ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY

INTERSECTION GRAPHS OF FINITE GROUPS

Ph.D. THESIS

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Department of Mathematical Engineering

Mathematical Engineering Programme

JUNE 2016

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FOREWORD

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TABLE OF CONTENTS

Page

ABBREVIATIONS

BNCT **:** Burnside Normal Complement Theorem
CFSG **:** Classification of Finite Simple Groups

SYMBOLS

$\boldsymbol{\Gamma(G)}$	: Intersection graph of the group G
$\boldsymbol{L(G)}$	: Lattice of subgroups of G
$\boldsymbol{N_G(H)}$	: Normalizer of H in G
$\boldsymbol{C_G(H)}$	: Centralizer of H in G
$\boldsymbol{\Phi(G)}$	: Frattini subgroup of G
$\boldsymbol{O_p(G)}$	: p-core of the finite group G
$\mathbf{soc}(\boldsymbol{G})$	: Socle of G
$\mathbb{Z}_{\boldsymbol{n}}$	: Cyclic group of order n
$\boldsymbol{D_{2n}}$	: Dihedral group of order $2n$
$\boldsymbol{Q_{2^n}}$	: (Generalized) quaternion group of order 2^n
$\boldsymbol{S_n}$	: Symmetric group of order n
$\boldsymbol{\kappa(G)}$	: Connectivity of $\Gamma(G)$
$\boldsymbol{C_n}$	: Cycle graph with n vertices
$\boldsymbol{P_n}$	: Path graph with n vertices
$\boldsymbol{K_n}$	: Complete graph with n vertices
$\boldsymbol{K_{m,n}}$	: Complete bipartite graph with $m+n$ vertices

LIST OF TABLES

Page

LIST OF FIGURES

Page

INTERSECTION GRAPHS OF FINITE GROUPS

SUMMARY

Let G be a group. The intersection graph $\Gamma(G)$ of G is an undirected graph without loops and multiple edges defined as follows: the vertex set is the set of all proper non-trivial subgroups of G, and there is an edge between two distinct vertices X and Y if and only if $X \cap Y \neq 1$ where 1 denotes the trivial subgroup of G. The purpose of this thesis is to study the intersection graphs of finite groups. Particular emphasis was put on the graph theoretical invariants of those objects.

In general, two non-isomorphic groups may have isomorphic intersection graphs. However, finite abelian groups can almost be distinguished by their intersection graphs. We prove that for any two abelian groups A and B, their intersection graphs are isomorphic if and only if (i) the product of the non-cyclic Sylow subgroups of A is isomorphic to the product of the non-cyclic Sylow subgroups of B, and (ii) exponents of the orders of the cyclic Sylow subgroups of A and of B are equal up to a permutation.

We classified all finite groups whose intersection graphs are planar. There are a few abelian groups with planar intersection graphs and the only non-abelian nilpotent groups with planar intersection graph are the dihedral group D_8 of order eight and the quaternion group Q_8. The rest of the list consists of some semi-direct products. In particular, there is no non-solvable group whose intersection graphs is planar. By Kuratowski's Theorem a graph is planar if and only if it does not contain the complete graph K_5 over five vertices and the complete bipartite graph $K_{3,3}$ as a minor. We further determine the finite groups whose intersection graphs contains a K_5 but not $K_{3,3}$ as a subgraph.

We studied the connectivity of intersection graphs of finite groups. Intuitively, intersection graphs should be highly connected graphs and if there are some examples of such graphs with 'low' connectivity, they must be exceptional. We classified finite solvable groups whose intersection graphs are *not* 2-connected and finite nilpotent groups whose intersection graphs are *not* 3-connected.

SONLU GRUPLARIN KESİŞİM ÇİZGELERİ

ÖZET

G ile bir grup temsil edilmek üzere G'nin kesişim çizgesi $\Gamma(G)$ ile şu şekilde tanımlanan döngü ve de çoklu kenar içermeyen yönsüz çizge kastedilmektedir: köşe kümesi G'nin trivial olmayan özalt gruplarının kümesidir ve birbirinden farklı iki köşe X ve Y arasında ancak ve ancak $X \cap Y \neq 1$ ise bir kenar vardır. Burada 1 ile trivial grup kastedilmektedir. Bu tez çalışmasının amacı sonlu grupların kesişim çizgelerini araştırmaktır. Daha özel olarak bu nesnelerin çizge kuramsal değişmezleri üzerinde durulmuştur.

Genel olarak, izomorf olmayan iki grubun kesişim çizgeleri izomorf olabilirler. Mesela kuaternion grup ile mertebesi bir asal sayının beşinci dereceden kuvveti olan döngüsel grubu gözönüne alalım. Her iki grubun da kesişim çizgeleri dörder adet köşeye sahiptir. Dahası her iki grubun tek bir minimal altgrubu mevcuttur ve bu minimal altgrup diğer bütün trivial olmayan özaltgruplar tarafından içerilir. Dolayısıyla bu iki grubun kesişim çizgeleri izomorftur. Ancak abelyen grupların sınıfı göz önüne alındığında kesişim çizgelerinin bu grupları birbirinden ayırmada neredeyse yeterli olduklarını gösterdik. Daha net ifade edecek olursak şu sonucu ispatladık: A ve B sonlu iki abelyen grup olmak üzere bu grupların kesişim çizgeleri yalnız ve yalnız şu şartlar sağlandığı takdirde izomorftur: (i) A'nın döngüsel olmayan Sylow altgruplarının çarpımı B'nin döngüsel olmayan Sylow altgruplarının çarpımına izormorftur, ve (ii) A'nın döngüsel Sylow altgruplarının mertebelerinin üsleri gerekiyorsa bir permutasyondan sonra B'nin döngüsel Sylow altgruplarının mertebelerinin üslerine eşittir.

Eğer bir çizge düzlem (yada küre) üzerine kenarları birbirini kesmeyecek şekilde tasvir edilebiliyorsa bu çizgeye düzlemsel çizge denilir. Kesişim çizgeleri düzlemsel olan sonlu grupları sınıflandırdık. Diyelim ki p, q, ve r birbirinden farklı asal sayıları temsil etsinler. Kesişim çizgesi düzlemsel olan abelyen gruplar şunlardan ibarettir:

$$\mathbb{Z}_{pqr},\ \mathbb{Z}_{p^2q},\ \mathbb{Z}_{pq},\ \mathbb{Z}_{p^i}\,(0 \le i \le 5),\ \mathbb{Z}_4 \times \mathbb{Z}_2,\ \mathbb{Z}_p \times \mathbb{Z}_p,\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_p\,(p \neq 2).$$

Abelyen olmayan ama nilpotent olan gruplar ise yalnızca mertebesi sekiz olan dihedral grup D_8 ve kuaternion grup Q_8'dir. Ayrıca aşağıdaki yarı-direkt çarpımların kesişim çizgeleri de düzlemseldir ve böylece liste tamamlanır:

- Yarı-direkt çarpımlar $\mathbb{Z}_q \rtimes \mathbb{Z}_{p^2}\ (p^2 \mid q-1)$ ve $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q\ (q \mid p+1)$,
- Yarı-direkt çarpım $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_{q^2}\ (q^2 \mid p+1)$,
- Yarı-direkt çarpım $\mathbb{Z}_r \rtimes \mathbb{Z}_{pq}\ (pq \mid r-1)$,
- Yarı-direkt çarpım $\mathbb{Z}_p \rtimes \mathbb{Z}_q\ (q \mid p-1)$.

Bu grupların prezentasyonları elde edilmiştir. Ayrıca çözülebilir olmayan grupların kesişim çizgelerinin düzlemsel olamayacağı sonlu basit grupların sınıflandırılması (CFSG) kullanılmadan ispatlanmıştır.

K_n ile n adet köşesi olan ve herhangi iki ayrı köşe arasında bir kenar bulunan yönsüz basit çizge, $K_{m,n}$ ile ise köşe kümesi $V_m \sqcup V_n$ şeklinde eleman sayıları m ve n olan iki kümenin ayrık birleşimi şeklinde yazılabilen öyle ki iki köşe arasında ancak ve ancak biri V_m'nin elemanı diğeri ise V_n'nin elemanı ise kenar bulunan yönsüz basit çizge temsil edilsin. Kuratowski'nin karakterizasyonu bir çizgenin ancak ve ancak hem K_5'i hem de $K_{3,3}$'ü minör olarak içermiyorsa düzlemsel olacağını söyler. İspatlarımız incelendiği vakit görülecektir ki bir grubun kesişim çizgesi ancak ve ancak K_5'i yada $K_{3,3}$'ü altçizge olarak içeriyorsa düzlemsel değildir. Biz bu çalışmada kesişim çizgeleri K_5'i altçizge olarak içeren ama $K_{3,3}$'ü altçizge olarak içermeyen grupları belirledik. Bu gruplar şunlardır:

$$\mathbb{Z}_{p^6}, \mathbb{Z}_{p^3} \times \mathbb{Z}_q, \mathbb{Z}_9 \times \mathbb{Z}_3, (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3, \mathbb{Z}_9 \rtimes \mathbb{Z}_3,$$

$$\mathbb{Z}_3 \rtimes \mathbb{Z}_4, D_{18}, \mathbb{Z}_q \rtimes \mathbb{Z}_{p^3}\,(p^3 \mid q-1).$$

Γ köşe sayısı k'dan fazla bağlantılı bir çizge olmak üzere eğer k'dan daha az sayıda köşeyi kaldırarak Γ'yı bağlantısız hale getirmek mümkün değil ise Γ'ya k-bağlantılıdır denir. Γ'nın k-bağlantılı olduğu en küçük değer ise Γ'nın bağlantılılık sayısıdır. Bu bağlamda bağlantısız çizgeler 0-bağlantılı çizgeler olarak görülebilirler. Kesişim çizgeleri bağlantısız olan gruplar halihazırda sınıflandırılmışlardır:

1. $\mathbb{Z}_p \times \mathbb{Z}_p$, yada $\mathbb{Z}_p \times \mathbb{Z}_q$;
2. $G \cong N \rtimes A$ öyle ki $N \cong \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$, $A \cong \mathbb{Z}_q$, $N_G(A) = A$, ve N altgrubu G'nin minimal normal altgrubudur.

Bu sonucu en azından bir trivial olmayan normal özaltgrup içerme faraziyesi altında ispatlamak çok zor değildir. Sezgisel olarak kesişim çizgeleri bağlantılılık sayıları yüksek çizgelerdir ve eğer bağlantılılık değerleri düşük çizgeler varsa bunlar istisnai durumlar olmalıdırlar. Menger Teoremi bir çizgenin ancak ve ancak herhangi iki köşe arasında birbirinden bağımsız en az k adet patika bulunması durumunda k-bağlantılı olacağını söyler. Ancak bir grubun kesişim çizgesinin k-bağlantılı olduğunu iddia edebilmek için belli şartları sağlayan birçok altgrubun mevcudiyetini gösterebilmeliyiz. Bu bakımdan yüksek k değerleri için daha katı faraziyeler sunmak kaçınılmaz olmuştur. Bu çalışmada kesişim çizgeleri 2-bağlantılı olmayan sonlu çözülebilir grupları sınıflandırdık. $\Phi(P)$ ile P grubunun Frattini altgrubu temsil edilmek üzere bu G grupları şu şekilde nitelenebilirler:

1. $|G| = p^\alpha\,(0 \le \alpha \le 2)$;
2. $|G| = p^3$ öyle ki $G \ncong Q_8$ ve $G \ncong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$;
3. $|G| = p^2 q$ öyle ki G'nin Sylow p-grubu P için
   (a) $P \cong \mathbb{Z}_{p^2}$, veya
   (b) $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$ ve G'nin mertebesi p olan ve normal olmayan bir altgrubu mevcuttur;

4. $G = PQ$ mertebesi $p^{\alpha}q$ ($\alpha \geq 3$) olan ve Sylow p-altgrubu P normal bir altgrup olan bir gruptur öyle ki

   (a) P elemanter abelyen olup Q'nun P üzerine etkisi indirgenemezdir ve $N_G(Q)$'nin mertebesi en fazla pq'dir, veya

   (b) $N := \Phi(P)$ elemanter abelyen olup Q'nun hem N hem de P/N üzerine etkisi indirgenemezdir, ayrıca ya $N_G(Q) = Q$ gerçeklenir yada $N_G(Q) = NQ \cong \mathbb{Z}_p \times \mathbb{Z}_q$'dir.

Özel olarak, mertebesi üç farklı asal sayı tarafından bölünebilen herhangi bir çözülebilir grup 2-bağlantılıdır. Ayrıca kesişim çizgeleri 3-bağlantılı olmayan sonlu nilpotent grupları sınıflandırdık. Bu G grupları şu şekilde nitelenebilirler:

1. $|G| = p^{\alpha}$ ($0 \leq \alpha \leq 3$), $G \ncong Q_8$ ve $G \ncong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$;

2. G mertebesi p^4 olan bir gruptur öyle ki

   (a) $G \cong \mathbb{Z}_{p^4}$, veya

   (b) $\Phi(G) \cong \mathbb{Z}_{p^2}$ ve $G \ncong Q_{16}$, veya

   (c) $\Phi(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $Z(G) < \Phi(G)$ ve
   $G \ncong \langle a,b,c \mid a^9 = b^3 = 1, ab = ba, a^3 = c^3, bcb^{-1} = c^4, aca^{-1} = cb^{-1} \rangle$;

3. $G \cong \mathbb{Z}_{p^3q}$, $G \cong \mathbb{Z}_{p^2q}$, $G \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \times \mathbb{Z}_q$, veya $G \cong \mathbb{Z}_{pqr}$.

Dahası, mertebesi dört farklı asal sayı tarafından bölünebilen herhangi bir çözülebilir grup 3-bağlantılıdır.

