalization method. To transform the considered equation into Fourier space and find out a nonlinear integral equation coupled to an algebraic equation is the main idea of this method. Implementation procedure of this method to the nonlinear Schrödinger equation explained in Chapter 2.

1.1 Purpose of Thesis

In this thesis, we aim to investigate the effect of the external potential and fourth-order dispersion term on the soliton solutions of the 4OD cubic-quintic nonlinear Schrödinger equation with $\mathscr{PT}$-symmetric potential and find an exact soliton type solution to this model equation. The cases of 4OD cubic-quintic nonlinear Schrödinger equation without a potential and with a special type of $\mathscr{PT}$-symmetric potential are compared to understand this effect.

1.2 Literature Review

Since their applications to telecommunication and ultrafast signal routing systems, optical solitons have been the objects of extensive theoretical and experimental studies in recent years [15]. They evolve from a nonlinear change in the refracive index of a material induced by the light intensity distribution [16]. Nonlinear Schrödinger equation (NLS) is the main nonlinear equation governing the pulse evolution in the picosecond regime [17].

Nonlinear Schrödinger (NLS) equation is usually denoted by the nonlinear evolution of short pulses in an optical fiber. This equation represents the mathematical models of different physical problems [18]. The mechanism and structure of the soliton interaction are explained both analytically and numerically in [19]. In [19, 20], decay problem of the ps degenerate soliton and the effect of the frequency down shift are investigated. In addition to this, wave propagation in nonlinear media [21], surface waves on deep waters [22] are denoted by nonlinear Schrödinger equation.

In fibers, NLS 4OD equation represent the mathematical model of the optical pulses in a picosecond time scale [6]. To find out analytical and numerical solutions of the considered equation, many studies and researches have been done. Biswas an coworkers used the solitary wave ansatz to produce exact solution of the NLS 4OD equation in [23]. To obtain exact solutions, the method of *tanh* and the method of $sine - cos$ are applied to the considered 4OD NLS equation [3]. Also in [24], the nonlinear Schrödinger equation with third and fourth order dispersion terms is investigated and analytical results are obtained. Various type of exact solitons for the fourth-order dispersive cubic-quintic nonlinear Schrödinger equation are given in [25].

In [26], the effect of nonlinearity in novel $\mathscr{PT}$-symmetric potential for 4OD cubic nonlinear Schrödinger equation are investigated. Numerical solutions of 4OD cubic-quintic nonlinear Schrödinger equation with a $\mathscr{PT}$-symmetric potential are investigated by means of spectral renormalization method in [27]. This method is essentially a Fourier iteration method and in this thesis, the method is modified so that it can be applied to the (1+1)-dimensional 4OD cubic-quintic NLS equation. This method can be effectively used to obtain localized solution of KDV equation [28], dispersion-managed systems [29], discrete diffraction-managed systems [30, 31] and NLS equation [32]. Also, (2+1)D and (1+1)D NLS equation with an external potential was solved by using the spectral renormalization method and the produced results are shown in [33, 34] and [27], respectively.

1.3 Hypothesis

The effect of the external potential and its type on the existence and stability of fundamental solitons is crucial. Higher order dispersion affects the maximum amplitude and the shape of the soliton solution of the model equation. The existence and stability of the soliton solutions, are greatly affected by this higher order dispersive term.

2. NUMERICAL METHODS

2.1 Spectral Renormalization Method

It is known that various techniques have been used to compute localized solutions (i.e., soliton solutions) to nonlinear evolution equations. Numerical solutions to Eq. (1.1) are investigated by using the Spectral renormalization method. The method is essentially a Fourier iteration method that was proposed by Petviashvili in [35].

Later, Ablowitz and Musslimani advanced this method [36] a generalized numerical scheme for computing solitons in nonlinear wave guides (SR). To transform the governing equation Fourier space and find a nonlinear nonlocal integral equation coupled to an algebraic equation is the essence of the method. The coupling gets under control the numerical scheme from diverging.

The optical mode is then obtained from an iteration scheme, which converges rapidly. This method is useful to apply to a large class of problems which include higher order nonlinear terms with different homogenetic.

In this section, we have given the numerical solution to the NLS 4OD cubic-quintic equation with an external potential in Eq. (1.2) and this solution will be obtained by the spectral renormalization method.

The method is modified so that it can be applied mainly to the (1+1)D NLS 4OD cubic-quintic equation with $\mathscr{PT}$-symmetric potential as follows:

$$iu_z + u_{xx} + \alpha|u|^2u + \gamma u_{xxxx} + \beta|u|^4u + V_{PT}u = 0. \tag{2.1}$$

Using the ansatz $u(x,z) = f(x)e^{i\mu z}$ where $f(x)$ is a complex-valued function and μ is the propagation constant (or eigenvalue), we have following expressions:

$$\begin{aligned}
&u_z = i\mu f e^{i\mu z} \\
&u_{xx} = f_{xx}e^{i\mu z} \\
&u_{xxxx} = f_{xxxx}e^{i\mu z} \\
&u^* = f e^{-i\mu z} \\
&|u|^2 = |f|^2 \\
&|u|^4 = |f|^4.
\end{aligned} \tag{2.2}$$

Substituting the set of the terms in Eq. (2.2) into Eq. (2.1), the following nonlinear equation for f is obtained

$$-\mu f e^{i\mu z} + f_{xx}e^{i\mu z} + \alpha|f|^2 f e^{i\mu z} + \gamma f_{xxxx}e^{i\mu z} + \beta|f|^4 f e^{i\mu z} + V_{PT} f e^{i\mu z} = 0. \tag{2.3}$$

After simplifying these equations we get

$$-\mu f + f_{xx} + \alpha|f|^2 f + \gamma f_{xxxx} + \beta|f|^4 f + V_{PT} f = 0. \tag{2.4}$$

After applying Fourier transformation to Eq. (2.4)

$$\begin{aligned}
&\mathscr{F}\{-\mu f\} + \mathscr{F}\{f_{xx}\} + \mathscr{F}\{\alpha|f|^2 f\} + \\
&\mathscr{F}\{\gamma f_{xxxx}\} + \mathscr{F}\{\beta|f|^4 f\} + \mathscr{F}\{V_{PT} f\} = \mathscr{F}\{0\}.
\end{aligned} \tag{2.5}$$

where $\mathscr{F}$ denotes Fourier transformation and considering the properties of this transformation, we have Eq. (2.6)

$$\begin{aligned}
&-\mu \hat{f} + (-ik_x)^2 \hat{f} + \alpha\mathscr{F}\{|f|^2 f\} + \gamma(-ik_x)^4 \hat{f} \\
&+\beta\mathscr{F}\{|f|^4 f\} + \mathscr{F}\{(V + iW)f\} = 0
\end{aligned} \tag{2.6}$$

where $\mathscr{F}(f) = \hat{f}$ and k_x are Fourier variables. Solving Eq. (2.6) for the $\hat{f}$ yields

$$\hat{f} = \frac{\alpha\mathscr{F}\{|f|^2 f\} + \beta\mathscr{F}\{|f|^4 f\} + \mathscr{F}\{(V + iW)f\}}{[\mu + {k_x}^2 - \gamma {k_x}^4]} \tag{2.7}$$

In order to find $f(x)$, this equation could be indexed and utilized but the scheme does not converge. At this point, we should make acquainted with a new field variable $f(x) = \lambda w(x)$ with $\lambda \in R^+$ where λ is a parameter to be determined. The system with the new variable can be written as

$$\lambda \hat{w} = \frac{\alpha\mathscr{F}\{|w|^2|\lambda|^2 w\lambda + \beta\mathscr{F}\{|w|^4|\lambda|^4 w\lambda + \mathscr{F}\{(V + iW)\lambda w\}}{\mu + {k_x}^2 - \gamma {k_x}^4} \tag{2.8}$$

simplifying this equation, we get

$$\hat{w} = \frac{\alpha\mathscr{F}\{|w|^2|\lambda|^2 w\} + \beta\mathscr{F}\{|w|^4|\lambda|^4 w\} + \mathscr{F}\{(V+iW)w\}}{\mu + {k_x}^2 - \gamma {k_x}^4} \tag{2.9}$$

For finding out w, Eq. (2.9) can be utilized in an iterative method. In order to succeed this, we can calculate $\hat{w}$ using the following iteration approach:

$$\hat{w}_{n+1} = \frac{\alpha|\lambda|^2\mathscr{F}\{|w_n|^2 w_n\} + \beta|\lambda|^4\mathscr{F}\{|w_n|^4 w_n\} + \mathscr{F}\{(V+iW)w_n\}}{\mu + {k_x}^2 - \gamma {k_x}^4}, \quad n \in N \tag{2.10}$$

with the initial condition taken as a Gaussian type function

$$w_0 = e^{-x^2} \tag{2.11}$$

where our convergence criterions are $|w_{n+1} - w_n| < 10^{-12}$. Multiplying both sides of Eq. (2.9) by $(\mu + {k_x}^2 - \gamma {k_x}^4)$ and we obtain

$$(\mu + {k_x}^2 - \gamma {k_x}^4)\hat{w} = |\lambda|^2\alpha\mathscr{F}\{|w|^2 w\} + |\lambda|^4\beta\mathscr{F}\{|w|^4 w\} + \mathscr{F}\{(V+iW)w\}. \tag{2.12}$$

When we take all terms of Eq. (2.12) to the left side, we lead to following equation

$$(\mu + {k_x}^2 - \gamma {k_x}^4)\hat{w} - |\lambda|^2\alpha\mathscr{F}\{|w|^2 w\} - |\lambda|^4\beta\mathscr{F}\{|w|^4 w\} - \mathscr{F}\{(V+iW)w\} = 0. \tag{2.13}$$

Multiplying Eq. (2.13) by the conjugate of $\hat{w}$, i.e. by $\hat{w}^*$ yields

$$(\mu + {k_x}^2 - \gamma {k_x}^4)|w|^2 - |\lambda|^2\alpha\mathscr{F}\{|w|^2 w\}\hat{w}^* - |\lambda|^4\beta\mathscr{F}\{|w|^4 w\}\hat{w}^* - \mathscr{F}\{(V+iW)w\}\hat{w}^* = 0. \tag{2.14}$$

Furthermore, integrating Eq. (2.14) leads to

$$\begin{aligned} &\int_{-\infty}^{\infty}(\mu + {k_x}^2 - \gamma {k_x}^4)|w|^2 dk - |\lambda|^2\int_{-\infty}^{\infty}\alpha\mathscr{F}\{|w|^2 w\}\hat{w}^* dk \\ -|\lambda|^4&\int_{-\infty}^{\infty}\beta\mathscr{F}\{|w|^4 w\}\hat{w}^* dk - \int_{-\infty}^{\infty}\mathscr{F}\{(V+iW)w\}\hat{w}^* dk = 0 \end{aligned} \tag{2.15}$$

or in a more compact form

$$\begin{aligned} &-\int_{-\infty}^{\infty}\left[-\mathscr{F}\{(V+iW)w\}\hat{w}^* + (\mu + {k_x}^2 - \gamma {k_x}^4)|w|^2\right]dk \\ +|\lambda|^2&\int_{-\infty}^{\infty}\alpha\mathscr{F}\{|w|^2 w\}\hat{w}^* dk + |\lambda|^4\int_{-\infty}^{\infty}\beta\mathscr{F}\{|w|^4 w\}\hat{w}^* dk = 0. \end{aligned} \tag{2.16}$$

Eq. (2.16) is a fourth order polynomial of λ in the form $P(\lambda) = a\lambda^4 + b\lambda^2 + c$ then λ can be calculated exactly by the imposing following formula:

$$\lambda_{1;2} = \pm\sqrt{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \tag{2.17}$$

where

$$a = \beta \int_{-\infty}^{\infty} \mathscr{F}\{|w|^4 w\}\hat{w}^* dk \tag{2.18}$$

$$b = \alpha \int_{-\infty}^{\infty} \mathscr{F}\{|w|^2 w\}\hat{w}^* dk \tag{2.19}$$

$$c = -\int_{-\infty}^{\infty} [-\mathscr{F}\{(V+iW)w\}\hat{w}^* + (\mu + {k_x}^2 - \gamma {k_x}^4)|w|^2]dk. \tag{2.20}$$

The required soliton will be $f(x) = \lambda(wx) = \lambda\mathscr{F}^{-1}(\hat{w})$ when the iteration convergence.

In Fig. 2.1, the soliton obtained by the method described above is plotted on top of the real and the imaginary parts of the specific $\mathscr{P}\mathscr{T}$-symmetric potential which is derived in Chapter 4.

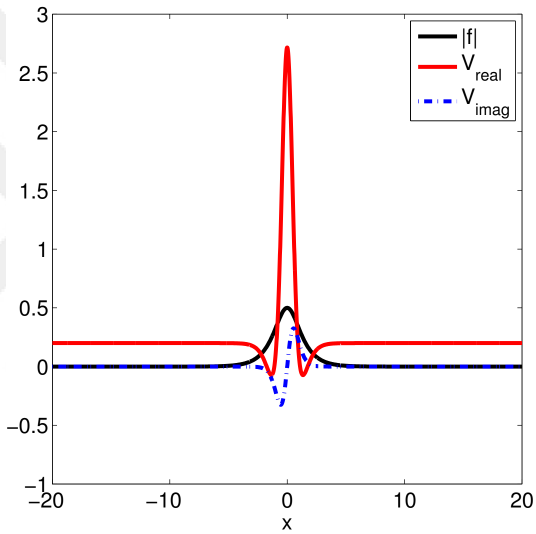

Figure 2.1 : Numerically obtained soliton on top of the real and imaginary parts of $\mathscr{P}\mathscr{T}$-symmetric potential.

2.2 Nonlinear Stability Analysis

If a soliton is considered as nonlinearity stable, then it should preserve its shape, location and maximum amplitude during direct simulations. To analyse the nonlinear stability of solitons, we directly compute Eq. (1.2) over a long distance. In order to do this, split-step Fourier method is employed to advance in z [37].

3. CQNLS 4OD EQUATION WITHOUT AN EXTERNAL POTENTIAL

3.1 Exact and Numerical Solutions

3.1.1 Exact solution

Current fiber manufacturing strategies provide experimentalists with fibers having an intensive range of dispersive behavior. We realize that not only the second order dispersion is important, but also its curvature (fourth order dispersion, 4OD) and slope (third order dispersion, 3OD) become significant when studying ultrashort pulses in optic fibers. In particular, at the frequency of ω_0 of minimum/maximum group velocity dispersion, the third order dispersion (3OD) vanishes and 4OD turns into the subsequent higher-order dispersion. This phenomenon is studied by Karlsson and Höök [5] for positive third order dispersion and it was discovered that pulses in such media will always loose power with the aid of radiation. But, the case of negative 4OD dispersion leads one to new solitary wave structures. The exact stationary solution of 4OD NLS with negative fourth order dispersion is given as well in above stated work.

In this chapter, we will use the spectral renormalization method to solve 4OD cubic/cubic-quintic NLS equation without an external potential and investigate the soliton properties in order to compare the analytical solution given in [5] with the numerical solution we obtain. We also inspect the effect of the fourth order dispersion term via the usage of the numerical method (SR).

The cubic-quintic nonlinear Schrödinger equation with a fourth order dispersion term (4OD) is given as follows:

$$iu_z + \alpha u_{xx} + |u|^2 u + \gamma u_{xxxx} + \beta |u|^4 u = 0 \tag{3.1}$$

where $u = u(x,z)$ is a complex-valued function and $x, z \in R$. Eq. (3.1) for only cubic nonlinearity case, (without the quintic term, corresponding to ($\beta = 0$) was introduced in [5] and exact stationary solution of this equation is given as follows

$$u(x,z) = \sqrt{\frac{3\alpha^2}{10\gamma}} \sec h^2 \left(\frac{x}{\sqrt{20\gamma/\alpha}} \right) \exp \left(i \frac{4\alpha^2}{25\gamma} z \right) \tag{3.2}$$

When this solution is compared to the NLS-soliton, it is seen that this solution has no free parameter, and it cannot be given a relative velocity (frequency offset). It should be considered that, this type of fixed parameter solutions have been discovered earlier [38, 39]. Moreover, the particular solution Eq. (3.2) could thoroughly belong to a class of solutions with an amplitude-width relation similar to that of the NLS soliton [5]. These solutions should have the equal $\sec h^2$-shape as the pulse in Eq. (3.2).

Eq. (3.1) is also investigated in [40] and [41] in connection with the theory of optical solitons in gyro-tropic media and the nonlinear fiber optics.

3.2 Numerical Illustrations

We will numerically investigate the soliton solution of Eq. (3.1) for various values of γ (40D term's coefficient) and the propagation constant μ in this subsection.

In Fig. 3.1, we show the exact soliton solution and the numerically obtained soliton solution of Eq. (3.1) and it can clearly be seen from this figure that our numerical algorithm converges to the exact solution for the parameters $\gamma = -1$, $\mu = 1$.

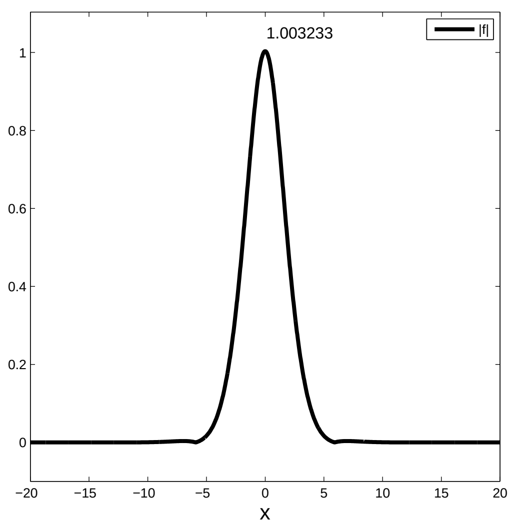

Figure 3.1 : Exact and numerical solutions of 4OD NLS equation on top of each other for $\gamma = -1, \mu = 1$

So as to investigate the effect of the eigenvalue and the fourth order dispersion on the soliton properties, we plotted solitons for diverse values of γ and μ.

To observe the effect of the eigenvalue μ, we fixed the coefficient of the 4OD dispersion as $\gamma = -1$ and then increased the eigenvalue to $\mu = 4$ in Fig. 3.2. From this figure, we can clearly see that the maximum amplitude of the soliton comparing to $\gamma = -1, \mu = 1$ case increases, namely to $max|f| = 1.003$ and the soliton becomes more vertical. Another interesting result is revealed by this figure: the tails of the soliton differs from the previous case. This phenomenon was also observed by Karpman in [40]. They called these solitons as "solitons with oscillating tails".

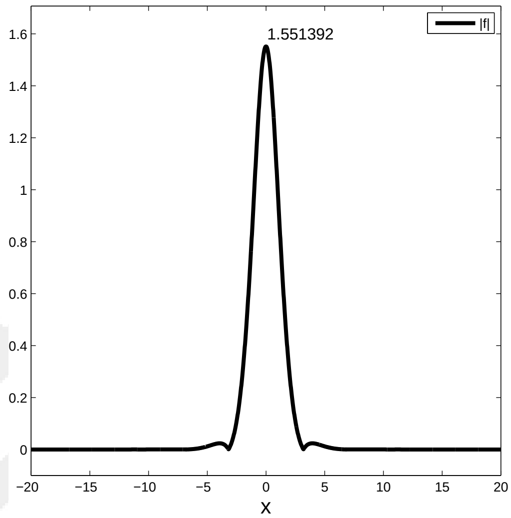

Figure 3.2 : Numerically obtained higher order mode for $\gamma = -1$, $\mu = 4$.

In Fig. 3.3, we still increase the eigenvalue to μ and this time we take $\mu = 7$. This figure shows that, the tails become more pronounced and the maximum amplitude increases to $max|f| = 1.82$.

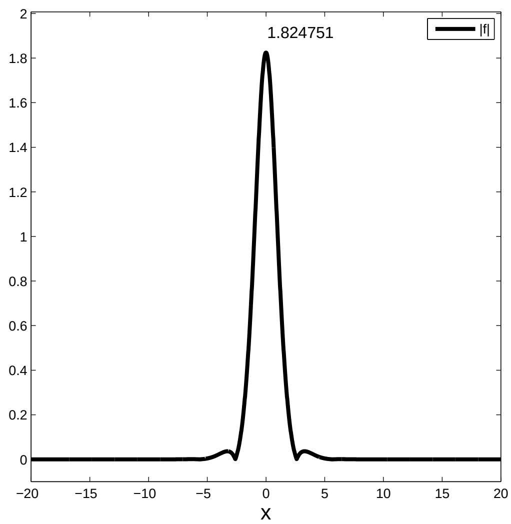

Figure 3.3 : Numerically obtained higher order mode for $\gamma = -1$, $\mu = 7$.