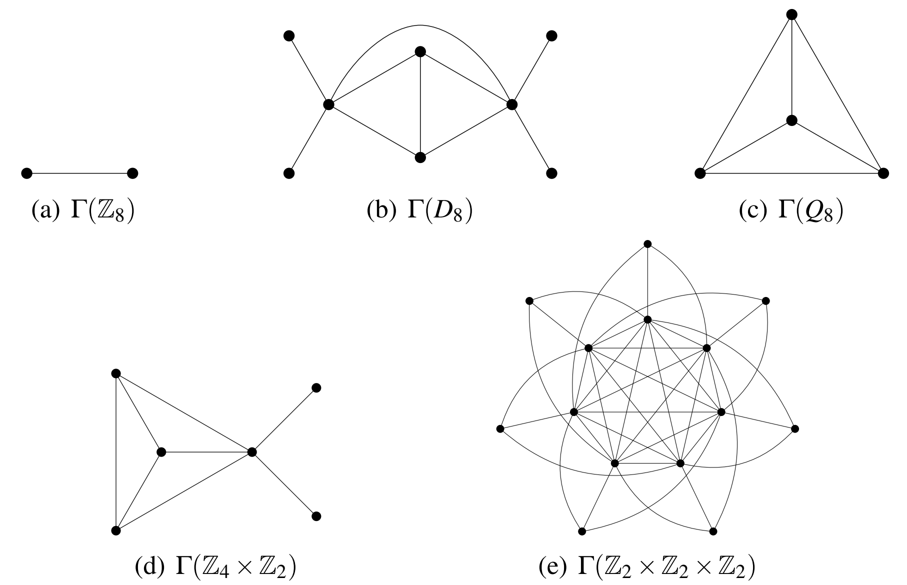
1. INTRODUCTION

Let $\mathscr{F}$ be the set of proper subobjects of an object with an algebraic structure. We define the *intersection graph of* $\mathscr{F}$ in the following way [1]: there is a vertex for each subobject in $\mathscr{F}$ other than the zero object, where the zero object is the object having a unique endomorphism, and there is an edge between two vertices whenever the intersection of the subobjects representing the vertices is not the zero object. In particular, if $\mathscr{F}$ is the set of proper subgroups of a group G, then the zero object is the trivial subgroup. The intersection graph of (the proper subgroups of) G will be denoted by $\Gamma(G)$.

Intersection graphs first defined for semigroups by Bosák in [2]. Let S be a semigroup. The intersection graph of the semigroup S is defined in the following way: the vertex set is the set of proper subsemigroups of S and there is an edge between two distinct vertices A and B if and only if $A \cap B \neq \varnothing$. It is interesting to note that this definition is not in the scope of the abstract generalization given in the preceding paragraph.

Afterwards, in [3] Csákány and Pollák adapted this definition into groups in the usual way. Still there are analogous definitions such as intersection graphs of the proper subspaces of a finite dimensional vector space over finite field, certain affine subspaces, and the proper ideals of a commutative ring. For example, in [4] authors studied the intersection graphs of ideals of a ring. In particular, they determine the values of n for which the intersection graph of the ideals of $\mathbb{Z}_n$ is connected, complete, bipartite, planar or has a cycle. For the corresponding literature the reader may also refer to [5–9] and some of the references therein. In [1], Yaraneri studied intersection graph of the proper submodules of any module over any ring, therefore most of the results of some of the above papers are easy consequences of this study. Notice that his results are also applicable to abelian groups.

It is easy to observe that the intersection graphs of the trivial group and the groups of prime order are empty graphs, i.e. the corresponding vertex sets are empty. Let p, q, and r be some pairwise distinct prime numbers. It is also easy to see that intersection



(a) $\Gamma(\mathbb{Z}_8)$ (b) $\Gamma(D_8)$ (c) $\Gamma(Q_8)$

(d) $\Gamma(\mathbb{Z}_4 \times \mathbb{Z}_2)$ (e) $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

Figure 1.1 : Intersection graphs of groups of order 2^3.

graphs of groups of order p^2 or of order pq consist of isolated vertices, so the first interesting examples emerge when the order of the group is p^3, p^2q, or pqr.

In Figure 1.1, we present the intersection graphs of the groups of order 2^3. It is well-known that if G is a finite cyclic group, then there is exactly one subgroup of order n for each divisor n of $|G|$ and for any pair of subgroups H and K, we have $H \leq K$ if and only if $|H| \mid |K|$. Hence the intersection graph of the cyclic group $\mathbb{Z}_8$ of order eight consists of two connected vertices: one for the subgroup of order two and the other for the subgroup of order four. Observe that any automorphism of the group induces an automorphism of the intersection graph. For the group $\mathbb{Z}_8$, there are exactly $\varphi(2^3) = 2^3 - 2^2 = 4$ automorphisms each inducing the trivial automorphism of the graph. Notice that the map interchanging the two vertices is an automorphism of the graph which is not induced by an automorphism of $\mathbb{Z}_8$.

The dihedral group D_8 of order eight contains three maximal subgroups of order four, one of them is cyclic and the other two are not. Those three maximal subgroups intersects at a subgroup of order two, hence together with this subgroup they form a complete graph K_4 in the intersection graph as a subgraph. In Figure 1.1(b), the leftmost two vertices represents two conjugate subgroups of order two which can be swapped by an automorphism of the graph induced by an inner automorphism of the group. Notice that those automorphisms of the graph induced by the inner automorphisms of the group form a subgroup.

Maybe the most interesting example is the intersection graph of the elementary abelian group of order eight which is depicted in Figure 1.1(e). Here the vertices on the outer circle represents the minimal subgroups and the vertices on the inner circle are the maximal subgroups. Considered as a vector space over the field of two elements, those minimal subgroups become 1-dimensional subspaces and the maximal subgroups become the hyperplanes. Since the whole space is 3-dimensional, any two hyperplane intersects at a 1-dimensional subspace. Therefore, the vertices in the inner circle form a complete subgraph. Also, induced by a change of basis of the vector space, any three element subset of the vertices at the outer circle can be mapped to any other three element subset of the outer circle by a graph automorphism. Accordingly, $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is symmetrical enough to reflect the vector space structure of the group.

Subgroups of a group form a lattice ordered by set inclusion. Some of the structural properties of a group may be inferred by studying its subgroup structure and those parts of the group theory form a part of the lattice theory. Intersection graphs of groups are natural objects and are intimately related with subgroup lattices. Let $L(G)$ denote the subgroup lattice of the group G. One can recover $\Gamma(G)$ from $L(G)$ by cutting off the uppermost vertex G from the maximal subgroups and also cutting off the trivial subgroup 1 from the minimal subgroups, and then by connecting each pair of vertices that still have a meet but are not linked by an edge. In other words, $L(G)$ collects more information than $\Gamma(G)$ in the sense that if $L(G)$ is given, then we can recover $\Gamma(G)$ but not vice versa in general.

As an example consider the quaternion group Q_8 which has three maximal subgroups, say $\langle i \rangle, \langle j \rangle$, and $\langle k \rangle$, of order four intersecting at the unique minimal subgroup $\{-1, 1\}$. By cutting off Q_8 itself and the trivial subgroup 1 from $L(Q_8)$, we obtain a partially ordered set (poset) of four elements and clearly its Hasse diagram, as a graph, is isomorphic to the star $K_{1,3}$. Adding the necessary edges, we see that $\Gamma(Q_8)$ is isomorphic to the complete graph K_4. Another example is the cyclic group $\mathbb{Z}_{p^5}$, where p denotes a prime number. After deleting the group itself and the trivial subgroup from $L(\mathbb{Z}_{p^5})$, we obtain a poset with its Hasse diagram isomorphic to the path graph P_4. Observe that $\Gamma(Q_8) \cong K_4 \cong \Gamma(\mathbb{Z}_{p^5})$.

Recall that a *(abstract) simplicial complex* is a collection $\mathscr{S}$ of finite non-empty sets, such that if σ is an element of $\mathscr{S}$, so is every non-empty subset of σ. The element

σ of $\mathscr{S}$ is called a simplex of $\mathscr{S}$ and each non-empty subset of σ is called a face of σ. The underlying set of $\mathscr{S}$ is the union of one-point elements (singletons) of $\mathscr{S}$. The *k-skeleton* of $\mathscr{S}$ is the subcollection of elements of $\mathscr{S}$ having cardinality at most $k+1$. For a group G, we may construct a simplicial complex $K(G)$ in the following way: the underlying set of $K(G)$ is the vertex set of $\Gamma(G)$ and for each vertex H in $\Gamma(G)$ there is an associated simplex σ_H in $K(G)$ which is defined as the set of proper subgroups of G containing H. Observe that the common face of σ_H and σ_K is $\sigma_{\langle H,K\rangle}$. Moreover, as a graph the 1-skeleton of $K(G)$ is isomorphic to the intersection graph $\Gamma(G)$. We call $K(G)$ the intersection complex of G. This notion is somewhat between the two other notions in literature, namely the order complex and the clique complex. In the first case, we begin with a poset and construct its order complex by declaring chains of the poset as the simplices. For example, the order complex of the poset of $\mathbb{Z}_{p^5}$ is the tetrahedron, whereas the order complex of the poset of Q_8 is isomorphic to $K_{1,3}$ as a graph. Since the intersection complex of Q_8 is tetrahedron, we see that order complexes and intersection complexes are not the same. In the latter case, we begin with a graph and define the corresponding clique complex by simply declaring its cliques as simplices. For example, the clique complex of $\Gamma(Q_8)$ is the tetrahedron. In Figure 1.1(e), the vertices in the inner circle do not form a simplex in the intersection complex whereas in the clique complex they do. Thus, intersection complexes and clique complexes are not the same in general.

In the previous paragraph we remarked that order complexes and intersection complexes are different in general. However, they are equivalent up to homotopy. The following argument is due to Volkmar Welker: Consider the face poset of $K(G)$, i.e. the poset of simplices ordered by inclusion. By the identification $H \mapsto \sigma_H$, the poset of proper non-trivial subgroups of G becomes a subposet (after reversing the order relation) of the face poset of $K(G)$. The order complex of the face poset of a simplicial complex is the barycentric subdivision of the simplicial complex and therefore they are homeomorphic. We want to show that the poset of the proper non-trivial subgroups of G and the face poset of $K(G)$ are of the same homotopy type as order complexes. Let f be the map taking σ_H to σ_K, where K is the intersection of all maximal subgroups containing H. Then f is a closure operator on the face poset of $K(G)$. Let g be the map taking H to K, where K is the intersection of all maximal subgroups containing

H. Then g is a closure operator on the poset of proper non-trivial subgroups of G. Since closure operations on posets preserve the homotopy type of the order complex and since the images of f and g are isomorphic by the identification $K \mapsto \sigma_K$, we are done. We shall remark that order complexes of subgroup posets are widely studied in literature, see for example [10–12].

By defining intersection graphs we attach a graph to a group, like in the case of Cayley graphs. So, there are two natural directions we may follow. First, we may study the graph theoretical properties of intersection graphs by means of group theoretical arguments. This is straightforward. For example we may ask for which groups their intersection graphs are connected. This thesis study particularly focuses in this direction. In particular, we've studied planarity and connectivity of intersection graphs. And second, we may study the algebraic properties of groups by means of combinatorial arguments applied to the intersection graphs. This part seems to require more ingenuity. In the case of Cayley graphs a nice illustrative example for both directions is the Gromow's landmark 'polynomial growth theorem'. It states that a finitely generated group is virtually nilpotent (which is an algebraic property of the group) if and only if its growth function is polynomial (which is a combinatorial property of the Cayley graph).

As was mentioned previously, for any prime number p the intersection graph of $\mathbb{Z}_{p^5}$ is isomorphic to K_4. More generally, the intersection graphs of the cyclic groups $\mathbb{Z}_m$ and $\mathbb{Z}_n$ are isomorphic if in the prime number decomposition of $m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}$ and $n = q_1^{\beta_1} q_2^{\beta_2} \dots q_t^{\beta_t}$, the multiset of the exponents α_i, $i \in \{1,2,\dots,s\}$ is same with the multiset of the exponents β_j, $j \in \{1,2,\dots,t\}$; since their lattices are isomorphic in that case. In Chapter 2, we prove that apart from this situation abelian groups can be distinguished by their intersection graphs. For an abelian group D, we denote by D_c the product of the cyclic Sylow subgroups of D and by D_{nc} we denote the product of the non-cyclic Sylow subgroups of D. To be more precise, we prove that intersection graphs of two abelian groups A and B are isomorphic if and only if (i) $A_{nc} \cong B_{nc}$ and (ii) $L(A_c) \cong L(B_c)$.

In Chapter 3, we classify the finite groups whose intersection graphs are planar. By Kuratowski's characterization, a graph is planar if and only if it contains neither K_5 nor

$K_{3,3}$ as a minor. In Chapter 4, we further determine finite groups whose intersection graphs contain a K_5 but not $K_{3,3}$ as a subgraph.

Finite groups with disconnected intersection graphs was determined by Shen in [8]. In an earlier work [13], Lucido classified finite groups whose poset of proper non-trivial subgroups are connected. Obviously, $\Gamma(G)$ is connected if and only if the poset of proper non-trivial subgroups of G is connected. In Chapter 5, we further elaborate in the previous results and classify finite solvable groups whose intersection graphs are not 2-connected and finite nilpotent groups whose intersection graphs are not 3-connected.

In the remaining part of this chapter we recollect some group theoretical results that we shall use later. As a preliminary remark, for any pair of subgroups H and K of the group G, their (set-theoretic) product $HK := \{hk \in G \colon h \in H, k \in K\}$ is a subgroup provided one of them is a normal subgroup. Accordingly, we may say that the construction of $\Gamma(G)$ is easier if there are normal subgroups of the group G. Consider the intersection graph of the dihedral group $D_8 = \langle a, b \mid a^4 = b^2 = 1, bab = a^3 \rangle$ of order 8. There are five involutions, namely a^2, b, ab, a^2b, and a^3b with $\langle a^2 \rangle$ being the center of the group. Here $\langle b \rangle$ and $\langle a^2b \rangle$ form a pair of permutable subgroups generating a subgroup of order four. On the other hand $\langle b, ab \rangle$ is the whole group D_8, hence the distance between $\langle b \rangle$ and $\langle ab \rangle$ in the intersection graph must be greater than two (compare with Figure 1.1(b)).