In the next figures, to observe the effect of the increasing eigenvalue γ, we decreased the effect of the 4OD dispersion by setting $\mu = 1$ and slowly increased the coefficient γ to $\gamma = -0.9$, $\gamma = -0.2$ and $\gamma = 0$ respectively.

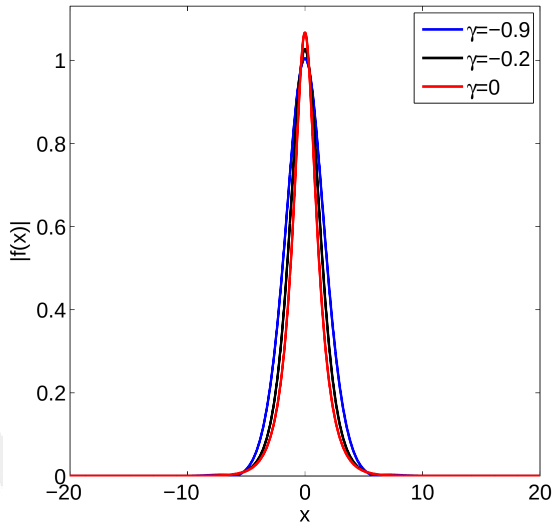

Figure 3.4 : Numerically obtained higher order mode for $\gamma = -0.9$, $\gamma = -0.2$ and $\gamma = 0$.

In Fig. 3.4, we observe a small increment in the maximum amplitude of the soliton comparing to larger 4OD effect case shown in Fig. 3.1 but the shape of the soliton is nearly the same with that figure and we observe the effect of β to the maximum amplitude of the 4OD equation. For this we choose gradually increased the coefficient γ to $\gamma = -0.9$, $\gamma = -0.2$ and $\gamma = 0$ respectively.

3.3 Nonlinear Stability

In this section, we will numerically see how the shape and the maximum amplitude of a higher order soliton effect its nonlinear stability properties. In order to investigate this, obtained solitons are computed over a long distance and the shapes, maximum amplitudes and locations during the evolution is monitored. We investigated the previously obtained solitons in the same order.

First we took the soliton solution that is shown in Fig. 3.1 and evolved it for $z = 1$. The results are demonstrated in Fig. 3.5. From this figure, one can clearly conclude that this soliton is nonlinearly unstable as it does not preserve its shape and maximum amplitude during the evolution.

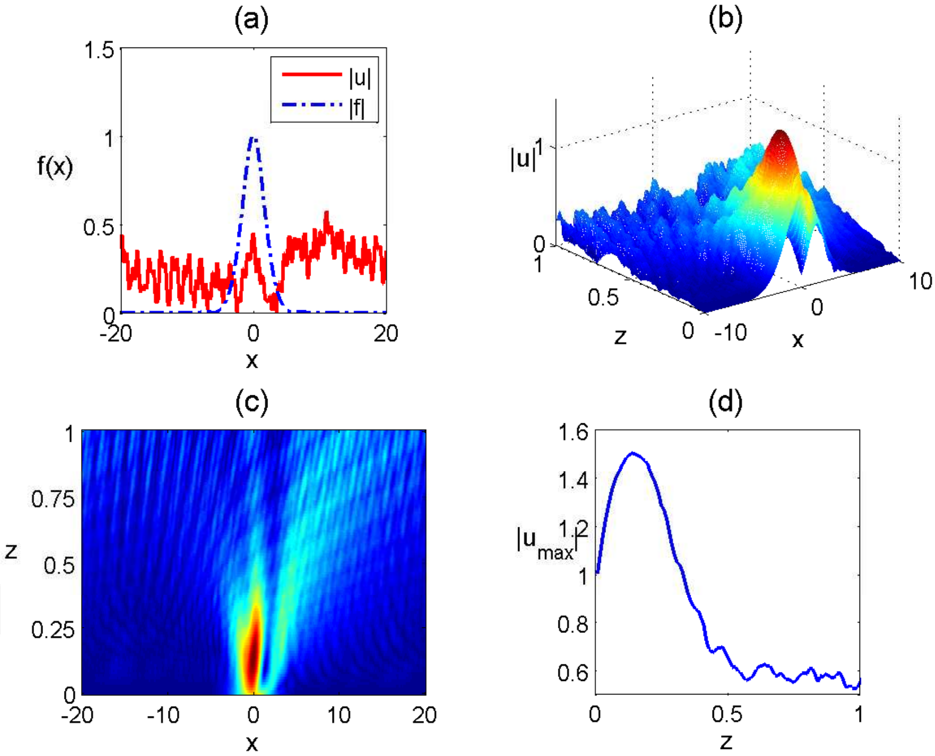

Figure 3.5 : Nonlinear instability of a higher order soliton for $\gamma = -1, \mu = 1$; (a) Numerically produced higher order soliton (blue dashes) on top of the solution after the evolution (red solid), (b) Nonlinear evolution of the soliton, (c) The view from top and (d) Maximum amplitude as a function of the propagation distance z.

In Figure 3.6, we show a nonlinearly unstable higher order soliton obtained for $\gamma = -1$, $\mu = 4$. It is seen that, around $z = 0.2$, the maximum amplitude starts to decrease swiftly and then starts to increase slowly around $z = 0.6$ as a result of the deterioration in the shape of the soliton.

In Figure 3.7, we show a nonlinearly unstable higher order soliton obtained for $\gamma = -1, \mu = 9$. It is seen from the figure that, around $z = 0.1$, a sharp decrease in the maxiumum amplitude is observed and it is revelaed from the figure that the soliton couldn't preserve its shape and is subjected to deterioration immediately after the evolution starts.

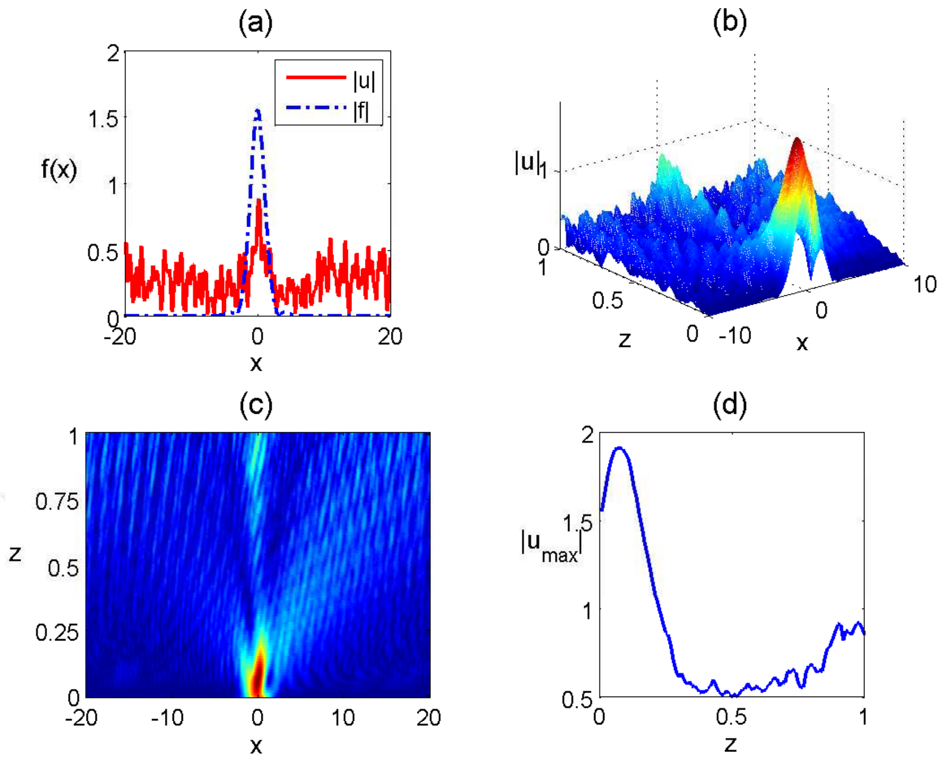

Figure 3.6 : Nonlinear instability of a higher order soliton for $\gamma = -1, \mu = 4$;(a) Numerically produced higher order soliton (blue dashes) on top of the solution after the evolution (red solid), (b) Nonlinear evolution of the soliton, (c) The view from top and (d) Maximum amplitude as a function of the propagation distance z.

In this section, we also investigate the effect of γ (the coefficient of the higher order dispersion term) to the higher order soliton. To obtain that, we will fix the eigenvalue to $\mu = 1$ and change the constant γ to see how it effects the maximum amplitude of the higher order soliton.

In Figure 3.8, we investigated the case given in Figure 3.5 for the constant $\gamma = -0.9$. When we change the constant γ form -1 to -0.9, the maximum amplitude increases slightly during the evolution and it can clearly seen from the graphic that this soliton is still nonlinearly unstable.

In Figure 3.9, we take the constant $\gamma = -0.5$ and continue to investigate how the soliton changes during the evolution. We can observe from the graphic that the maximum amplitude of higher order soliton increase swiftly.

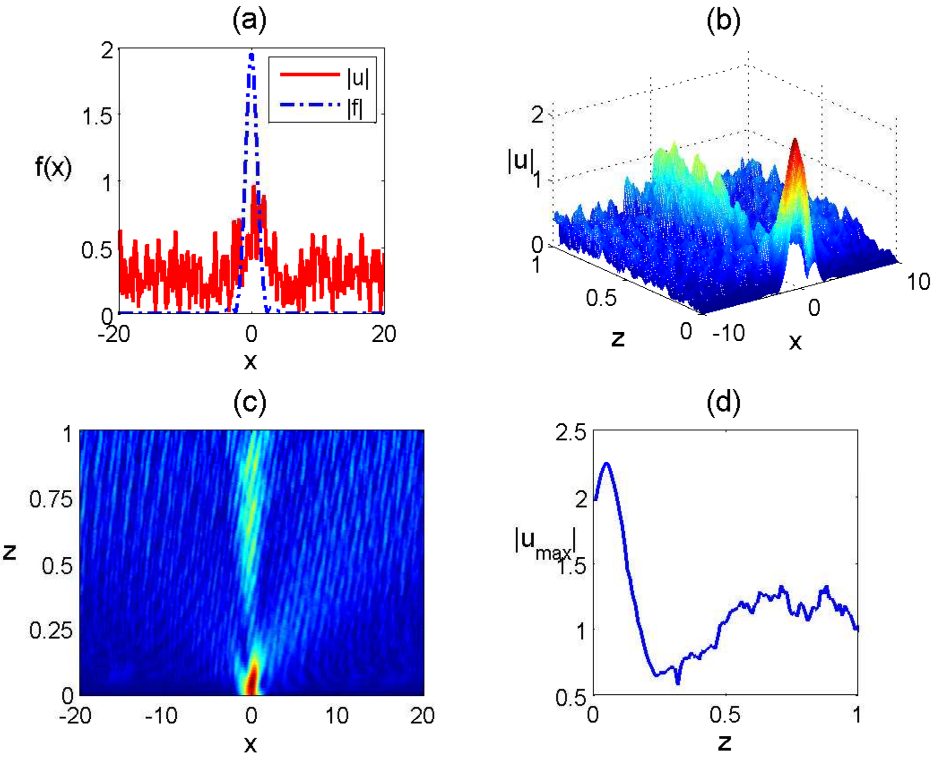

Figure 3.7 : Nonlinear instability of a higher order soliton $\gamma = -1, \mu = 9$; (a) Numerically produced higher order soliton (blue dashes) on top of the solution after the evolution (red solid), (b) Nonlinear evolution of the soliton, (c) The view from top and (d) Maximum amplitude as a function of the propagation distance z.

In Figure 3.10, when the constant γ approach to zero, the maximum amplitude of the soliton continues to increase and it starts to change its location during the evolution.

When we come to the Figure 3.11 for final, we take the constant $\gamma = 0$ to observe soliton's shape and the maximum amplitude. We have seen from the previous figures that the value of constant γ increases the maximum amplitude of the soliton when it approaches to zero. If we take $\gamma = 0$, the maximum amplitude keeps increasing during the evolution and this time, this indicates blow up which we can be clearly seen in Figure 3.11.

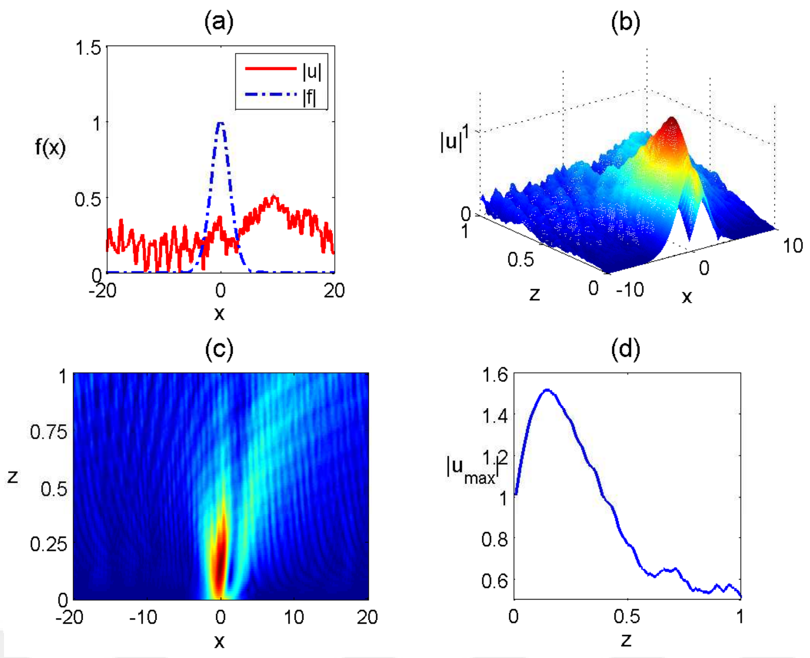

Figure 3.8 : Nonlinear stability of a higher order soliton for $\gamma = -0.9, \mu = 1$; (a) Numerically produced higher order soliton (blue dashes) on top of the solution after the evolution (red solid), (b) Nonlinear evolution of the soliton, (c) The view from top and (d) Maximum amplitude as a function of the propagation distance z.

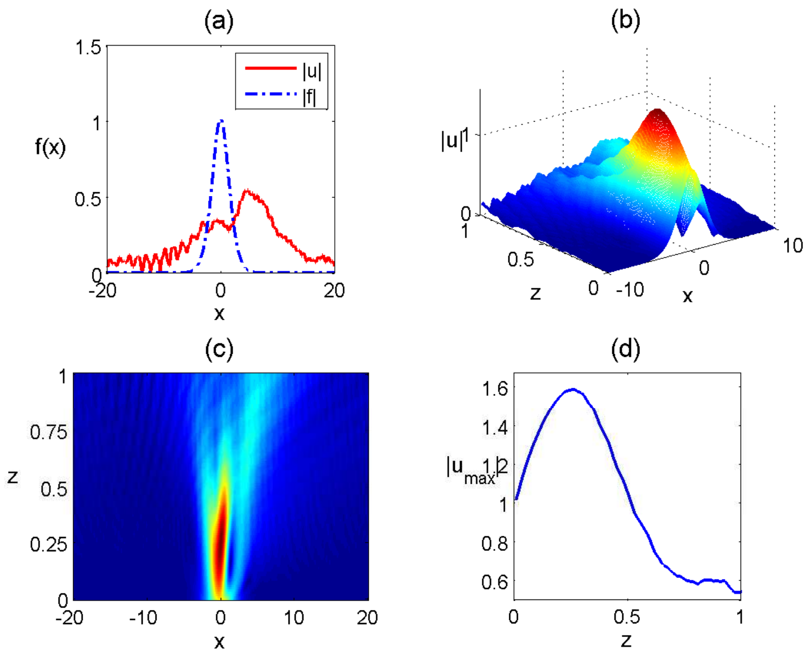

Figure 3.9 : Nonlinear instability of a higher order soliton for $\gamma = -0.5, \mu = 1$; (a) Numerically produced higher order soliton (blue dashes) on top of the solution after the evolution (red solid), (b) Nonlinear evolution of the soliton, (c) The view from top and (d) Maximum amplitude as a function of the propagation distance z.

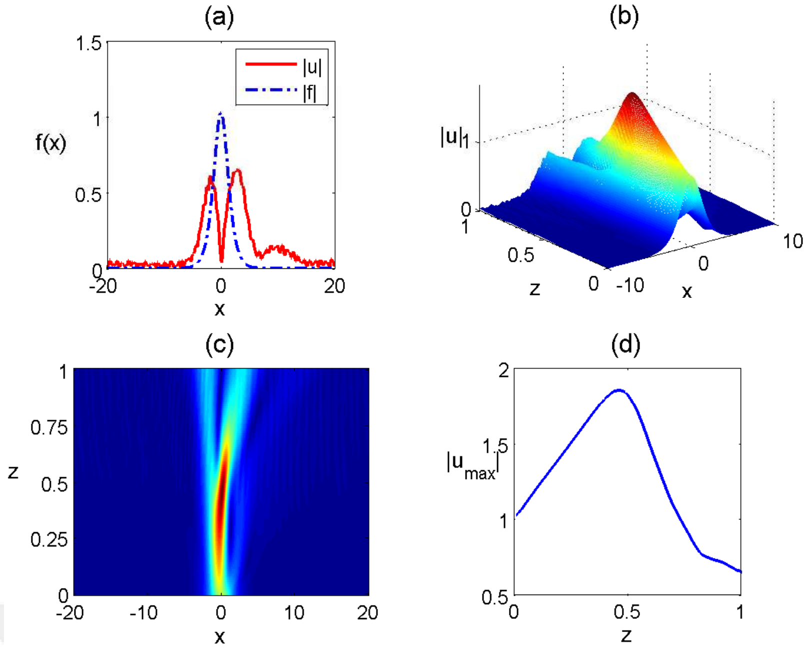

Figure 3.10 : Nonlinear instability of a higher order soliton for $\gamma = -0.2, \mu = 1$; (a) Numerically produced higher order soliton (blue dashes) on top of the solution after the evolution (red solid), (b) Nonlinear evolution of the soliton, (c) The view from top and (d) Maximum amplitude as a function of the propagation distance z.

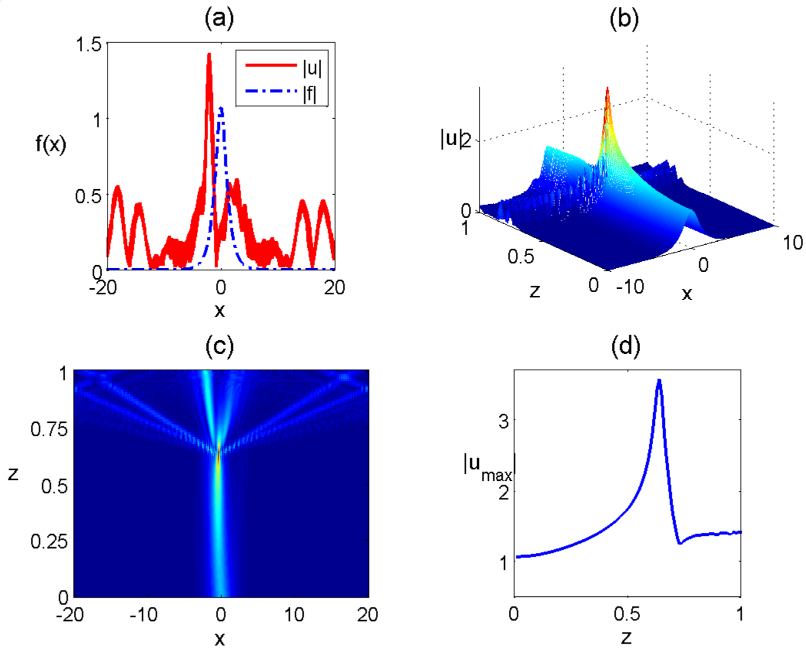

Figure 3.11 : Nonlinear instability of a higher order soliton for $\gamma = 0, \mu = 1$; (a) Numerically produced higher order soliton (blue dashes) on top of the solution after the evolution (red solid), (b) Nonlinear evolution of the soliton, (c) The view from top and (d) Maximum amplitude as a function of the propagation distance z.