Theorem 1.1 (Product Formula, see [14, Theorem 2.20])**.** *If X and Y are subgroups of a finite group G, then*

$$|XY||X \cap Y| = |X||Y|.$$

Since we are dealing exclusively with finite groups, it is not surprising that the Product Formula is an important tool in our investigations. Naturally, another important result is the Sylow Theorems. Let p be a prime and G be a group. If $|G| = p^n s$ and $p \nmid s$, then a *Sylow p-subgroup* of G is a subgroup of order p^n.

Theorem 1.2 (Sylow, see [15, p. 7, Theorem 2.9])**.** *If G is a finite group, then any p-subgroup is contained in a Sylow p-subgroup. Moreover, any two Sylow p-subgroups are conjugate.*

Furthermore, Sylow Theorems states that the number of Sylow p-subgroups is $1+mp$ for some integer m. However, this fact is valid in a more general setting.

Theorem 1.3 (Sylow, see [16, p. 30, Exercise 9])**.**

(i) *Let G be a group of order p^n and $k<n$. Then the number of subgroups of order p^k in G is* $\equiv 1 \pmod{p}$.

(ii) *Let G be a group of order $p^n s$, $p \nmid s$, $k \leq n$. Then the number of subgroups of order p^k in G is* $\equiv 1 \pmod{p}$.

Sylow Theorems are important not only for counting the subgroups but they also claim their existence. In this regard, Correspondence Theorem is another powerful tool. It is also useful when we want to derive structural results about the interrelations between the intersection graph $\Gamma(G)$ of the group G and the intersection graph $\Gamma(G/N)$ of the quotient group G/N, where $N \lhd G$.

Theorem 1.4 (Correspondence Theorem, see [14, Theorem 2.28])**.** *Let $N \unlhd G$ and let $\nu\colon G \to G/N$ be the canonical morphism. Then $S \mapsto \nu(S) = S/N$ is a bijection from the family of all those subgroups S of G which contain N to the family of all the subgroups of G/N.*

Moreover, if we denote S/N by S^*, then:

(i) *$T \leq S$ if and only if $T^* \leq S^*$, and then $[S:T] = [S^*:T^*]$; and*

(ii) *$T \unlhd S$ if and only if $T^* \unlhd S^*$, and then $S/T \cong S^*/T^*$.*

The description of intersection graphs is easier if we impose some constraints onto the groups such as being abelian. Actually, whenever we ask a question about the graph theoretical invariants of the intersection graphs, we tend to answer it step by step for the classes of groups ordered in the following way:

- cyclic groups
- abelian groups
- p-groups
- nilpotent groups

- solvable groups
- non-solvable groups

Notice that in the easier class of abelian groups all subgroups are normal whereas in the most difficult class of non-solvable groups we claim neither the existence of normal subgroups nor the existence of some subgroups of specified order. However, even in the class of solvable groups we have some strong results. Recall that a chief series for a group is a normal series which is maximal, i.e. there is no normal subgroup of the group which lies between the two successive terms of the series.

Theorem 1.5 (see [15, p. 24, Theorem 4.2])**.** *In a finite solvable group G, the factors of every chief series are elementary abelian of prime power order. In particular, every minimal normal subgroup of G is elementary abelian.*

An important property of Sylow Theorems is that they are valid for all finite groups. In the case of solvable groups we have more refined results. If π is a set of primes, recall that a Hall π-subgroup is a subgroup whose order is a product of primes in π, and whose index is coprime to its order.

Theorem 1.6 (see [15, p. 231, Theorem 4.1])**.** *If G is a finite solvable group, then any π-subgroup is contained in a Hall π-subgroup. Moreover, any two Hall π-subgroups are conjugate.*

Following two theorems give sufficient conditions for solvability.

Theorem 1.7 (Burnside p^aq^b Theorem, see [15, p.131, Theorem 3.3])**.** *Any group of order p^aq^b is solvable where p,q are prime numbers and a,b are natural numbers.*

Theorem 1.8 (Hölder's Theorem, see [14, Corollary 7.54])**.** *Any finite group of square-free order is solvable.*

Once we answered a question in the case of solvable groups, then we may invoke the Classification of Finite Simple Groups (CFSG for short) to answer it for non-solvable groups.

Subgroups defined by some property unambiguously, such as the center $Z(G)$ or the derived subgroup G' of the group G, are important in group theory. There are two such 'characteristic' subgroups which will appear frequently in our later arguments.

The *Frattini subgroup* $\Phi(G)$ of a group G is the intersection of all maximal subgroups of G. The following result is standard in finite group theory.

Theorem 1.9 (see [15, p. 174, Theorem 1.3])**.** *The Frattini factor group $G/\Phi(G)$ of a p-group G is elementary abelian. Furthermore, $\Phi(G) = 1$ if and only if G is elementary abelian.*

The p-core $O_p(G)$ of a finite group G is the intersection of all Sylow p-subgroups of G. It is the unique largest normal p-subgroup of G. In a finite solvable group G, the factors of every chief series are elementary abelian of prime power order. In particular, every minimal normal subgroup of G is elementary abelian (see Theorem 1.5). Hence, for a non-trivial solvable group G, there exists a prime p dividing the order of G such that $O_p(G)$ is non-trivial.

In our context, it is useful to know when a normal subgroup is complemented, i.e. when the group is the semidirect product of the normal subgroup by some subgroup.

Theorem 1.10 (Schur-Zassenhaus Lemma, see [14, Theorem 7.41])**.** *A normal Hall subgroup H of a finite group G has a complement, i.e. G is the semidirect product of H by G/H.*

Theorem 1.11 (Gaschütz, see [14, Theorem 7.43])**.** *Let K be a normal abelian p-subgroup of a finite group G, and let P be a Sylow p-subgroup of G. Then K has a complement in G if and only if K has a complement in P.*

It is also useful to know when a subgroup complemented by a normal subgroup. The following theorem is known as the Burnside Normal Complement Theorem in literature which we refer to as BNCT for short.

Theorem 1.12 (BNCT, see [14, Theorem 7.50])**.** *Let G be a finite group and P be a Sylow p-subgroup of G. If P is contained in the center of its normalizer $N_G(P)$ in G then there is a normal subgroup Q of G such that $P \cap Q = 1$ and $G = PQ$.*

Theorem 1.13 (see [14, Theorem 7.51])**.** *Let G be a finite group and p be the smallest prime divisor of $|G|$. If a Sylow p-subgroup P of G is cyclic, then there is a normal subgroup N of G such that $P \cap N = 1$ and $G = PN$.*

We finish this introduction with some further results for later references.

Theorem 1.14 (Frattini Argument, see [14, Theorem 4.18])**.** *Let K be a normal subgroup of a finite group G. If P is a Sylow p-subgroup of K (for some prime p), then*

$$G = KN_G(P).$$

The following lemma is an easy consequence of the Frattini Argument.

Lemma 1.15 (see [14, Exercise 4.11])**.** *Let P be a Sylow p-subgroup of a finite group G. If $N_G(P) \leq H \leq G$, then H is self-normalizing, i.e. $N_G(H) = H$.*

Theorem 1.16 (N/C Lemma, see [14, Theorem 7.1])**.** *If $H \leq G$, then $C_G(H) \trianglelefteq N_G(H)$ and $N_G(H)/C_G(H)$ can be imbedded in* $\mathrm{Aut}(H)$.

2. INTERSECTION GRAPHS OF ABELIAN GROUPS

The aim of this chapter is to show that the following conjecture raised in [9] is almost true.

Conjecture (see [9]). Two finite abelian groups with isomorphic intersection graphs are isomorphic.

This conjecture was already studied in [17] whose main result is one half of our main result. However, the proof in [17] contains some mistakes and inaccuracies. Here we use a different approach. Our main result in this chapter is the following

Theorem 2.1. *Let A and B be two finite abelian groups. Then, the intersection graphs of A and B are isomorphic if and only if the following two conditions hold:*

(i) The product of non-cyclic Sylow subgroups of A is isomorphic to the product of non-cyclic Sylow subgroups of B.

(ii) There is a bijection θ between the set of cyclic Sylow subgroups of A and the set of cyclic Sylow subgroups of B such that if $\theta(S) = T$ then the number of divisors of $|S|$ is equal to the number of divisors of $|T|$.

Let $\alpha_1, \dots, \alpha_r$ be some positive integers. By a Theorem of R. Baer [18], a group G is cyclic of order $p_1^{\alpha_1} \dots p_r^{\alpha_r}$ with distinct primes p_i if and only if $L(G)$ is a direct product of chains of lengths $\alpha_1, \dots, \alpha_r$. Hence, we have the following

Corollary 2.2 (see [19, 1.2.8 Corollary])**.** *Let $\alpha_1, \dots, \alpha_r$ be some positive integers and let $p_1, \dots, p_r$ be distinct primes. If G is a cyclic group of order $p_1^{\alpha_1} \dots p_r^{\alpha_r}$ and $\overline{G}$ is any group, then $L(\overline{G}) \cong L(G)$ if and only if $\overline{G}$ is cyclic of order $q_1^{\alpha_1} \dots q_r^{\alpha_r}$ with distinct primes $q_1, \dots, q_r$.*

Let A_c and B_c be the product of cyclic Sylow subgroups of A and B respectively. By Corollary 2.2, the second condition of the Theorem 2.1 is equivalent to the condition $L(A_c) \cong L(B_c)$.

We shall explain some conventions we adopt in this chapter. It is clear that the intersection graph of the trivial group or a group of prime order are empty graphs (that is, it has no vertex). To distinguish groups of prime order from the trivial group, whenever we mention the intersection graph of a group we implicitly assume that the group is a non-trivial group. By a Sylow subgroup of G we mean a Sylow p-subgroup of G for some prime number p dividing the order of G. Therefore, according to this convention, G has no Sylow q-subgroups for prime numbers q not dividing the order of G. Finally, we denote by $V(G)$ the set of all proper non-trivial subgroups of G, i.e. the vertex set of $\Gamma(G)$.

2.1 Preliminaries

In this section we recall the definitions of some basic notions, and also recall some preliminary results from [9]. We state some of the results of [9] in slightly different forms which are more convenient for our purposes.

Given two graphs Γ_1 and Γ_2, by a graph isomorphism $\phi : \Gamma_1 \to \Gamma_2$ we mean a bijective map ϕ from the set of vertices of Γ_1 to the set of vertices of Γ_2 such that, for any two vertices u and v of Γ_1, there is an edge in Γ_1 between u and v if and only if there is an edge in Γ_2 between $\phi(u)$ and $\phi(v)$. Note that the inverse of a graph isomorphism is a graph isomorphism. Therefore, a graph isomorphism from the intersection graph $\Gamma(G)$ of a group G to the intersection graph $\Gamma(H)$ of a group H is a bijective map $\psi \colon V(G) \to V(H)$ satisfying for any X and Y in $V(G)$ the condition: $X \cap Y \neq 1$ if and only if $\psi(X) \cap \psi(Y) \neq 1$.

Let Γ be a graph. A subset $\mathscr{A}$ of the set of vertices of Γ is called an independent set in Γ if there is no edge between any two elements of $\mathscr{A}$. It is obvious that a graph isomorphism maps independent sets to independent sets.

Let G be a finite group. Since a finite group X has a subgroup of order p for any prime divisor p of $|X|$, we see that any element of an independent set of maximum possible cardinality in $\Gamma(G)$ must have a unique minimal subgroup. Moreover, assuming that $|G|$ is not a prime number, the set of all minimal subgroups of G is an independent set of maximum possible cardinality in $\Gamma(G)$. Conversely, for any proper subgroup X of G, if X has a unique minimal subgroup, then X together with all the minimal subgroups of G different from the minimal subgroup of X form an independent set of maximum possible cardinality in $\Gamma(G)$. Thus, a graph isomorphism between intersection graphs of groups maps a subgroup with a unique minimal subgroup to a subgroup with a unique minimal subgroup. Since a finite abelian group having a unique minimal subgroup must be a cyclic p-group for some prime p, we have the following result of [9, Corollary of Lemma 2].

Remark 2.1. Let A and B be two finite abelian groups, and let $\phi\colon \Gamma(A) \to \Gamma(B)$ be a graph isomorphism. Let p be a prime number. Then, for any proper non-trivial cyclic p-subgroup X of A there is a prime number q depending on X such that $\phi(X)$ is a proper non-trivial cyclic q-subgroup of B. More to the point, the numbers of minimal subgroups of A and B are equal.

We will observe in Proposition 2.4 that the prime number q in the above result does not depend on the choice of the cyclic p-subgroup X. We first need a lemma whose proof contains ideas from the proof of [9, Lemma 3].

Lemma 2.3. *Let A and B be two finite abelian groups, and let $\phi\colon \Gamma(A) \to \Gamma(B)$ be a graph isomorphism. Let p be a prime number. Then, for any two proper non-trivial cyclic p-subgroups X and Y of A, if $X \cap Y = 1$ then there is a prime number q such that both $\phi(X)$ and $\phi(Y)$ are proper non-trivial cyclic q-subgroups of B.*

Proof. Let X and Y be two proper non-trivial cyclic p-subgroups of A such that $X \cap Y = 1$. There is a subgroup of A isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, implying that A and hence B has more than 2 proper non-trivial subgroups. It follows from Remark 2.1 that $\phi(X)$ is a proper non-trivial cyclic q_1-subgroup and $\phi(Y)$ is a proper non-trivial cyclic q_2-subgroup where q_1 and q_2 are some prime numbers. Moreover, $\phi(X) \cap \phi(Y) = 1$. We need to show that $q_1 = q_2$.