In this chapter, as a result of the numerical observations, we conclude that:

(i) The higher order solitons obtained for 4OD NLS equation may have oscillating tails and these solitons are found to be nonlinearly *unstable*;

(ii) The higher order solitons obtained for 4OD cubic-quintic NLS equation are found to be nonlinearly *unstable* for the cases when the effect of the 4OD effect is either small or the eigenvalue is μ is small (even if the 4OD effect is large).

4. CQNLS 4OD EQUATION WITH AN EXTERNAL POTENTIAL

4.1 Exact and Numerical Solutions

4.1.1 Exact solution

Exact solutions provide to understand the structure of the complex nonlinear physical phenomena which are related to wave propagation in a higher-order CQNLS equation with $\mathscr{PT}$-symmetric potential.

Consider the following (1+1)D 4OD cubic-quintic NLS equation with a $\mathscr{PT}$-symmetric potential:

$$iu_z + u_{xx} + \alpha|u|^2 u + \gamma u_{xxxx} + \beta|u|^4 u + V_{PT} u = 0. \tag{4.1}$$

If we take $u = 0$ then we find a trivial solution of Eq. (4.1). For finding non-zero solutions, set $u \neq 0$. Dividing Eq. (4.1) by u and by use of Eq. (1.3) leads to

$$i\frac{u_z}{u} + \frac{u_{xx}}{u} + \alpha|u|^2 + \gamma\frac{u_{xxxx}}{u} + \beta|u|^4 + V + iW = 0. \tag{4.2}$$

The following ansatz is used to get non-zero stationary solitons:

$$u(x,z) = f(x)e^{i(\mu z + g(x))} \tag{4.3}$$

where $f(x)$ and $g(x)$ are real-valued functions different than zero, u is a function of x and z to be determined and μ is the propagation constant. Taking derivatives of Eq. (4.3) with respect to z and x, results in following equations,

$$u_z = f(x)i\mu e^{i(\mu z + g(x))} \tag{4.4}$$

$$u_{xx} = [f''(x) + 2if'(x)g'(x) + if(x)g''(x) - f(x)(g'(x))^2]e^{i(\mu z + g(x))} \tag{4.5}$$

$$|u|^2 = f(x)e^{i(\mu z + g(x))}f(x)e^{-i(\mu z + g(x))} = (f(x))^2 \tag{4.6}$$

$$\begin{aligned} u_{xxxx} = [&f''''(x) + 4if'''(x)g'(x) + 6if''(x)g''(x) + 4if'(x)g'''(x) + if(x)g''''(x) \\ &-6f''(x)(g'(x))^2 - 12f'(x)g'(x)g''(x) - 3f(x)(g''(x))^2 - 4f(x)g'(x)g'''(x) \\ &-4if'(x)(g'(x))^3 - 6if(x)(g'(x))^2g''(x) + f(x)(g'(x))^4]. \end{aligned} \tag{4.7}$$

Substituting Eq. (4.4)-Eq. (4.7) into Eq. (4.2) yields

$$\begin{aligned}&[-\mu+\alpha f^2(x)+\frac{f''(x)}{f(x)}+\gamma\frac{f''''(x)}{f}-(g'(x))^2-6\gamma\frac{f''(x)}{f(x)}(g'(x))^2+\gamma(g'(x))^4\\&-3\gamma(g''(x))^2-12\gamma\frac{f'(x)}{f(x)}g'(x)g''(x)-4\gamma g'(x)g'''(x)+\beta(f(x))^4+V(x)]\\&+i[-4\gamma\frac{f'(x)}{f(x)}(g'(x))^3+4\gamma\frac{f'(x)}{f(x)}g'''(x)+2\frac{f'(x)}{f(x)}g'(x)+6\gamma\frac{f''(x)}{f(x)}g''(x)\\&+4\gamma\frac{f'''(x)}{f(x)}g'(x)+g''(x)-6\gamma(g'(x))^2g''(x)+\gamma g''''(x)+W(x)]=0.\end{aligned} \tag{4.8}$$

To obtain soliton solutions, we used the following ansatz

$$f(x)=f_0\sec h^p(x),\quad g'(x)=g_0\sec h^q(x) \tag{4.9}$$

where f_0 and g_0 are non-zero real constants and $p\in N$. We need to calculate the derivatives of the functions f and g to equate simple form of Eq. (4.8). By using Eq. (4.9) we obtain

$$f'(x)=-f_0p\tanh(x)\text{sech}^p(x) \tag{4.10}$$

$$f''(x)=-f_0p(1+p)\sec h^{p+2}(x)+fop^2\sec h^p(x) \tag{4.11}$$

$$f'''(x)=f_0[-p^3+(p^3+3p^2+2p)\sec h^2(x)]\sec h^p(x)\tanh(x) \tag{4.12}$$

$$\begin{aligned}f''''(x)=&f_0[p^4-(2p^4+6p^3+8p^2+4p)\sec h^2(x)\\&+(p^4+6p^3+11p^2+6p)\sec h^4(x)]\sec h^p(x)\end{aligned} \tag{4.13}$$

$$g'(x)=g_0\text{sech}^q(x) \tag{4.14}$$

$$g''(x)=g_0q\sec h^q(x)\tanh(x) \tag{4.15}$$

$$g'''(x)=g_0q^2\sec h^q(x)-g_0(q^2+q)\sec h^{q+2}(x) \tag{4.16}$$

$$g''''(x)=-g_0q^3\sec h^q(x)\tanh(x)+g_0(q^3+3q^2+2p)\sec h^{q+2}(x)\tanh(x). \tag{4.17}$$

Substituting Eq. (4.10)-Eq. (4.17) into Eq. (4.8) we obtain

$$\begin{aligned}&-\mu+p^2+\gamma p^4+\sec h^2(x)[-p^2-p-2\gamma p^4-6p^3-8p^2-4p]\\&+\sec h^4(x)[\gamma(p^4+6p^3+11p^2+6p)]+\sec h^{2p}(x)(\alpha f_0{}^2)+\sec h^{4p}(x)(\beta f_0{}^4)\\&+\sec h^{2q}(x)[-g_0{}^2-6\gamma g_0{}^2p^2-7\gamma g_0{}^2q^2-12\gamma g_0{}^2pq]+\sec h^{4q}(x)[\gamma g_0{}^4]\\&+\sec h^{2q+2}(x)[6\gamma g_0{}^2p^2+6\gamma g_0{}^2p+7\gamma g_0{}^2q^2+4\gamma g_0{}^2q+12\gamma g_0{}^2pq]+V\\&+i[\sec h^q(x)\tanh(x)[-2pg_0-qg_0-4\gamma p^3g_0-6\gamma p^2qg_0-\gamma p^3g_0]\\&+\sec h^{q+2}(x)\tanh(x)[4\gamma(p^3+3p^2+2p)g_0+6\gamma(p^2+p)qg_0+4\gamma p(q^2+q)g_0\\&+\gamma(p^3+3p^2+2p)g_0]+\sec h^{3q}(x)\tanh(x)[4\gamma pg_0{}^3+6\gamma qg_0{}^3]+W]=0.\end{aligned} \tag{4.18}$$

When we split Eq. (4.18) into real and imaginary parts, we get the expressions for the real and imaginary parts of the $\mathscr{PT}$-symmetric potential as we can see below:

Real Part

The real part of the Eq. (4.18) can be written as,

$$\begin{aligned}
&(-\mu+p^2+\gamma p^4)+\text{sech}^2(x)[-p^2-p-\gamma(2p^4+6p^3+8p^2+4p)] \\
&+\text{sech}^4(x)[\gamma(p^4+6p^3+11p^2+6p)]+\text{sech}^{2p}(x)(\alpha {f_0}^2)+\text{sec}\,h^{4p}(x)(\beta {f_0}^4) \\
&+\text{sech}^{2q}(x)[-{g_0}^2-6\gamma {g_0}^2p^2-7\gamma {g_0}^2q^2-12\gamma {g_0}^2pq] \\
&+\text{sech}^{4q}(x)[\gamma {g_0}^4]+\text{sech}^{2q+2}(x)[6\gamma {g_0}^2p^2+6\gamma {g_0}^2p \\
&+7\gamma {g_0}^2q^2+4\gamma {g_0}^2q+12\gamma {g_0}^2pq]+V=0.
\end{aligned} \tag{4.19}$$

The real part of the $\mathscr{PT}$-symmetric potential is found as

$$\begin{aligned}
V(x)=&V_0+V_1\text{sech}^2(x)+V_2\text{sech}^4(x)+V_3\text{sech}^{2p}(x)+V_4\text{sech}^{4p}(x) \\
&+V_5\text{sech}^{2q}(x)+V_6\text{sech}^{4q}(x)+V_7\text{sech}^{2q+2}(x)
\end{aligned} \tag{4.20}$$

where

$$V_0=-\mu+p^2+\gamma p^4 \tag{4.21}$$

$$V_1=-p^2-p-\gamma(2p^4+6p^3+8p^2+4p) \tag{4.22}$$

$$V_2=\gamma(p^4+6p^3+11p^2+6p) \tag{4.23}$$

$$V_3=\alpha {f_0}^2 \tag{4.24}$$

$$V_4=\beta {f_0}^4 \tag{4.25}$$

$$V_5=-{g_0}^2-6\gamma {g_0}^2p^2-7\gamma {g_0}^2q^2-12\gamma {g_0}^2pq \tag{4.26}$$

$$V_6=\gamma {g_0}^4 \tag{4.27}$$

$$V_7=6\gamma {g_0}^2p^2+6\gamma {g_0}^2p+7\gamma {g_0}^2q^2+4\gamma {g_0}^2q+12\gamma {g_0}^2pq \tag{4.28}$$