Assume for a moment that $q_1 \neq q_2$. For any non-trivial cyclic group C of prime power order, we let $m(C)$ denote the unique minimal subgroup of C. We note that $m(C)$ is of prime order. Let $U = m(\phi(X))m(\phi(Y))$. As $q_1 \neq q_2$, the proper non-trivial subgroups of U are precisely $m(\phi(X))$ and $m(\phi(Y))$. Therefore, U is a proper subgroup of B, and, in the graph $\Gamma(B)$, any vertex adjacent to U must be adjacent or equal to either $\phi(X)$ or $\phi(Y)$. Consequently, letting Z be the proper non-trivial subgroup of A such that $\phi(Z) = U$, it follows that, in the graph $\Gamma(A)$, any vertex adjacent to Z must be adjacent or equal to either X or Y. We will observe that this is not possible. Firstly, as U is adjacent to both of $\phi(X)$ and $\phi(Y)$, it follows that Z is adjacent to both of X and Y. This implies that $m(X)m(Y) \leq Z$. Now, as both of $m(X)$ and $m(Y)$ are of order p, we may take a group isomorphism $\eta : m(X) \to m(Y)$ and consider $T := \{x\eta(x) : x \in m(X)\}$. Then, T is a subgroup of $m(X)m(Y)$ of order p. Furthermore, although T is adjacent to Z, as $X \cap Y = 1$ it follows that T is not adjacent (and not equal) to X and to Y. □

We now explain the idea of the proof of the main result of [9]. Let A and B be two finite abelian groups, and $\phi : \Gamma(A) \to \Gamma(B)$ be a graph isomorphism. Let p be a prime number. Suppose that X is a proper non-trivial p-subgroup of A. Let $\mathscr{A}$ be the set of all minimal subgroups of A, and let $\mathscr{A}_p$ be the set of all the minimal subgroups of A of order p. We know from the explanation given before Remark 2.1 that $\phi(\mathscr{A})$ is an independent set in $\Gamma(B)$ of maximal possible cardinality. Moreover, we know from Lemma 2.3 that there is a prime number q such that each element of $\phi(\mathscr{A}_p)$ is a q-group. We want to observe that $\phi(X)$ is a q-group. Otherwise, $\phi(X)$ has a subgroup U of order r for some prime number r different from q. As $\phi(\mathscr{A})$ is an independent set in $\Gamma(B)$ of maximal possible cardinality, either U must belong to $\phi(\mathscr{A})$ or else U must be adjacent to an element of $\phi(\mathscr{A})$. In any case, $U \cap \phi(Y) \neq 1$ for some $Y \in \mathscr{A}$. As U is a subgroup of $\phi(X)$, it follows that $\phi(X) \cap \phi(Y) \neq 1$. This implies that $X \cap Y \neq 1$, and hence $Y \in \mathscr{A}_p$ (because X is p-group and $Y \in \mathscr{A}$). But then $\phi(Y) \in \phi(\mathscr{A}_p)$ is a q-group intersecting U non-trivially. This is impossible, because $|U|$ is a prime number different from q. Therefore, we justified the first part of the following result. The rest is easy because the inverse of the map ϕ is a graph isomorphism from $\Gamma(B)$ to $\Gamma(A)$.

Proposition 2.4 (see [9, Theorem]). *Let A and B be two finite abelian groups, and let*

$$\phi : \Gamma(A) \to \Gamma(B)$$

be a graph isomorphism. Let p be a prime divisor of $|A|$, and $S_p(A)$ be the Sylow p-subgroup of A. Then, there is a prime divisor q of $|B|$ such that $\phi(X) \leq S_q(B)$ for any $X \leq S_p(A)$ with $1 \neq X \neq A$, where $S_q(B)$ is the Sylow q-subgroup of B. In particular, the numbers of subgroups of $S_p(A)$ and $S_q(B)$ are equal. Moreover, there is a bijection from the set of Sylow subgroups of A to the set of Sylow subgroups of B.

In [9, Theorem], it is further claimed that the intersection graph of any Sylow subgroup of an abelian group A is determined by the intersection graph of A. This is the only point in [9] we disagree with. Using the given arguments there (which are explained here before Proposition 2.4), one can only determine p-subgroups of A, but among these p-subgroups one cannot determine which one is the Sylow p-subgroup. That is, assuming the notations of Proposition 2.4, it is further claimed in [9, Theorem] that the restriction of ϕ to the proper non-trivial subgroups of $S_p(A)$ induces a graph isomorphism $\phi\colon \Gamma(S_p(A)) \to \Gamma(S_q(B))$. However, it may happen, for instance, that ϕ may map a proper subgroup of $S_p(A)$ to $S_q(B)$. Fortunately, we remedy this situation in Remark 2.5 by showing that if the intersection graphs of A and B are isomorphic and if $|A|$ is not a prime power, then $|B|$ is not a prime power and there is a graph isomorphism $\psi\colon \Gamma(A) \to \Gamma(B)$ such that for each prime divisor p of $|A|$ there is a prime divisor of q of $|B|$ satisfying $\psi(S_p(A)) = S_q(B)$. Moreover, in this case, the restriction of ψ to the proper non-trivial subgroups of $S_p(A)$ induces a graph isomorphism $\psi\colon \Gamma(S_p(A)) \to \Gamma(S_q(B))$.

The paper [9] ends with the conjecture: Two finite abelian groups with isomorphic intersection graphs are isomorphic.

For any two distinct primes p and q, and for any natural number n, it is clear that the intersection graphs of $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_{q^n}$ are isomorphic (because both are complete graphs on $n-1$ vertices). Therefore, we assume that in the above conjecture of [9] it was implicitly assumed that the abelian groups are not cyclic.

As remarked in [9], it follows from Proposition 2.4 (and Remark 2.5) that it suffices to prove the conjecture for abelian groups whose orders are powers of prime numbers.

2.2 An Equivalence Relation

We begin by introducing some notations. Let G be an abelian group. We denote by $\operatorname{soc}(G)$ the product of all minimal subgroups of G, which coincides with the socle of G considered as a $\mathbb{Z}$-module. For any proper non-trivial subgroup X of G we define the notations $\mathscr{N}_G(X)$ and $\overline{\mathscr{N}_G}(X)$ as follows:

$$\mathscr{N}_G(X) := \{Y \in V(G) \colon Y \cap X \neq 1\} \quad \text{and} \quad \overline{\mathscr{N}_G}(X) := V(G) - \mathscr{N}_G(X).$$

So, in graph theoretical terminology, $\mathscr{N}_G(X)$ is the closed neighborhood of the vertex X of the graph $\Gamma(G)$. We next define a relation $\approx_G$ on $V(G)$ as follows: for any U and V in $V(G)$, $U \approx_G V$ if and only if $\mathscr{N}_G(U) = \mathscr{N}_G(V)$. The following remark is obvious.

Remark 2.2. Let G be an abelian group, and let U and V be elements of $V(G)$. Then:

(1) $\mathscr{N}_G(U) = \mathscr{N}_G(V)$ if and only if $\mathrm{soc}(U) = \mathrm{soc}(V)$.

(2) $\approx_G$ is an equivalence relation.

For an abelian group G and a proper non-trivial subgroup X of G, we denote by $[X]_G$ the $\approx_G$ equivalence class of X. Therefore, $[X]_G = \{Y \in V(G) \colon \mathrm{soc}(X) = \mathrm{soc}(Y)\}$ and $[X]_G = [\mathrm{soc}(X)]_G$. The following result is immediate.

Remark 2.3. Let G be an abelian group. Then, $\approx_G$ equivalence classes in $V(G)$ are precisely $[X]_G$ where $X \in V(G)$ with $X \leq \mathrm{soc}(G)$. Moreover, for any two distinct elements $Y, Z \in V(G)$ with $Y, Z \leq \mathrm{soc}(G)$, the equivalence classes $[Y]_G$ and $[Z]_G$ are distinct. In particular, the number of $\approx_G$ equivalence classes in $V(G)$ is equal to the number of non-trivial subgroups of $\mathrm{soc}(G)$ which are different from G.

Since any two elements lying in the same equivalence class have the same socle (or equivalently, have the same closed neighborhoods), we have the following obvious observation.

Remark 2.4. Let G be an abelian group, and let $X \in V(G)$. Then:

1. There is an edge in $\Gamma(G)$ between any two distinct elements of $[X]_G$.
2. For any $Y \in V(G)$, if there is an edge in $\Gamma(G)$ between Y and an element of $[X]_G$ then there is an edge in $\Gamma(G)$ between Y and every element of $[X]_G$ other than Y.

In the next result we observe that a graph isomorphism maps an equivalence class to an equivalence class.

Lemma 2.5. *Let A and B be two abelian groups, and let*

$$\phi : \Gamma(A) \to \Gamma(B)$$

be a graph isomorphism. Then, for any X in $V(A)$, the restriction of ϕ to $[X]_A$ induces a bijection from $[X]_A$ to $[\phi(X)]_B$.

Proof. Let U and V be in $V(A)$. Since a graph isomorphism and its inverse map adjacent vertices to adjacent vertices, we see that: $\mathscr{N}_A(U) = \mathscr{N}_A(V)$ if and only if $\mathscr{N}_B(\phi(U)) = \mathscr{N}_B(\phi(V))$. In other words, $U \approx_A V$ if and only if $\phi(U) \approx_B \phi(V)$. Therefore, for any $Y \in V(A)$, the element Y belongs to $[X]_A$ if and only if the element $\phi(Y)$ belongs to $[\phi(X)]_B$. This proves the result. $\square$

Let A and B be two finite abelian groups with isomorphic intersection graphs. It follows from Remark 2.3 and Lemma 2.5 that the numbers of subgroups of $\mathrm{soc}(A)$ and $\mathrm{soc}(B)$ are equal. We next show that more is true. That is, the intersection graphs of $\mathrm{soc}(A)$ and $\mathrm{soc}(B)$ are isomorphic. We first need two lemmas.

Note that the existence of a bijection σ in the following result is guaranteed by Lemma 2.5.

Lemma 2.6. *Let A and B be two abelian groups, and let*

$$\phi\colon \Gamma(A) \to \Gamma(B)$$

be a graph isomorphism. Then, for any X in $V(A)$, *and for any bijection* $\sigma\colon [X]_A \to [\phi(X)]_B$, *the map* $\varphi\colon \Gamma(A) \to \Gamma(B)$ *defined for any* $U \in V(A)$ *as*

$$\varphi(U) = \begin{cases} \sigma(U), & \text{if } U \in [X]_A \\ \phi(U), & \text{if } U \notin [X]_A \end{cases}$$

is a graph isomorphism. In particular, there is a graph isomorphism $\Gamma(A) \to \Gamma(B)$ *mapping* $\mathrm{soc}(X)$ *to* $\mathrm{soc}(\phi(X))$.

Proof. It follows from Lemma 2.5 that the map $\varphi\colon V(A) \to V(B)$ is a bijection. Take any two distinct elements V and W from $V(A)$. To finish the proof, we need to show that: there is an edge in $\Gamma(A)$ between V and W if and only if there is an edge in $\Gamma(B)$ between $\varphi(V)$ and $\varphi(W)$.

Since ϕ is a graph isomorphism, what we need to show is already true if both of V and W are not in $[X]_A$. Hence we have to check two cases: the case in which both of V and W are in $[X]_A$, and the case in which exactly one of V and W is in $[X]_A$.

Suppose that both of V and W are in $[X]_A$. Then, $\sigma(V)$ and $\sigma(W)$ are distinct elements of $[\phi(X)]_B$. The first part of Remark 2.4 implies that there is an edge in $\Gamma(A)$ between V and W, and that there is an edge in $\Gamma(B)$ between $\varphi(V) = \sigma(V)$ and $\varphi(W) = \sigma(W)$.

Suppose for the rest of the proof that $V \in [X]_A$ but $W \notin [X]_A$. Then, using Lemma 2.5 we see that $\phi(V) \in [\phi(X)]_B$ but $\phi(W) \notin [\phi(X)]_B$.

Assume that there is an edge in $\Gamma(A)$ between V and W. Since ϕ is a graph isomorphism, there must be an edge in $\Gamma(B)$ between $\phi(V)$ and $\phi(W)$. The second part of Remark 2.4 implies that there is an edge in $\Gamma(B)$ between $\phi(W)$ and any element of $[\phi(X)]_B$, in particular between $\phi(W)$ and $\sigma(V)$, which is in $[\phi(X)]_B$. Therefore, there is an edge in $\Gamma(B)$ between $\varphi(W) = \phi(W)$ and $\varphi(V) = \sigma(V)$.

Assume finally that there is an edge in $\Gamma(B)$ between $\varphi(V)$ and $\varphi(W)$. So there is an edge in $\Gamma(B)$ between $\sigma(V)$ and $\phi(W)$. Since $\sigma(V) \in [\phi(X)]_B$ and $\phi(W) \notin [\phi(X)]_B$, it follows from the second part of Remark 2.4 that there is an edge in $\Gamma(B)$ between $\phi(W)$ and any element of $[\phi(X)]_B$, in particular between $\phi(W)$ and $\phi(V)$, which is in $[\phi(X)]_B$. As ϕ is a graph isomorphism, there must be an edge in $\Gamma(A)$ between W and V.

To justify the last sentence of the lemma, we note that $\mathrm{soc}(X) \in [X]_A$ and $\mathrm{soc}(\phi(X)) \in [\phi(X)]_B$, and so we choose a bijection σ such that $\sigma(\mathrm{soc}(X)) = \mathrm{soc}(\phi(X))$. □

Remark 2.5. Let A and B be two finite abelian groups with isomorphic intersection graphs. Then, $|A|$ is not a prime power if and only if $|B|$ is not a prime power. Moreover, in the case $|A|$ is not a prime power, there is a graph isomorphism $\psi\colon \Gamma(A) \to \Gamma(B)$ mapping Sylow subgroups of A to Sylow subgroups of B. Moreover, the restriction of ψ to the proper non-trivial subgroups of any Sylow subgroup S of A induces a graph isomorphism $\psi\colon \Gamma(S) \to \Gamma(\psi(S))$.

Proof. Let $\phi : \Gamma(A) \to \Gamma(B)$ be a graph isomorphism. It follows from Proposition 2.4 that the number of Sylow subgroups of A and B are the same. This justifies the first part of the result.

Assume for the rest that Sylow subgroups of A (and hence of B) are proper subgroups. Let p be a prime number dividing $|A|$, and let P be the Sylow p-subgroup of A. It follows from Proposition 2.4 that $\phi(P)$ is a proper non-trivial q-subgroup of B for some prime number q. Take any non-trivial q-subgroup Y of B. Applying Proposition 2.4 to the inverse of ϕ^{-1} we see that $Y = \phi(X)$ for some non-trivial p-subgroup X of A. Now, either $P = X$ or else there is an edge in $\Gamma(A)$ between P and X. Hence, either $\phi(P) = Y$ or else there must be an edge in $\Gamma(B)$ between $\phi(P)$ and Y. Since Y is an arbitrary q-subgroup, the intersection of $\phi(P)$ and any non-trivial q-subgroups of B is non-trivial. This shows that $\mathrm{soc}(Q) \le \phi(P)$ where Q is the Sylow q-subgroup of B.

It is clear from $\mathrm{soc}(Q) \leq \phi(P)$ that $Q \in [\phi(P)]_B$. Letting $\sigma\colon [P]_A \to [\phi(P)]_B$ be any bijection sending P to Q, which is in $[\phi(P)]_B$, we apply Lemma 2.6 to find a graph isomorphism from $\Gamma(A)$ to $\Gamma(B)$ mapping P to Q.

Note that $\approx_A$ equivalence classes of Sylow subgroups of A are mutually distinct (and hence disjoint). The map ψ is obtained by applying the above procedure for each Sylow subgroup of A. Finally, knowing that ψ maps Sylow subgroups to Sylow subgroups, it follows from Proposition 2.4 that the restriction of ψ to the proper non-trivial subgroups of any Sylow subgroup S of A induces a graph isomorphism $\psi\colon \Gamma(S) \to \Gamma(\psi(S))$. □

Lemma 2.7. *Let A and B be two abelian groups with isomorphic intersection graphs. Suppose that* $\mathrm{soc}(A)$ *is a proper subgroup of* A. *Then,* $\mathrm{soc}(B)$ *is a proper subgroup of* B, *and there is a graph isomorphism from* $\Gamma(A)$ *to* $\Gamma(B)$ *mapping* $\mathrm{soc}(A)$ *to* $\mathrm{soc}(B)$.

Proof. Let G be an abelian group, and H be a subgroup of G. It is clear that $H \cap K \neq 1$ for every non-trivial subgroup K of G if and only if $\mathrm{soc}(G) \leq H$.

Let $\phi\colon \Gamma(A) \to \Gamma(B)$ be a graph isomorphism. Suppose that $\mathrm{soc}(A)$ is a proper subgroup of A. Then, by the explanation in the above paragraph, there is an edge in $\Gamma(A)$ between $\mathrm{soc}(A)$ and every proper non-trivial subgroup of A different from $\mathrm{soc}(A)$. Thus, there is an edge in $\Gamma(B)$ between $\phi(\mathrm{soc}(A))$ and every proper non-trivial subgroup of B different from $\phi(\mathrm{soc}(A))$. In other words, $\phi(\mathrm{soc}(A))$ is a proper subgroup of B and $\phi(\mathrm{soc}(A)) \cap Z \neq 1$ for any non-trivial subgroup Z of B. Hence, $\mathrm{soc}(B) \leq \phi(\mathrm{soc}(A))$. This shows that $\mathrm{soc}(B)$ is a proper subgroup of B, and shows that the socles of $\mathrm{soc}(B)$ and $\phi(\mathrm{soc}(A))$ are the same so that $[\mathrm{soc}(B)]_B = [\phi(\mathrm{soc}(A))]_B$. Letting $\sigma\colon [\mathrm{soc}(A)]_A \to [\phi(\mathrm{soc}(A))]_B$ be any bijection sending $\mathrm{soc}(A)$ to $\mathrm{soc}(B)$, which is in $[\phi(\mathrm{soc}(A))]_B$, we apply Lemma 2.6 to find a graph isomorphism from $\Gamma(A)$ to $\Gamma(B)$ mapping $\mathrm{soc}(A)$ to $\mathrm{soc}(B)$. □

Proposition 2.8. *Let A and B be two abelian groups with isomorphic intersection graphs. Then, there is a graph isomorphism*

$$\psi\colon \Gamma(A) \to \Gamma(B)$$

mapping proper non-trivial semisimple $\mathbb{Z}$*-submodules of the* $\mathbb{Z}$*-module A to proper non-trivial semisimple* $\mathbb{Z}$*-submodules of the* $\mathbb{Z}$*-module B. Moreover, if* $\mathrm{soc}(A)$ *is a* *proper subgroup of* A, *then* $\mathrm{soc}(B)$ *is a proper subgroup of* B *and* $\psi(\mathrm{soc}(A)) = \mathrm{soc}(B)$. *In particular, the restriction of* ψ *to* $V(\mathrm{soc}(A))$ *induces a graph isomorphism*

$$\psi\colon \Gamma(\mathrm{soc}(A)) \to \Gamma(\mathrm{soc}(B)).$$

Proof. Let $\phi: \Gamma(A) \to \Gamma(B)$ be a graph isomorphism. Applying Lemma 2.6 for each $X \in V(\mathrm{soc}(A))$ and for each bijection $[X]_A \to [\phi(X)]_B$ mapping X to $\mathrm{soc}(\phi(X))$, we obtain a graph isomorphism $\varphi\colon \Gamma(A) \to \Gamma(B)$ mapping each $X \in V(\mathrm{soc}(A))$ to $\mathrm{soc}(\phi(X))$, which is a non-trivial subgroup of $\mathrm{soc}(B)$. If $\mathrm{soc}(A) = A$ then Lemma 2.7 implies that $\mathrm{soc}(B) = B$, and hence $\mathrm{soc}(\phi(X)) \neq \mathrm{soc}(B)$. Therefore, in this case, φ maps proper non-trivial semisimple submodules of A to proper non-trivial semisimple submodules of B. We may let $\psi := \varphi$ in this case.

Assume for the rest that $\mathrm{soc}(A) \neq A$. It follows from Lemma 2.7 and its proof that $\mathrm{soc}(B) \neq B$ and that $\mathrm{soc}(B) \in [\varphi(\mathrm{soc}(A))]_B$. Letting $\sigma\colon [\mathrm{soc}(A)]_A \to [\varphi(\mathrm{soc}(A))]_B$ be any bijection sending $\mathrm{soc}(A)$ to $\mathrm{soc}(B)$, which is in $[\varphi(\mathrm{soc}(A))]_B$, we apply Lemma 2.6 to φ to find a graph isomorphism $\psi\colon \Gamma(A) \to \Gamma(B)$ mapping $\mathrm{soc}(A)$ to $\mathrm{soc}(B)$. Take any non-trivial subgroup Y of $\mathrm{soc}(A)$ different from $\mathrm{soc}(A)$. As $\psi\colon V(A) \to V(B)$ is a bijection satisfying $\psi(\mathrm{soc}(A)) = \mathrm{soc}(B)$, we must have that $\psi(Y) \neq \mathrm{soc}(B)$. Note that the classes $[Y]_A$ and $[\mathrm{soc}(A)]_A$ are different. In particular, $Y \notin [\mathrm{soc}(A)]_A$ so that $\psi(Y) = \varphi(Y) = \mathrm{soc}(\phi(Y))$ is semisimple. Consequently, ψ maps proper non-trivial semisimple submodules of A to proper non-trivial semisimple submodules of B. □

Corollary 2.9. *Let* A *and* B *be two finite abelian groups with isomorphic intersection graphs. Then, there is a graph isomorphism*

$$\psi\colon \Gamma(A) \to \Gamma(B),$$

whose restriction to $V(\mathrm{soc}(A))$ *induces a graph isomorphism* $\psi\colon \Gamma(\mathrm{soc}(A)) \to \Gamma(\mathrm{soc}(B))$ *satisfying the following conditions for any* $X \in V(A)$:

1. *X is a minimal subgroup of A if and only if $\psi(X)$ is a minimal subgroup of B. (That is, $|X|$ is prime if and only if $|\psi(X)|$ is prime).*

2. *X is a maximal subgroup of* $\mathrm{soc}(A)$ *if and only if $\psi(X)$ is a maximal subgroup of* $\mathrm{soc}(B)$.

Proof. Let $\psi\colon \Gamma(A) \to \Gamma(B)$ be a graph isomorphism satisfying the conclusion of Proposition 2.8.

Let $\mathscr{A}$ be the set of all proper non-trivial cyclic subgroups of A of prime power orders, and let $\mathscr{B}$ be the set of all proper non-trivial cyclic subgroups of B of prime power orders. It follows from Remark 2.1 that ψ is a bijection from $\mathscr{A}$ to $\mathscr{B}$. Also it follows from Proposition 2.8 that ψ is a bijection from $V(\mathrm{soc}(A))$ to $V(\mathrm{soc}(B))$. Hence, ψ is a bijection from $\mathscr{A} \cap V(\mathrm{soc}(A))$ to $\mathscr{B} \cap V(\mathrm{soc}(B))$. It is clear that $\mathscr{A} \cap V(\mathrm{soc}(A))$ (respectively, $\mathscr{B} \cap V(\mathrm{soc}(B))$) is the set of all proper minimal subgroups of A (respectively, of B). So, ψ must satisfy the condition (i).

Let C be an abelian group, and M be a proper subgroup of $\mathrm{soc}(C)$. It is clear that M is a maximal $\mathbb{Z}$-submodule of $\mathrm{soc}(C)$ if and only if there is no non-simple $\mathbb{Z}$-submodule of $\mathrm{soc}(C)$ intersecting M trivially. That is, M is a maximal subgroup of $\mathrm{soc}(C)$ if and only if $\overline{\mathscr{N}_{\mathrm{soc}(C)}}(M)$ consists entirely of minimal subgroups of $\mathrm{soc}(C)$.

Since $\psi\colon \Gamma(\mathrm{soc}(A)) \to \Gamma(\mathrm{soc}(B))$ is a graph isomorphism, for any $Y \in V(\mathrm{soc}(A))$ we have

$$\psi(\mathscr{N}_{\mathrm{soc}(A)}(Y)) = \mathscr{N}_{\mathrm{soc}(B)}(\psi(Y)) \quad \text{and} \quad \psi(V(\mathrm{soc}(A))) = \psi(V(\mathrm{soc}(B))),$$

implying that $\psi(\overline{\mathscr{N}_{\mathrm{soc}(A)}}(Y)) = \overline{\mathscr{N}_{\mathrm{soc}(B)}}(\psi(Y))$. As ψ satisfies the condition (i), $\overline{\mathscr{N}_{\mathrm{soc}(A)}}(Y)$ consists entirely of minimal subgroups of $\mathrm{soc}(A)$ if and only if $\overline{\mathscr{N}_{\mathrm{soc}(B)}}(\psi(Y))$ consists entirely of minimal subgroups of $\mathrm{soc}(B)$. Therefore, ψ satisfies the condition (ii). □

Let A and B be as in the following result. It follows from Remark 2.1 or (from the first part of Corollary 2.9) that the number of minimal subgroups of A is equal to the number of minimal subgroups of B. Unfortunately, as remarked in [9], this is not enough to deduce that $p = q$. However, in the next result we see that Corollary 2.9 implies $p = q$ for non-cyclic groups.

Proposition 2.10. *Let p and q be prime numbers. Let A be a finite abelian p-group and B be a finite abelian q-group. Suppose that the intersection graphs of A and B are isomorphic. Then:*

1. *A is cyclic if and only if B is cyclic. Moreover, in this case, there is a natural number n such that $A \cong \mathbb{Z}_{p^n}$ and $B \cong \mathbb{Z}_{q^n}$.*

2. *If A is not cyclic then* $\mathrm{soc}(A) \cong \mathrm{soc}(B)$. *In particular, in this case, $p = q$ and the ranks of A and B are equal.*

Proof. Let $\psi \colon \Gamma(A) \to \Gamma(B)$ be a graph isomorphism satisfying the conditions in Corollary 2.9.

(1) It is clear that an abelian group of prime power order is cyclic if and only if it has a unique minimal subgroup. As ψ satisfies the condition (i) in Corollary 2.9, the equivalence of being cyclic groups is easy. Moreover, as the number of subgroups of A and B must be equal, there must be a natural number n satisfying the mentioned conditions.

(2) Suppose that A is not cyclic. So, by the first part, B is not cyclic. Therefore, there are natural numbers $r > 1$ and $s > 1$ such that

$$\mathrm{soc}(A) \cong (\mathbb{Z}_p)^r := \underbrace{\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{r-\text{times}} \quad \text{and} \quad \mathrm{soc}(B) \cong (\mathbb{Z}_q)^s := \underbrace{\mathbb{Z}_q \times \mathbb{Z}_q \times \cdots \times \mathbb{Z}_q}_{s-\text{times}}.$$

Let M be a maximal subgroup of $\mathrm{soc}(A)$. We will compare the numbers

$$|\overline{\mathcal{N}_{\mathrm{soc}(A)}}(M)| \quad \text{and} \quad |\overline{\mathcal{N}_{\mathrm{soc}(B)}}(\psi(M))|$$

to deduce that $p = q$ and $r = s$. Since $\psi \colon \Gamma(\mathrm{soc}(A)) \to \Gamma(\mathrm{soc}(B))$ is a graph isomorphism, it is clear that

$$|\overline{\mathcal{N}_{\mathrm{soc}(A)}}(M)| = |\overline{\mathcal{N}_{\mathrm{soc}(B)}}(\psi(M))|.$$

Let X be a proper non-trivial subgroup of $\mathrm{soc}(A)$ such that $X \cap M = 1$. Since M is a maximal subgroup of $\mathrm{soc}(A)$, the order of X must be p so that $X = \langle x \rangle$ for some $x \in \mathrm{soc}(A) - M$. Conversely, for any element $y \in \mathrm{soc}(A) - M$ the cyclic group $\langle y \rangle$ is a proper non-trivial subgroup of $\mathrm{soc}(A)$ satisfying $\langle y \rangle \cap M = 1$. As a cyclic group of order p has $p-1$ generators, the number of proper non-trivial subgroups of $\mathrm{soc}(A)$ intersecting M trivially is

$$|\overline{\mathcal{N}_{\mathrm{soc}(A)}}(M)| = (|\mathrm{soc}(A)| - |M|)/(p-1) = (p^r - p^{r-1})/(p-1) = p^{r-1}.$$

As ψ satisfies the condition (ii) in Corollary 2.9, it follows that $\psi(M)$ is a maximal subgroup of $\mathrm{soc}(B)$. Therefore, arguing as in the previous paragraph, we may calculate that

$$|\overline{\mathcal{N}_{\mathrm{soc}(B)}}(\psi(M))| = q^{s-1}.$$

Hence, $p^{r-1} = q^{s-1}$, which implies that $p = q$ and $r = s$ (because p and q are primes, and because $r-1$ and $s-1$ are not zero). □

We finish this section with the following result. We use it in the next section to show that two non-cyclic finite abelian p-groups, where p is a prime number, with isomorphic intersection graphs are isomorphic. To facilitate reading we first introduce some notations.