We set $\mu=p^2+\gamma p^4$ to get rid of coefficient V_0 for the sake of simplicity and we can see in the following form that $V(x)$ is indeed an even function

$$\begin{aligned}
V(-x)=&V_1\text{sech}^2(-x)+V_2\text{sech}^4(-x)+V_3\text{sech}^{2p}(-x)+V_4\text{sech}^{4p}(-x) \\
&+V_5\text{sech}^{2q}(-x)+V_6\text{sech}^{4q}(-x)+V_7\text{sech}^{2q+2}(-x) \\
=&V_1\text{sech}^2(x)+V_2\text{sech}^4(x)+V_3\text{sech}^{2p}(x)+V_4\text{sech}^{4p}(x) \\
&+V_5\text{sech}^{2q}(x)+V_6\text{sech}^{4q}(x)+V_7\text{sech}^{2q+2}(x) \\
=&V(x).
\end{aligned} \tag{4.29}$$

Now, $V(x)$ can be simplified by equating the powers of $sech(x)$. Considering the case of $p = q = 1$, then Eq. (4.20) can be rewritten as following form,

$$\begin{aligned} V(x) = &[2+20\gamma-\alpha f_0{}^2+25\gamma g_0{}^2+g_0{}^2]\text{sech}^2(x) \\ &-(24\gamma+\beta f_0{}^4+35\gamma g_0{}^2+\gamma g_0{}^4)\text{sech}^4(x) \end{aligned} \tag{4.30}$$

Then we take $\gamma = -0.2$ to find out even function $V(x)$, we get

$$V(x) = -(2+\alpha f_0{}^2+4g_0{}^2)\text{sech}^2(x)+(0.2g_0{}^4-\beta f_0{}^4+7g_0{}^2+4.8)\text{sech}^4(x). \tag{4.31}$$

where

$$V_1 = -2-\alpha f_0{}^2-4g_0{}^2 \tag{4.32}$$

$$V_2 = 0.2g_0{}^4-\beta f_0{}^4+7g_0{}^2+4.8. \tag{4.33}$$

Considering the case of $f_0 = 1$, $g_0 = 1$ we finally obtain the real part of the $\mathscr{PT}$-symmetric potential as,

$$V(x) = -(\alpha+6)\text{sech}^2(x)+(-\beta+12)\text{sech}^4(x). \tag{4.34}$$

Imaginary Part

The complex part of the Eq. (4.18) can be written as

$$\begin{aligned} &\text{sech}^q(x)\tanh(x)[-2pg_0-qg_0-4\gamma p^3g_0-6\gamma p^2qg_0-4\gamma pq^2g_0-\gamma q^3g_0] \\ &+\text{sech}^{q+2}(x)\tanh(x)[4\gamma(p^3+3p^2+2p)g_0+6\gamma(p^2+p)qg_0+4\gamma p(q^2+q)g_0 \\ &+\gamma(p^3+3q^2+2q)g_0]+\text{sech}^{3q}(x)\tanh(x)[4\gamma pg_0{}^3+6\gamma qg_0{}^3]+W(x)=0 \end{aligned} \tag{4.35}$$

Then the imaginary part of the $\mathscr{PT}$-symmetric potential is obtained as

$$W(x) = W_0\text{sech}^q(x)\tanh(x)+W_1\text{sech}^{q+2}(x)\tanh(x)+W_2\sec h^{3q}(x)\tanh(x) \tag{4.36}$$

where

$$W_0 = g_0[2p+q+\gamma(4p^3+6p^2q+4pq^2+q^3)] \tag{4.37}$$

$$W_1 = -\gamma g_0[4p^3+12p^2+8p+6p^2q+10pq+4pq^2+q^3+3q^2+2q] \tag{4.38}$$

$$W_2 = -\gamma g_0{}^3[4p+6q]. \tag{4.39}$$

We can see in the following form that W(x) is indeed an odd function.

$$\begin{aligned} W(-x) = &W_0\text{sech}^q(-x)\tanh(-x)+W_1\text{sech}^{q+2}(-x)\tanh(-x) \\ &+W_2\text{sech}^{3q}(-x)\tanh(-x) \\ = &W_0\text{sech}^q(x)(-\tanh(x))+W_1\text{sech}^{q+2}(x)(-\tanh(x)) \\ &+W_2\text{sech}^{3q}(x)(-\tanh(x)) \\ = &-W(x). \end{aligned} \tag{4.40}$$

Considering the case of $p=q=1$, then we can rewritten Eq. (4.36) as following form,

$$W(x) = 3g_0(5\gamma+1)\sec h(x)\tanh(x) - \gamma g_0(10{g_0}^2+50)\sec h^3(x)\tanh(x) \qquad (4.41)$$

by taking $\gamma = -0.2$ we get

$$W(x) = 2g_0({g_0}^2+5)\text{sech}^3(x)\tanh(x). \qquad (4.42)$$

where

$$W_3 = 2g_0({g_0}^2+5). \qquad (4.43)$$

If we choose note that $g_0 = 1$ into Eq. (4.42) leads one to

$$W(x) = 12\text{sech}^3(x)\tanh(x). \qquad (4.44)$$

Attention should be paid in case of $p=q=1$, by considering Eq. (4.30) and Eq. (4.41) the analytical solution of the problem can begin with

$$u(x,z) = f_0 \sec h(x) e^{i[\mu z + g_0 \arctan h(x)\sinh(x)]}. \qquad (4.45)$$

Consequently, Eq. (1.2), with the real and the imaginary parts in Eq. (4.31) and Eq. (4.42) can be given as

$$V_{PT} = [V_1\text{sech}^2(x) + V_2\text{sech}^4(x)] + i[W_3\text{sech}^3(x)\tanh(x)]. \qquad (4.46)$$

Eq. (4.46) can be seen as an extension of the so-called complexified Scarf II potential $[V_0\text{sech}^2(x) + iW_0\text{sech}(x)\tanh(x)]$ for Kerr media with cubic nonlinearity.

4.1.2 Numerical illustrations

In this section, we will show obtained results with graphics for various values of μ, γ, f_0 and g_0 respectively. As we see in Fig. 4.1 and Fig. 4.2, one of the most important things that affects the maximum amplitude of the soliton is the selection of f_0. In Fig. 4.1, we plot the real and imaginary parts of the functions f and V under the parameters $f_0 = 1.5$, $g_0 = 1$, $\gamma = -0.2$ and $\mu = 1$. In order to show the effect of the f_0 on the maximum amplitude of the soliton, in Fig. 4.2 and Fig. 4.3, we use the same parameters as in the Fig. 4.1 for $0 \leq f_0 \leq 1.55$. The relation between the considered quantities is almost linear as it is seen in Fig. 4.2.

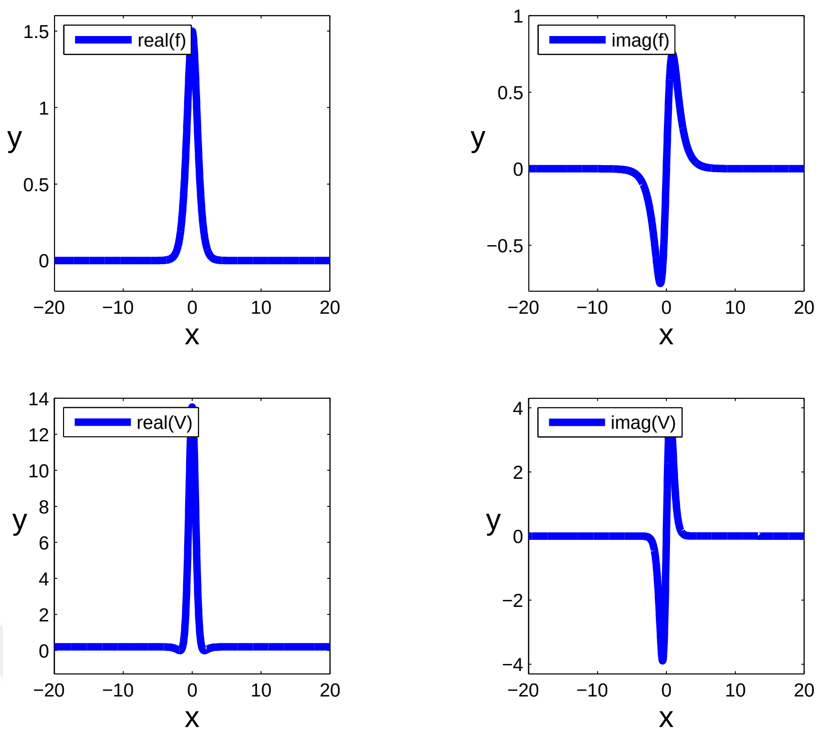

Figure 4.1 : Real and imaginary part of the soliton and potential for $f_0 = 1.5$, $g_0 = 1$, $\mu = 1$ and $\gamma = -0.2$.

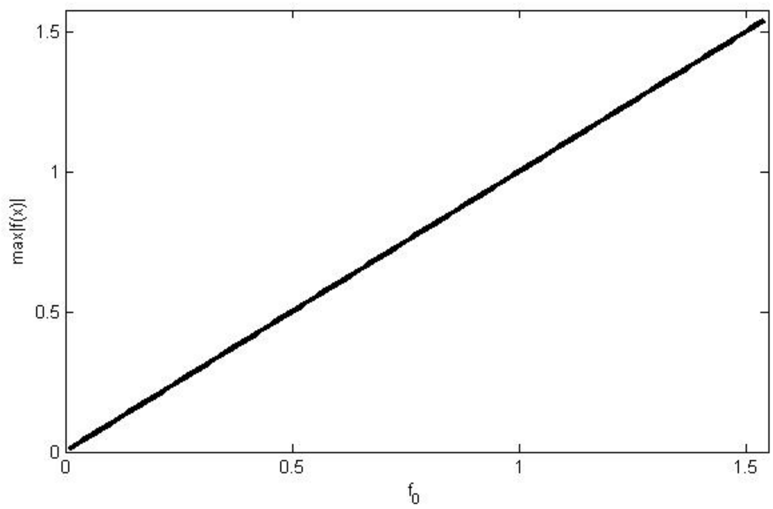

Figure 4.2 : Numerically obtained solitons for various values of f_0 for $\mu = 1$, $\gamma = -0.2$ and $g_0 = 1$.

In Fig. 4.4 , the effect of the 4OD constant γ to the maximum amplitude and to the shape of the $\mathscr{PT}$-symmetric potential are revealed.

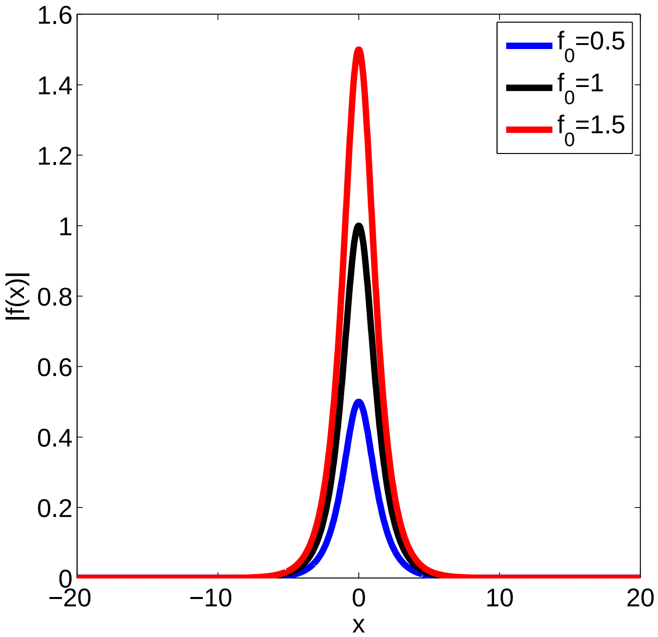

Figure 4.3 : Numerically obtained solitons for various values of f_0 for $\mu = 1$, $\gamma = -0.2$ and $g_0 = 1$.

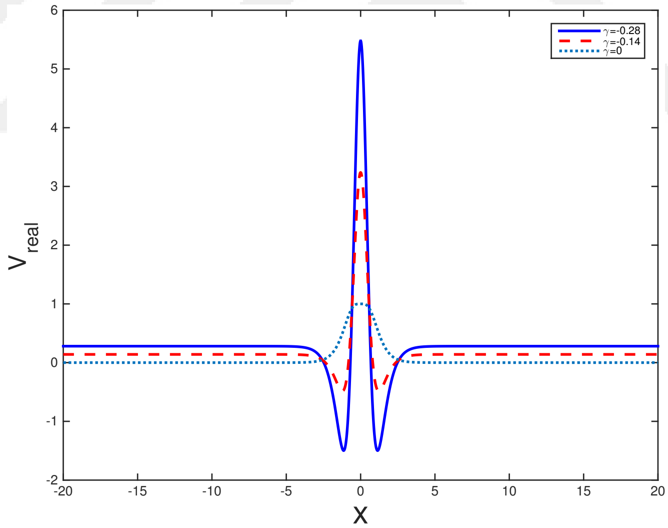

Figure 4.4 : Numerically obtained solitons for various values of γ for $\mu = 1$, $f_0 = 1$ and $g_0 = 1$.

In Fig. 4.5, we demonstrate the effect of the eigenvalue μ on the numerical solution f and $\mathscr{PT}$-symmetric potential.

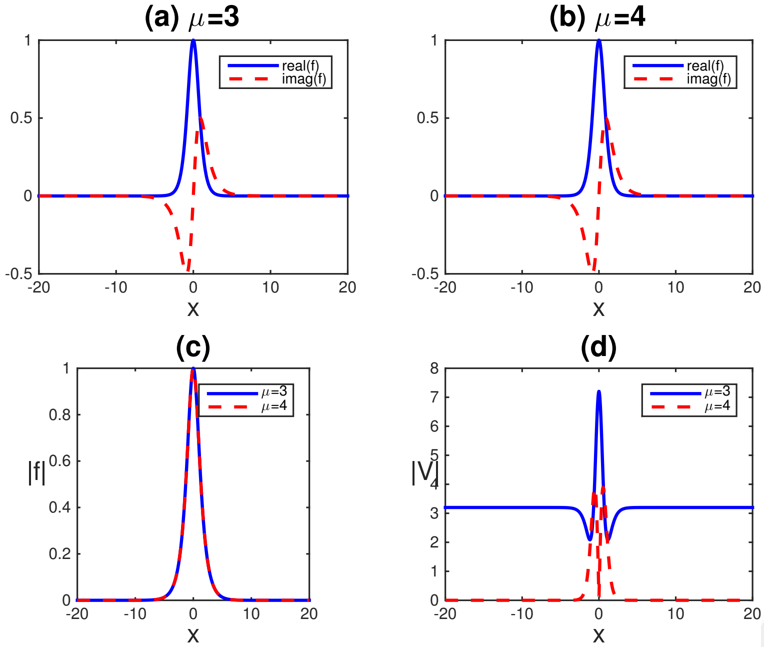

Figure 4.5 : (a) Real and imaginary parts of numerical solution f with $\mu = 3$, (b) real and imaginary parts of numerical solution f with $\mu = 4$, (c) $|f|$ for $\mu = 3$ and $\mu = 4$, (d) absolute values of $\mathscr{PT}$-symmetric potential for $\mu = 3$ and $\mu = 4$, by considering $\gamma = -0.2$, $f_0 = 1$ and $g_0 = 1$.

4.2 Nonlinear Stability

If solitons preserve their shape, position and maximum amplitude during direct simulations then they are called nonlinearly stable in the field of optics. We evolved solitons over a long distance to search out their nonlinear stability. To do this, we advanced in z with split-step Fourier method.

Numerically generated soliton of the (1+1)D 4OD NLS equation with a $\mathscr{PT}$-symmetric potential, nonlinear evolution of the soliton, the view from top to $|u(x,z)|$ and maximum values of $|u|$ along with the z is plotted in this figures, respectively. It can be easily seen that the solitons shape and the maximum amplitude decay with variable z. To plot Fig. 4.6 we used the parameters $\mu = 1$, $\gamma = -0.9$, $f_0 = 1$ and $g_0 = 1$. In this figure, $|u_{max}| = 0.99$ is the maximum amplitude of soliton and its getting decays up.

For the stability of the acquired solutions, the dispersion coefficient γ is an another important parameter. Dispersion coefficient γ is extremely important for decaying constant and amplitude of the $|u_{max}|$ and they depends exceedingly on γ. In Fig. 4.7 we demonstrate the case of $\gamma = -0.5$ and we fixed the other parameters are as in the Fig. 4.6. We also deal with the effect of potential depth on the maximum amplitude. We can clearly see in Fig. 4.8 the maximum amplitude of the soliton more than the case of Fig. 4.6. When plotting Fig. 4.8, we use the parameter $\mu = 1$, $\gamma = -0.2$, $f_0 = 1$ and $g_0 = 1$. In Fig. 4.9, with taking $\gamma = 0$, we demonstrate the effects of dispersion coefficient on stability of the soliton solution. As it is shown in Fig. 4.9, if the change in the maximum amplitude is less than or equal to 10^{-2}, then we can say that the soliton is more conservative than the case of Fig.4.7 in maximum amplitude sense.

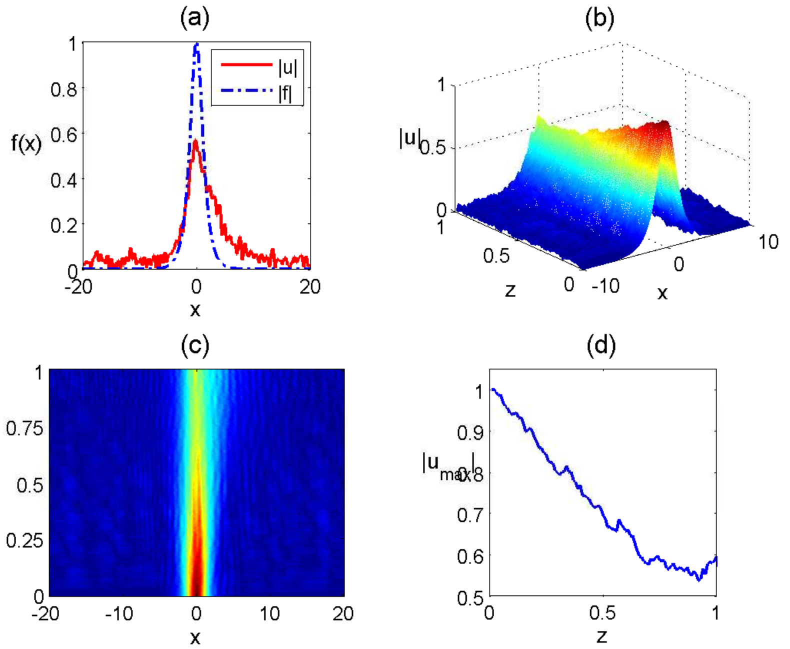

Figure 4.6 : Nonlinear instability of a higher order soliton for $\gamma = -0.9, \mu = 1$ $f_0 = 1$ and $g_0 = 1$ with a $\mathcal{PT}$-symmetric potential; (a) Numerically produced higher order nonlinear soliton (blue dashes) on top of the solution after the evolution (green solid), (b) Nonlinear evolution of the soliton, (c)The view from top and (d) Maximum amplitude as a function of the propagation distance z.

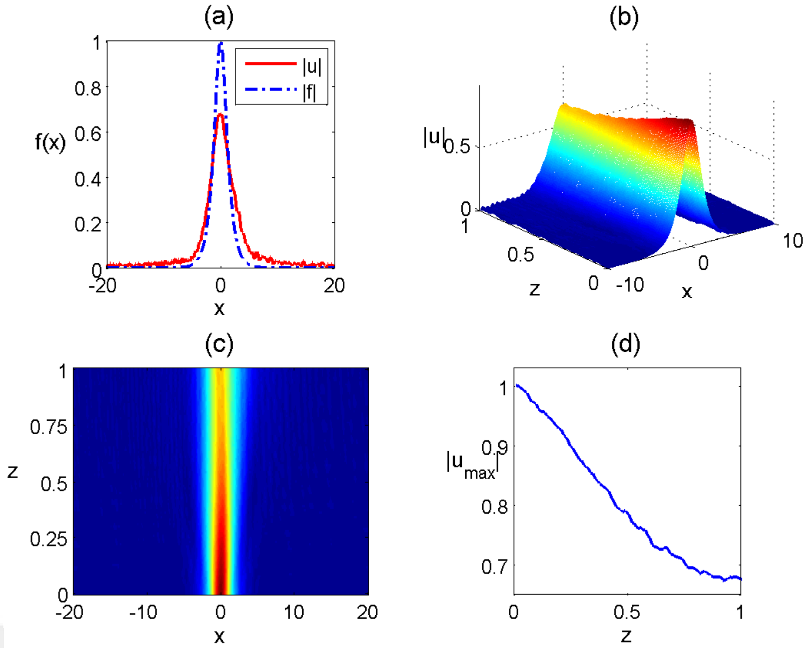

Figure 4.7 : Nonlinear instability of a higher order soliton for $\gamma = -0.5, \mu = 1$, $f_0 = 1$ and $g_0 = 1$ with a $\mathscr{P}\mathscr{T}$-symmetric potential; (a) Numerically produced higher order nonlinear soliton (blue dashes) on top of the solution after the evolution (green solid), (b) Nonlinear evolution of the soliton, (c)The view from top and (d) Maximum amplitude as a function of the propagation distance z.