Let p be a prime number. For any finite abelian p-group G and any minimal subgroup S of G we let $c_G(S)$ denote the number of proper cyclic subgroups of G containing S. As S is a minimal subgroup of G, any subgroup of G intersecting S non-trivially must contain S. Therefore, $c_G(S)$ is the number of cyclic groups in $\mathscr{N}_G(S)$, or equivalently it is the number of cyclic groups in the equivalence class $[S]_G$. We have the finite list of numbers $c_G(S)$ where S is ranging in the set of all minimal subgroups of G. We form the sequence $\operatorname{seq}(G)$ by writing all the distinct numbers $c_G(S)$ in this list in increasing order. Note that although the list of numbers $c_G(S)$ may contain equal numbers, the sequence $\operatorname{seq}(G)$ does not.

Lemma 2.11. *Let p be a prime number, and let A and B be two finite abelian p-groups with isomorphic intersection graphs. Then,* $\operatorname{seq}(A)$ *and* $\operatorname{seq}(B)$ *are the same.*

Proof. Let $\psi\colon \Gamma(A) \to \Gamma(B)$ be a graph isomorphism satisfying the conditions in Corollary 2.9. Since ψ satisfies the condition (i) in Corollary 2.9, it induces a bijection from the set of all minimal subgroups of A to the set of all minimal subgroups of B. Moreover, it is clear for any proper non-trivial subgroup X of A that $\psi(\mathscr{N}_A(X)) = \mathscr{N}_B(\psi(X))$. The result follows from Remark 2.1 saying that any graph isomorphism $\Gamma(A) \to \Gamma(B)$ maps proper non-trivial cyclic subgroups of A to proper non-trivial cyclic subgroups of B. □

2.3 Cyclic Subgroups

The aim of this section is to calculate the number of cyclic subgroups of an abelian group with a given fixed minimal subgroup. In other words we calculate the sequence $\operatorname{seq}(G)$, defined in the previous section, of an abelian group G of prime power order.

Throughout this section, let p be a prime number, and let G be a finite abelian p-group of rank $r > 1$. So, there are natural numbers α_i such that

$$G \cong \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p^{\alpha_r}} \quad \text{where} \quad 1 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_r.$$

Therefore, there are cyclic subgroups G_i of G such that G is the internal direct product of them as follows:

$$G = G_1 G_2 \ldots G_r \quad \text{where} \quad G_i \cong \mathbb{Z}_{p^{\alpha_i}}.$$

For any natural number m and any minimal subgroup S of G, we let $c_G^m(S)$ denote the number of cyclic subgroups of G of order p^m that contain S. Note that cyclic subgroups of G are proper (because $r > 1$), implying that $c_G(S) = \sum_{m=1}^{\infty} c_G^m(S)$, which is a finite sum. For any element g of G we define the following notations:

$$I := \{1, 2, \ldots, r\}, \quad J_g := \{i \in I \colon \pi_i(g) = 1\}, \quad \text{and} \quad I_g := I - J_g,$$

where $\pi_i \colon G \to G_i$ is the i-th projection. Note that, for any elements x and y of G, if $\langle x \rangle = \langle y \rangle$ then $J_x = J_y$ and $I_x = I_y$.

We begin with a trivial observation.

Remark 2.6. Let C be a cyclic p-group, and let a be an element of C of order p. For any natural number n with $p^n \leq |C|$, there are p^{n-1} elements c of C of order p^n satisfying the condition $c^{p^{n-1}} = a$.

Lemma 2.12. *Let g be an element of G of order p, and let $g_i := \pi_i(g)$ for each $i \in I$ so that $g = g_1 g_2 \ldots g_r$ with $g_i \in G_i$. Let m be a natural number, and let $S := \langle g \rangle$. Then:*

1. *$c_G^m(S) \neq 0$ if and only if $m \leq \alpha_j$ for any $j \in I_g$.*

2. *Suppose that $m \leq \alpha_j$ for any $j \in I_g$. Then, any cyclic subgroup of G of order p^m that contains S is generated by an element of G of the form*

$$a_1 a_2 \ldots a_r,$$

where if $j \in I_g$ then a_j is any element of G_j of order p^m satisfying the condition

$$a_j^{p^{m-1}} = g_j,$$

and where if $j \in J_g$ then a_j is any element of G_j of order less than p^m. Moreover, in each cyclic subgroup of G of order p^m that contains S there are exactly p^{m-1} such generators.

Proof. Suppose that there is a cyclic subgroup X of G of order p^m that contains S. Let $X = \langle x \rangle$ and $x_i := \pi_i(x) \in G_i$. As X has a unique minimal subgroup, the unique minimal subgroup of X must be equal to S. This gives that $\langle x^{p^{m-1}} \rangle = \langle g \rangle$. Therefore, there is an integer λ with $0 < \lambda < p$ such that

$$x_i^{\lambda p^{m-1}} = g_i$$

for any $i \in I$. We let $a_i := x_i^{\lambda} \in G_i$ for any $i \in I$. Note that

$$X = \langle x^{\lambda} \rangle = \langle a_1 a_2 \dots a_r \rangle \quad \text{and} \quad a_i^{p^{m-1}} = g_i.$$

If $i \in J_g$, then $1 = g_i = a_i^{p^{m-1}}$ so that the order of a_i is less than p^m. If $i \in I_g$, then g_i is of order p so that the order of a_i is p^m, implying that $|G_i| \geq p^m$ and hence $\alpha_i \geq m$. Moreover, if $i \in I_g$ then a_i is any element of G_i of order p^m satisfying the condition $a_i^{p^{m-1}} = g_i$.

So far we have observed that if there is a cyclic subgroup X of G of order p^m containing S then $m \leq \alpha_j$ for any $j \in I_g$ and X must be generated by an element described in the second part of this lemma.

Conversely, assume that $\alpha_i \geq m$ for any $i \in I_g$. Let $Y = \langle a_1 a_2 \dots a_r \rangle$ be any subgroup generated by an element described in this lemma. It is clear that $|Y| = p^m$, and that Y contains S because

$$(a_1 a_2 \dots a_r)^{p^{m-1}} = g.$$

To finish the proof, let C be a cyclic subgroup of G of order p^m that contains S. Then, C is generated by an element $u := a_1 a_2 \dots a_r$ of G described above. Note that $a := u^{p^{m-1}}$ is an element of C of order p, and note that generators of C satisfying the described conditions above are the elements c of C of order p^m satisfying $c^{p^{m-1}} = a$. Therefore, it follows from Remark 2.6 that C contains exactly p^{m-1} such generators. □

For any integers k and l, we denote by $\min(k,l)$ the minimum of k and l.

Proposition 2.13. *Let g be an element of G of order p, and let $g_i := \pi_i(g)$ for each $i \in I$ so that $g = g_1 g_2 \dots g_r$ with $g_i \in G_i$. Let m be a natural number, and let $S := \langle g \rangle$. Then:*

1. $c_G^m(S)$, *the number of cyclic subgroups of G of order* p^m *containing* S, *is given by*

$$c_G^m(S) = \begin{cases} \dfrac{P_G(m)}{p^{m-1}}, & \text{if } m \le \alpha_i \text{ for all } i \in I_g \\ 0, & \text{otherwise} \end{cases}$$

where $P_G(m) := \prod_{j=1}^{r} p^{\min(m-1,\alpha_j)}$.

2. $c_G(S)$, *the number of cyclic subgroups of G containing* S, *is given by*

$$c_G(S) = \sum_{m=1}^{\alpha_s} \frac{P_G(m)}{p^{m-1}}$$

where s is the smallest natural number in I_g.

Proof. (1) Because of the first part of Lemma 2.12, it is enough to consider the case in which $m \le \alpha_i$ for all $i \in I_g$. As a cyclic group of order p^α contains $p^{\min(m-1,\alpha)}$ elements of order less than p^m, we see that the number of generators

$$a_1 a_2 \ldots a_r$$

described in Lemma 2.12 is

$$\left(p^{m-1}\right)^{|I_g|} \prod_{j \in J_g} p^{\min(m-1,\alpha_j)}.$$

Here, the factor $\left(p^{m-1}\right)^{|I_g|}$ comes from Remark 2.6. As we noted in Lemma 2.12 that in any cyclic subgroup of G of order p^m that contains S there are exactly p^{m-1} such generators, which implies that $c_G^m(S)$ is the number of such generators divided by p^{m-1}. Thus,

$$c_G^m(S) = \left(p^{m-1}\right)^{|I_g|-1} \prod_{j \in J_g} p^{\min(m-1,\alpha_j)}.$$

As I is the disjoint union of I_g and J_g, and as $\min(m-1,\alpha_j) = m-1$ for all $j \in I_g$, we see that

$$P_G(m) = \left(p^{m-1}\right)^{|I_g|} \prod_{j \in J_g} p^{\min(m-1,\alpha_j)},$$

and so the result follows.

(2) Let s be the smallest natural number in I_g. Then, for all $i \in I_g$, we have $\alpha_s \le \alpha_i$, implying that $m \le \alpha_s$ if and only if $m \le \alpha_i$ for all $i \in I_g$. From the first part we then see that $c_G^m(S) \neq 0$ if and only if $m \le \alpha_s$. Therefore,

$$c_G(S) = \sum_{m=1}^{\infty} c_G^m(S) = \sum_{m=1}^{\alpha_s} c_G^m(S).$$

The result follows from the first part. □

We now aim to find the sequence $\text{seq}(G)$. We first need a technical lemma. We also use it later in an induction argument.

Lemma 2.14. *Let a, g, and h be elements of G of order p. Then:*

1. *If s is the smallest natural number in I_a then $c_G(\langle a_s\rangle) = c_G(\langle a\rangle)$, where $a_i = \pi_i(a) \in G_i$ for each $i \in I$ so that $a = a_1a_2\ldots a_r$.*

2. *Suppose that $I_g = \{k\}$ and $I_h = \{k+1\}$ for some $k \in \{1,2,3,\ldots,r-1\}$. Then,*

$$c_G(\langle h\rangle) - c_G(\langle g\rangle) = \begin{cases} \left(\prod_{j=1}^{k} p^{\alpha_j}\right)\left(\sum_{m=\alpha_k+1}^{\alpha_{k+1}} (p^{r-k-1})^{m-1}\right), & \text{if } \alpha_{k+1} \neq \alpha_k. \\ 0, & \text{if } \alpha_{k+1} = \alpha_k. \end{cases}$$

In particular, $c_G(\langle h\rangle) \geq c_G(\langle g\rangle)$.

Proof. (1) Follows from the second part of Proposition 2.13.

(2) Using the second part of Proposition 2.13 we see that

$$c_G(\langle h\rangle) - c_G(\langle g\rangle) = \begin{cases} \sum_{m=\alpha_k+1}^{\alpha_{k+1}} \frac{P_G(m)}{p^{m-1}}, & \text{if } \alpha_{k+1} \neq \alpha_k. \\ 0, & \text{if } \alpha_{k+1} = \alpha_k. \end{cases}$$

Assume for the rest that $\alpha_{k+1} \neq \alpha_k$. As $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_r$, for any m with $\alpha_k + 1 \leq m \leq \alpha_{k+1}$ we see that

$$\min(m-1, \alpha_j) = \begin{cases} \alpha_j, & \text{if } j \leq k \\ m-1, & \text{if } j > k \end{cases}$$

This shows that if $\alpha_k + 1 \leq m \leq \alpha_{k+1}$, then

$$P_G(m) = \left(\prod_{j=1}^{k} p^{\alpha_j}\right)\left(p^{m-1}\right)^{r-k}.$$

The result follows. □

Recall that the sequence $\text{seq}(G)$ is an increasing sequence whose terms are distinct numbers in the list of numbers $c_G(S)$ where S is ranging in the set of all minimal subgroups of G.

Lemma 2.15. *For each $i \in I$, let h_i be any element of G_i of order p so that $I_{h_i} = \{i\}$. Then, terms of the sequence $\text{seq}(G)$ are precisely the distinct terms of the following non-decreasing sequence of numbers:*

$$c_G(\langle h_1\rangle),\ c_G(\langle h_2\rangle),\ c_G(\langle h_3\rangle), \ldots, c_G(\langle h_{r-1}\rangle),\ c_G(\langle h_r\rangle).$$

Moreover, $c_G(\langle h_i\rangle) = c_G(\langle h_{i+1}\rangle)$ if and only if $\alpha_i = \alpha_{i+1}$.

Proof. Firstly, it follows from the second part of Lemma 2.14 that

$$c_G(\langle h_1\rangle) \le c_G(\langle h_2\rangle) \le c_G(\langle h_3\rangle) \le \cdots \le c_G(\langle h_{r-1}\rangle) \le c_G(\langle h_r\rangle),$$

and that $c_G(\langle h_i\rangle) = c_G(\langle h_{i+1}\rangle)$ if and only if $\alpha_i = \alpha_{i+1}$.

To finish the proof, it is enough to show that for any minimal subgroup S of G there is an $i \in I$ such that $c_G(S) = c_G(h_i)$. Indeed, given any minimal subgroup S of G, it follows from the first part of Lemma 2.14 and the second part of Proposition 2.13 that

$$c_G(S) = c_G(g_s) = c_G(h_s)$$

where g is any element of G such that $S = \langle g\rangle$ and s is the smallest natural number in I_g. □

Now we may state the main result of this section.