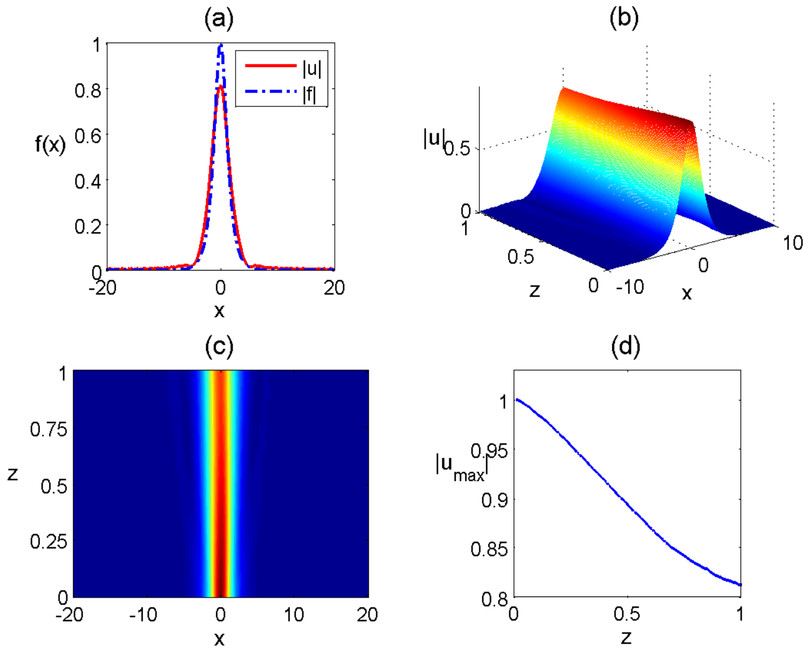

Figure 4.8 : Nonlinear instability of a higher order soliton for $\gamma = -0.2, \mu = 1$ $f_0 = 1$ and $g_0 = 1$ with a $\mathscr{P}\mathscr{T}$-symmetric potential; (a) Numerically produced higher order nonlinear soliton (blue dashes) on top of the solution after the evolution (green solid), (b) Nonlinear evolution of the soliton, (c) The view from top and (d) Maximum amplitude as a function of the propagation distance z.

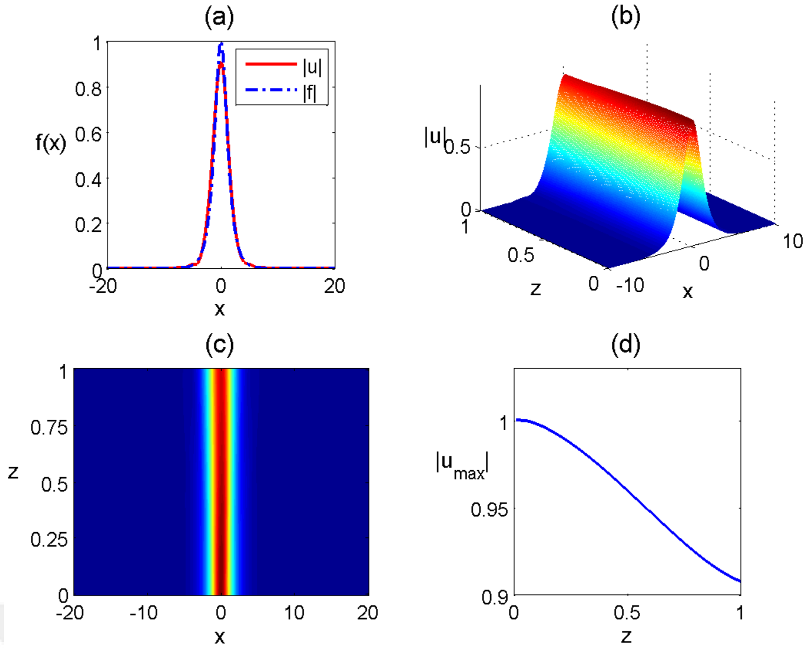

Figure 4.9 : Nonlinear instability of a higher order soliton for $\gamma = 0, \mu = 1$ $f_0 = 1$ and $g_0 = 1$ with a $\mathscr{PT}$-symmetric potential; (a) Numerically produced higher order nonlinear soliton (blue dashes) on top of the solution after the evolution (green solid), (b) Nonlinear evolution of the soliton, (c) The view from top and (d) Maximum amplitude as a function of the propagation distance z.

In this chapter, by using the analytical and the numerical results, we conclude that:

(i) When the dispersion coefficient term γ increases, the maximum amplitude of the $\mathscr{PT}$-symmetric potential decreases

(ii) Despite the change in the shape of the real and imaginary parts of the solitons, the maximum amplitude of the soliton is not affected from decrement or increment of eigenvalue μ,

(iii) The parameter f_0 can directly change the maximum amplitude of the solitons.

5. CONCLUSION

The aim of this thesis is to study (1+1)D 4OD cubic-quintic NLS equation with and without an external potential and investigate the existence and nonlinear stability properties of the soliton type solutions to this equation. First of all, spectral renormalization method is introduced to obtain numerical solutions and split-step Fourier method is explained to analyze stability of the fundamental solitons. We implemented the considered numerical approach to the (1+1)D 4OD cubic-quintic NLS equations both with and without $\mathscr{P}\mathscr{T}$-symmetric potential. In the second place, we show numerical illustration of the soliton solution of (1+1)D 4OD cubic-quintic NLS equation without an external potential for different values of γ (40D term's coefficient) and the propagation constant μ. The stability analysis of the soliton solutions is obtain by considering the effects of the γ and μ on the stability.
As a result of the numerical observations, we conclude that:
(*i*) The higher order solitons obtained for 4OD NLS equation may have oscillating tails and these solitons are found to be nonlinearly *unstable*
(*ii*) The higher order solitons obtained for 4OD cubic-quintic NLS equation are found to be nonlinearly *unstable* for the cases when the effect of the 4OD effect is either small or the eigenvalue is μ is small (even if the 4OD effect is large).

In the final chapter, we obtained numerical results for the case with a $\mathscr{P}\mathscr{T}$-symmetric potential and compared them with exact solutions. Using SR method we showed numerical illustrations and stability of the obtained solitons for various values of the problem parameters. The maximum amplitudes of the obtained solitons are discovered and illustrated with changing values of solitons parameters. We saw how the shape and maximum amplitude of the produced solitons changed in each case.

The effect of f_0, the eigenvalue μ and the dispersion coefficient term γ on the maximum amplitude of the solitons are depicted and as a result of the numerical studies, we conclude that:

(i) When the dispersion coefficient term γ increases, the maximum amplitude of the $\mathscr{P}\mathscr{T}$-symmetric potential decreases

(ii) Despite changing the shape of the real and imaginary parts of the solitons, the maximum amplitude of the soliton is not affected from decreases or increases of eigenvalue μ

(iii) The parameter f_0 can directly change the maximum amplitude of the solitons.

As we can see in illustrations and results, numerical studies are satisfactory in terms of accuracy and stability. That means we chose the suitable numerical method for the solution of (1+1)D 4OD cubic-quintic NLS equation and we reached physically acceptable solutions. In future, maybe someone try to find stable soliton solutions by taking different potential parameters V_1, V_2 and W_0 and we can discuss inclusion of fourth order dispersion to the problem.

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APPENDICES

APPENDIX A.1 : Fourier Transform

APPENDIX A.1

Fourier Transform

For a continuous, smooth and absolutely integrable function $f(x)$, the integral transform

$$F(k_x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(k_x)x} dx \tag{A.1}$$

is called *the Fourier transform of f (x)* and conversely, the transform

$$F(k_x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i(k_x)x} \mathrm{d}x \tag{A.2}$$

is called *the inverse Fourier transform of* $F(k_x)$.

The Fourier transform of f is denoted by $\mathscr{F}(f) = \hat{f}$, the inverse Fourier transform of $\hat{f}$ is denoted by $\mathscr{F}^{-1}(\hat{f})$ and clearly $\mathscr{F}^{-1}(\hat{f}) = \mathscr{F}^{-1}(\mathscr{F}(\hat{f}))$.
Integral transform methods are very useful for solving partial differential equations because of their properties such as linearity, shifting, scaling, etc.
Suppose that $f(x)$ tends to zero as x tends to infinity. Then,

$$\begin{aligned}\mathscr{F}\left(f'(x)\right) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\prime(x) e^{i(k_x)x} \mathrm{d}x = \frac{1}{\sqrt{2\pi}} f(x) e^{i(k_x)} \Big|_{-\infty}^{\infty} - ik_x \int_{-\infty}^{\infty} f(x) e^{i} k_x x \mathrm{d}x \\ &= -ik_x \mathscr{F}(f(x))\end{aligned} \tag{A.3}$$

This result can be extended to obtain the differentiation property of the Fourier transform:

$$\mathscr{F}(f'(x)) = (-ik_x)^n (f(x)) = (-ik_x)^n \hat{f}, \qquad n \in N \tag{A.4}$$

CURRICULUM VITAE

Name Surname : Ayşe Şebnem YAR

Place and Date of Birth : Kadıköy 07/08/1992

Adress :Ünalan Mh. Gülnar Sk. No:7/3 Üsküdar/İstanbul

E-Mail : yara@itu.edu.tr

B.Sc. : Istanbul University/ Faculty of Science/ Department of Mathematics

M.Sc. :Istanbul Technical University/Institute of Science and Technology/ Department of Mathematical Engineering