Proposition 2.16. *Let p be a prime number. Then, two non-cyclic finite abelian p-groups with isomorphic intersection graphs are isomorphic.*

Proof. Suppose that A and B are two non-cyclic finite abelian p-groups with isomorphic intersection graphs. By Proposition 2.10 the ranks of A and B are equal. So, there is a natural number $r > 1$, and there are natural numbers β_i and γ_i with

$$1 \le \beta_1 \le \beta_2 \le \cdots \le \beta_r \quad \text{and} \quad 1 \le \gamma_1 \le \gamma_2 \le \cdots \le \gamma_r$$

such that

$$A \cong \mathbb{Z}_{p^{\beta_1}} \times \mathbb{Z}_{p^{\beta_2}} \times \cdots \times \mathbb{Z}_{p^{\beta_r}} \quad \text{and} \quad B \cong \mathbb{Z}_{p^{\gamma_1}} \times \mathbb{Z}_{p^{\gamma_2}} \times \cdots \times \mathbb{Z}_{p^{\gamma_r}}.$$

Therefore, there are subgroups A_i of A and subgroups B_i of B such that A and B may be written as internal direct sums as follows:

$$A = A_1A_2\ldots A_r \quad \text{and} \quad B = B_1B_2\ldots B_r, \quad \text{where} \quad A_i \cong \mathbb{Z}_{p^{\beta_i}} \quad \text{and} \quad B_i \cong \mathbb{Z}_{p^{\gamma_i}}.$$

By comparing $\operatorname{seq}(A)$ and $\operatorname{seq}(B)$, we will show that $\beta_i = \gamma_i$ for each i.

Firstly, for each i, choose an element x_i of A_i of order p, and choose an element y_i of B_i of order p. It follows from Lemma 2.11 that $\operatorname{seq}(A)$ and $\operatorname{seq}(B)$ are the same.

Then, Lemma 2.15 implies that $c_G(\langle x_1\rangle) = c_G(\langle y_1\rangle)$. By using the second part of Proposition 2.13 we see that

$$c_A(\langle x_1\rangle) = \sum_{m=1}^{\beta_1} (p^{r-1})^{m-1} \quad \text{and} \quad c_B(\langle y_1\rangle) = \sum_{m=1}^{\gamma_1} (p^{r-1})^{m-1},$$

implying that $\beta_1 = \gamma_1$.

Using Lemma 2.15, we then see that $\beta_i = \beta_1$ for all $i \in I$ if and only if the sequence $\text{seq}(A)$ has only one term. As this observation is also true for B and as $\text{seq}(A) = \text{seq}(B)$, we conclude that $\beta_i = \beta_1$ for all $i \in I$ if and only if $\gamma_i = \gamma_1$ for all $i \in I$. Therefore, if $\beta_i = \beta_1$ for all $i \in I$ then $\gamma_i = \gamma_1$ for all $i \in I$, implying that $\beta_i = \gamma_i$ for all $i \in I$.

For the rest we assume that there is an $i \in I$ such that $\beta_i \neq \beta_1$. Let u be the smallest number in I such that $\beta_{u+1} \neq \beta_1$, and let v be the smallest number in I such that $\gamma_{v+1} \neq \gamma_1$. Note that

$$\beta_1 = \beta_2 = \cdots = \beta_u < \beta_{u+1} \quad \text{and} \quad \gamma_1 = \gamma_2 = \cdots = \gamma_v < \gamma_{v+1}.$$

Then, Lemma 2.15 implies that

$$c_A(\langle x_1\rangle) = c_A(\langle x_2\rangle) = \cdots = c_A(\langle x_u\rangle) < c_A(\langle x_{u+1}\rangle),$$

$$c_B(\langle y_1\rangle) = c_B(\langle y_2\rangle) = \cdots = c_B(\langle y_v\rangle) < c_B(\langle y_{v+1}\rangle).$$

The first two terms of the sequences $\text{seq}(A)$ and $\text{seq}(B)$ are

$$c_A(\langle x_u\rangle),\ c_A(\langle x_{u+1}\rangle) \quad \text{and} \quad c_B(\langle y_v\rangle),\ c_B(\langle y_{v+1}\rangle).$$

As $\text{seq}(A) = \text{seq}(B)$, we must have that

$$c_A(\langle x_{u+1}\rangle) - c_A(\langle x_u\rangle) = c_B(\langle y_{v+1}\rangle) - c_B(\langle y_v\rangle).$$

If we use the second part of Lemma 2.14 to calculate the above equal differences, we see that

$$\left(p^{r-1}\right)^{\beta_1} \sum_{m=0}^{\beta_{u+1}-\beta_u-1} \left(p^{r-u-1}\right)^m = \left(p^{r-1}\right)^{\gamma_1} \sum_{m=0}^{\gamma_{v+1}-\gamma_v-1} \left(p^{r-v-1}\right)^m.$$

As $\beta_1 = \gamma_1$,

$$-1 + \sum_{m=0}^{\beta_{u+1}-\beta_u-1} \left(p^{r-u-1}\right)^m = -1 + \sum_{m=0}^{\gamma_{v+1}-\gamma_v-1} \left(p^{r-v-1}\right)^m.$$

If the above equal numbers are zero, then from $\beta_u = \gamma_v$ we see that $\beta_{u+1} = \gamma_{v+1}$. Otherwise, comparing the highest powers of the prime p dividing the above equal numbers, we see that ($u = v$ and) $\beta_{u+1} = \gamma_{v+1}$. To see that $u = v$ in both cases, using Remark 2.1, Corollary 2.9, and Lemma 2.11, we note that the number of minimal subgroups S of A such that $c_A(S) > c_A(\langle x_u \rangle)$ is equal to the number of minimal subgroups T of B such that $c_B(T) > c_B(\langle y_v \rangle)$. It follows from Lemma 2.14 that the numbers of such minimal subgroups of A and B are equal to the numbers of minimal subgroups of the groups $A_{u+1}A_{u+2}\cdots A_r$ and $B_{v+1}B_{v+2}\cdots B_r$. Hence, being abelian p-groups, these groups must have isomorphic socles, implying that $u = v$.

We now continue as in the previous two paragraphs. We see that $\beta_i = \beta_{u+1}$ for all $i \geq u+1$ if and only if the sequence $\mathrm{seq}(A)$ has exactly two terms. As this observation is also true for B and as $\mathrm{seq}(A) = \mathrm{seq}(B)$, for the rest we assume that there is an $i \geq u+1$ such that $\beta_i \neq \beta_{u+1}$. Let u' be the smallest number such that $\beta_{u'+1} > \beta_{u+1}$, and let v' be the smallest number such that $\gamma_{v'+1} > \gamma_{v+1}$. Note that $u' > u = v < v'$ and

$$\beta_1 = \beta_2 = \cdots = \beta_u < \beta_{u+1} = \cdots = \beta_{u'} < \beta_{u'+1},$$

$$\gamma_1 = \gamma_2 = \cdots = \gamma_v < \gamma_{v+1} = \cdots = \gamma_{v'} < \gamma_{v'+1}.$$

The second and the third terms of the sequences $\mathrm{seq}(A)$ and $\mathrm{seq}(B)$ are

$$c_A(\langle x_{u'} \rangle),\ c_A(\langle x_{u'+1} \rangle) \quad \text{and} \quad c_B(\langle y_{v'} \rangle),\ c_B(\langle y_{v'+1} \rangle).$$

We must have that

$$c_A(\langle x_{u'+1} \rangle) - c_A(\langle x_{u'} \rangle) = c_B(\langle y_{v'+1} \rangle) - c_B(\langle y_{v'} \rangle).$$

By the second part of Lemma 2.14,

$$p^{\beta_1 u}\left(p^{r-u-1}\right)^{\beta_{u+1}} \sum_{m=0}^{\beta_{u'+1}-\beta_{u'}-1} \left(p^{r-u'-1}\right)^m = p^{\gamma_1 v}\left(p^{r-v-1}\right)^{\gamma_{v+1}} \sum_{m=0}^{\gamma_{v'+1}-\gamma_{v'}-1} \left(p^{r-v'-1}\right)^m.$$

As $u = v$ and $\beta_i = \gamma_i$ for all i with $1 \leq i \leq u+1$,

$$-1 + \sum_{m=0}^{\beta_{u'+1}-\beta_{u'}-1} \left(p^{r-u'-1}\right)^m = -1 + \sum_{m=0}^{\gamma_{v'+1}-\gamma_{v'}-1} \left(p^{r-v'-1}\right)^m.$$

As in the first part, from the above equal numbers we deduce that $\beta_{u'+1} = \gamma_{v'+1}$. To see that $u' = v'$ we may use the equality of the number of minimal subgroups S' of A

such that $c_A(S') > c_A(\langle x_{u'} \rangle)$ and the number of minimal subgroups T' of B such that $c_B(T') > c_B(\langle y_{v'} \rangle)$.

As I is a finite set, applying this procedure finitely many times we may prove that $\beta_i = \gamma_i$ for all $i \in I$. □

2.4 Proof of the Main Theorem

This section contains the main result this chapter. One half of the main result will follow from what we have proved in the previous sections. The other half will be the consequence of the following result. For any two finite groups K and L of coprime orders, the subgroups of $K \times L$ are all of the form $M \times N$ where M is a subgroup of K and N is a subgroup of L. Thus, the next result follows.

Lemma 2.17. *Let U_1, U_2 and V_1, V_2 be four groups such that $|U_1|$ and $|U_2|$ are coprime and that $|V_1|$ and $|V_2|$ are coprime. If the intersection graphs of U_i and V_i are isomorphic for each $i = 1, 2$, then the intersection graphs of the direct products $U_1 \times U_2$ and $V_1 \times V_2$ are isomorphic.*

Now we can prove our main theorem.

Proof of Theorem 2.1. ($\Rightarrow$): Suppose that the intersection graphs of A and B are isomorphic. If A is of prime power order than the result follows from Remark 2.5, Proposition 2.10, and Proposition 2.16. So, assume that $|A|$ is not a prime power. Let $P_1, P_2, \dots, P_a$ be a complete list of Sylow subgroups of A where $P_1, P_2, \dots, P_b$ are non-cyclic and all the others are cyclic. (Here, $a > 1$, and the cases $b = 0$ and $b = a$ are not excluded). Assume that each P_i is a p_i-group where p_i is a prime. It follows from Proposition 2.4 and Remark 2.5 that there is a graph isomorphism $\psi\colon \Gamma(A) \to \Gamma(B)$ such that $Q_1, Q_2, \dots, Q_a$ is a complete list of Sylow subgroups of B and $\psi\colon \Gamma(P_i) \to \Gamma(Q_i)$ is a graph isomorphism for each i, where $Q_i := \psi(P_i)$. Assume that each Q_i is a q_i-group where q_i is a prime.

The first part of Proposition 2.10 implies that $Q_{b+1}, Q_{b+2}, \dots, Q_a$ are all cyclic (and all the other Q_i are non-cyclic), and that there are natural numbers $n_{b+1}, n_{b+2}, \dots, n_a$ such that $P_i \cong \mathbb{Z}_{{p_i}^{n_i}}$ and $Q_i \cong \mathbb{Z}_{{q_i}^{n_i}}$ for each $i > b$. If we let θ be the map defined for any $i > b$ by $\theta(P_i) = Q_i$, then the condition (ii) is satisfied.

The second part of Proposition 2.10 implies that $Q_1, Q_2, \dots, Q_b$ are all non-cyclic and $p_i = q_i$ for each $i \leq b$. It then follows from Proposition 2.16 that $P_i \cong Q_i$ for each $i \leq b$. So, the condition (i) is satisfied.

($\Leftarrow$): Suppose that the conditions (i) and (ii) are satisfied. Let $S_1, S_2, \dots, S_r$ and $T_1, T_2, \dots, T_r$ be complete lists of cyclic Sylow subgroups of A and B, respectively, where $T_i = \theta(S_i)$. (Here, the case $r = 0$ is not excluded). Since the subgroup lattice of a cyclic group of prime power order is a chain, the intersection graphs of S_i and T_i are both complete graphs. As the number of divisors of $|S_i|$ and$|T_i|$ are the same, it follows that the graphs $\Gamma(S_i)$ and $\Gamma(T_i)$ are isomorphic.

For any finite abelian group G let us denote by G_{nc} the product of all non-cyclic Sylow subgroups of G. As A_{nc} and B_{nc} are isomorphic, the graphs $\Gamma(A_{nc})$ and $\Gamma(B_{nc})$ are isomorphic. Note that any two distinct Sylow subgroups of a finite abelian group have coprime orders. We now apply Lemma 2.17 r times to see that the intersection graphs of the groups $(A_{nc} \times S_1 \times S_2 \times \cdots \times S_r)$ and $(B_{nc} \times T_1 \times T_2 \times \cdots \times T_r)$ are isomorphic. As the first group is isomorphic to A and the second group is isomorphic to B, this finishes the proof. □

We finish this chapter with the following obvious consequence of Theorem 2.1, which is the main result of [17].

Corollary 2.18. *Let A and B be two finite abelian groups. Suppose that A has no cyclic Sylow subgroup. Then, if the intersection graphs of A and B are isomorphic, then A and B are isomorphic.*

3. PLANARITY OF INTERSECTION GRAPHS

In this chapter we characterize all finite groups whose intersection graphs are planar.

A graph is called planar if it can be drawn on the plane in such a way that its edges intersect only at their endpoints. Planarity of the subgroup lattice and the subgroup graph of a group were studied by Bohanon and Reid in [20] and by Schmidt in [21, 22] and by Starr and Turner III in [23], and planarity of the intersection graph of a module over any ring was studied in [1].

We call a group *planar* if its intersection graph is planar. Our main result in this chapter is the following

Theorem 3.1. *A finite group is planar if and only if it is isomorphic to one of the following groups:*

1. $\mathbb{Z}_{pqr}$, $\mathbb{Z}_{p^2q}$, $\mathbb{Z}_{pq}$, $\mathbb{Z}_{p^i}$, *where* p, q, r *are distinct primes and* $0 \leq i \leq 5$.
2. $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_p \times \mathbb{Z}_p$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_p$ $(p \neq 2)$, *where* p *is a prime.*
3. *The dihedral group* D_8 *of order* 8, *the quaternion group* Q_8 *of order* 8.
4. *The semidirect products* $\mathbb{Z}_q \rtimes \mathbb{Z}_{p^2}$ *with* $p^2 \mid q-1$, $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q$ *with* $q \mid p+1$, *where* p, q *are distinct primes. Presentations and the subgroup structures of these groups are given in Lemma 3.11.*
5. *The semidirect product* $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_{q^2}$ *with* $q^2 \mid p+1$, *where* p, q *are distinct primes. A presentation and the subgroup structure of this group are given in Lemma 3.12.*
6. *The semidirect product* $\mathbb{Z}_r \rtimes \mathbb{Z}_{pq}$ *with* $pq \mid r-1$, *where* p, q, r *are distinct primes. A presentation and the subgroup structure of this group are given in Lemma 3.13.*
7. *The semidirect product* $\mathbb{Z}_p \rtimes \mathbb{Z}_q$ *with* $q \mid p-1$, *where* $p > q$ *are distinct primes. A presentation and the subgroup structure of this group are given in Lemma 3.15.*

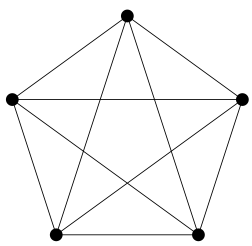
In the above theorem, up to isomorphism, the first item lists the finite cyclic planar groups, the second item lists the finite non-cyclic abelian planar groups, the third item lists the finite non-abelian nilpotent planar groups, and the remaining items list the finite non-nilpotent solvable planar groups. There are no finite non-solvable planar groups (this was proved without using CFSG).

It may be interesting to study connections between the subgroup lattice and the intersection graph of a group. It is clear that the subgroup lattice determines the intersection graph, but not conversely. Moreover, comparing our main result with the main results of [20,21], we see that there are groups whose subgroup lattices are planar but the intersection graphs are not planar, and vice versa.

3.1 Preliminaries

Let Γ be a graph. By replacing some of the edges of Γ (possibly none or all) by independent paths, we obtain another graph which is called a subdivision of Γ. Let Λ be another graph. We say Λ contains Γ as a minor if there is a subgraph of Λ which is a subdivision Γ.

Kuratowski's theorem characterizes planar graphs by means of forbidden minors: a finite graph is planar if and only if it does not contain a subdivision of either the complete graph K_5 or the complete bipartite graph $K_{3,3}$. The complete graph K_n is a simple undirected graph with n vertices in which every pair of distinct vertices is connected by a unique edge. The complete bipartite graph $K_{m,n}$ is a simple undirected graph with $m+n$ vertices and with two disjoint sets V_m and V_n containing exactly m and n vertices respectively such that there is an edge between two vertices if and only if one of them belongs to V_m and the other belongs to V_n. Figure 3.1 shows the forbidden minors of planar graphs.


(a) K_5

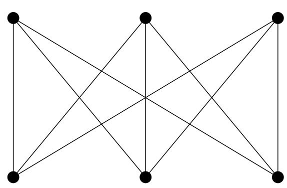

(b) $K_{3,3}$

Figure 3.1 : Forbidden minors of planar graphs.

We use Kuratowski's theorem for the planarity of intersection graphs. That is, if G is a finite group, to show that $\Gamma(G)$ is not planar, we typically try to find five proper non-trivial subgroups of G such that any pair of them intersect non-trivially and to show that $\Gamma(G)$ is planar, we simply draw it onto the plane without crossings of its edges apart from their end points. It turns out $\Gamma(G)$ is planar if and only if it does not contain K_5 and $K_{3,3}$ as a subgraph.

3.2 Solvable Groups

In this section we determine solvable groups which are planar. We first deal with abelian groups.

Modules over any ring whose intersection graphs are planar were already characterized in [1]. Notice that if $H \leq G$ and $\Gamma(G)$ does not contain a graph Γ as a subgraph/minor, then $\Gamma(H)$ also does not contain Γ as a subgraph/minor. By using this simple remark and the fundamental theorem of finite abelian groups (see [14, Theorem 6.5]), we may easily justify the following result. We will further use it in Chapter 4.

Proposition 3.2. *Let G be a finite abelian group. Then $\Gamma(G)$ does not contain $K_{3,3}$ as a subgraph if and only if G is isomorphic to one of the following groups*

$$\mathbb{Z}_{p^i}\ (0 \leq i \leq 6), \quad \mathbb{Z}_{p^3} \times \mathbb{Z}_q, \quad \mathbb{Z}_{p^2} \times \mathbb{Z}_q, \quad \mathbb{Z}_p \times \mathbb{Z}_q, \quad \mathbb{Z}_9 \times \mathbb{Z}_3, \quad \mathbb{Z}_4 \times \mathbb{Z}_2,$$

$$\mathbb{Z}_p \times \mathbb{Z}_p, \qquad \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r, \qquad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_p\ (p \neq 2)$$

where p, q, and r are distinct primes. Moreover, among those groups only

$$\mathbb{Z}_{p^6}, \qquad \mathbb{Z}_{p^3} \times \mathbb{Z}_q, \qquad \text{and} \qquad \mathbb{Z}_9 \times \mathbb{Z}_3$$

are non-planar.

Proof. Let G be a finite abelian group and p, q, and r be prime numbers.

Case I: G is a cyclic group. Then there is exactly one subgroup of G of order n for each divisor n of $|G|$. Observe that $\Gamma(G)$ does not contain $K_{3,3}$ if G is of order p^i $(0 \leq i \leq 6)$, p^2q, or pq; as the number of proper non-trivial subgroups of G would be less than six in such cases.

Case I (a): $|G| = p^i$ $(i > 6)$. Then $\Gamma(G)$ contains a K_6 and so $K_{3,3}$ as well.

Case I (b): $G \cong \mathbb{Z}_{p^3q}$. Let a and b be two elements of G of order p^3 and of order q respectively. There are exactly six proper non-trivial subgroups of G in this case and five of them, namely $\langle a^{p^2} \rangle$, $\langle a^p \rangle$, $\langle a \rangle$, $\langle a^{p^2}, b \rangle$, and $\langle a^p, b \rangle$ form a complete graph in $\Gamma(G)$ as all of them contain $\langle a^{p^2} \rangle$. The remaining vertex $\langle b \rangle$ has degree two in the intersection graph and, therefore, $\Gamma(G)$ does not contain $K_{3,3}$. Moreover, since $\Gamma(G)$ contains a K_5, it cannot be a proper subgroup of a group containing $K_{3,3}$. (Notice that in a larger group G becomes a vertex and so vertices containing $\langle a^{p^2} \rangle$ form a subgraph containing a K_6.)

Case I (c): $G \cong \mathbb{Z}_{p^2q^2}$. Let a and b be two elements of G of order p^2 and of order q^2 respectively. As in the previous case we have five subgroups forming a K_5, namely $\langle a^p \rangle$, $\langle a \rangle$, $\langle a^p, b^p \rangle$, $\langle a, b^p \rangle$, and $\langle a^p, b \rangle$. However, unlike the previous case, the subgroup $\langle b^p \rangle$ is linked by an edge with those last three subgroups forming the K_5. Therefore, $\Gamma(G)$ contains $K_{3,3}$ in this case.

Case I (d): $G \cong \mathbb{Z}_{pqr}$. Let a, b, and c be three elements of orders p, q, and r respectively. The vertices of $\Gamma(G)$ are $\langle a \rangle$, $\langle b \rangle$, $\langle c \rangle$, $\langle a, b \rangle$, $\langle a, c \rangle$, and $\langle b, c \rangle$. As the valency of $\langle a \rangle$ is two, $\Gamma(G)$ does not contain $K_{3,3}$ in this case. However, if G is a proper subgroup of a larger group, then maximal subgroups of G together with $\langle a \rangle$, $\langle b \rangle$, and G form a subgraph containing a $K_{3,3}$.

To sum up, the only possible values for the order of a cyclic group which does not contain a $K_{3,3}$ are

$$p^i \ (0 \leq i \leq 6), \qquad p^3q, \qquad p^2q, \qquad pqr, \qquad pq.$$

Case II: G is *not* a cyclic group. Let us make a useful observation. If G is an abelian group of order n that does not contain $K_{3,3}$, then n must be one of the above values. This is because for any pair of subgroups $A < B$ of $\mathbb{Z}_n$, there are corresponding subgroups $H < K$ of G such that $|A| = |H|$ and $|B| = |K|$.

Case II (a): $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. Observe that maximal subgroups of G are dimension 2 subspaces of G considering it as a vector space over $\mathbb{F}_p$. Then, by a counting argument the number of maximal subgroups of G is $[(p^3-1)(p^3-p)]/[(p^2-1)(p^2-p)] = p^2+p+1$. However, by the Product Formula (see Theorem 1.1), any pair of maximal subgroups intersects non-trivially; and hence, they form a complete graph in $\Gamma(G)$.

Thus, any group containing an elementary abelian subgroup of rank three contains $K_{3,3}$ in its intersection graph.

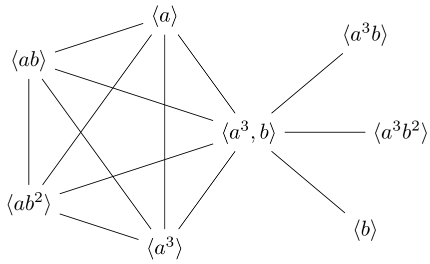
Case II (b): $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$. Let a and b be two generator elements for G of order p^2 and of order p respectively. Then, subgroups $\langle a^p \rangle$, $\langle a^p, b \rangle$, $\langle a \rangle$, $\langle ab \rangle, \ldots, \langle ab^{p-1} \rangle$ form a K_{p+2} in $\Gamma(G)$. Hence, the only possible values of p are 2 and 3 if G does not contain $K_{3,3}$. Actually, G is '$K_{3,3}$-free' for those primes. Intersection graph of $\mathbb{Z}_9 \times \mathbb{Z}_3$ is depicted in Figure 3.2(a).

Case II (c): G is an abelian p-group which is not considered in Cases II (a) and (b). If $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$ then $\Gamma(G)$ does not contain $K_{3,3}$, since the intersection graph of a group of order p^2 consists of isolated vertices. Otherwise, G has a proper subgroup H isomorphic to $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ with $p \in \{2, 3\}$ by the previous cases. Let a and b be two generator elements for H of order p^2 and of order p respectively. Since H is a proper subgroup of G, there exists an element $c \in G$ that does not lie in H. Now, if $c^p \in \langle a \rangle$, then $\langle a^p \rangle$, $\langle a^p, b \rangle$, $\langle a \rangle$, $\langle ab \rangle$, $\langle c \rangle$, and H form a K_6 in $\Gamma(G)$. And if $c^p \notin \langle a \rangle$, then $\langle a^p \rangle$, $\langle a^p, b \rangle$, $\langle a \rangle$, $\langle ab \rangle$, $\langle a^p, c \rangle$, and H form a K_6 in $\Gamma(G)$.

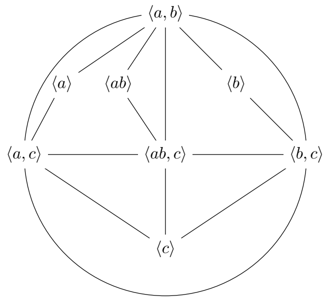
Case II (d): $|G| = p^3 q$. Since G is a non-cyclic abelian group, either $G \cong (\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p) \times \mathbb{Z}_q$ or $G \cong (\mathbb{Z}_{p^2} \times \mathbb{Z}_p) \times \mathbb{Z}_q$. However, the first case cannot occur if G does not contain $K_{3,3}$ in virtue of Case II (a); and in the latter case $\langle a^p \rangle$, $\langle a^p, b \rangle$, $\langle a \rangle$, $\langle ab \rangle$, $\langle a, b \rangle$, and $\langle a^p, c \rangle$ form a K_6 in the intersection graph where a, b, and c are some generators of G of orders p^2, p, and q respectively.

Case II (e): $G \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \times \mathbb{Z}_q$. Let a and b be two elements of G generating a subgroup of order p^2, and let c be an element of G of order q. As in Case II (b), subgroups $\langle c \rangle$, $\langle a, c \rangle$, $\langle ab, c \rangle, \ldots, \langle ab^{p-1}, c \rangle$, and $\langle b, c \rangle$ form a K_{p+2} in $\Gamma(G)$, and therefore, if G does not contain $K_{3,3}$ either $p = 2$ or $p = 3$. If $p = 2$, then G is planar and its intersection graph presented in Figure 3.2(b). However, if $p = 3$, then subgroups $\langle c \rangle$, $\langle a, c \rangle$, $\langle ab, c \rangle$, $\langle ab^2, c \rangle$, $\langle b, c \rangle$ together with $\langle a, b \rangle$ form a subgraph containing $K_{3,3}$.

As abelian groups of order pqr and of order pq are necessarily cyclic, this completes the proof of the first part. It is also easy to show that the intersection graphs of $\mathbb{Z}_{p^6}, \mathbb{Z}_{p^3} \times \mathbb{Z}_q$, and $\mathbb{Z}_9 \times \mathbb{Z}_3$ contains K_5 in their intersection graphs, hence they cannot be planar. The second part of the lemma can be justified by simply drawing the intersection graphs of the remaining groups. □



(a) $\Gamma(\mathbb{Z}_9 \times \mathbb{Z}_3)$



(b) $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_p)$

Figure 3.2 : $\Gamma(\mathbb{Z}_9 \times \mathbb{Z}_3)$ and $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_p)$.

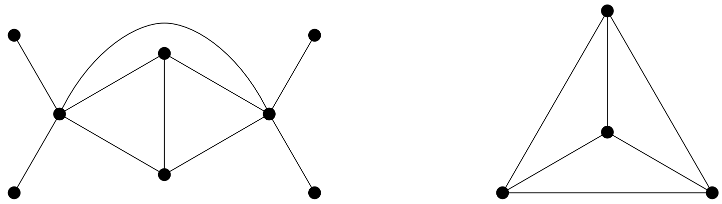
Lemma 3.3. *Let p be a prime number and G be a non-cyclic group of order p^4. Then G is not planar.*

Proof. Since a finite group having exactly one maximal subgroup must be cyclic, it follows from Theorem 1.3 that there are at least three maximal subgroups of G, say X_1, X_2 and X_3. Since G is a p-group of order p^4, each X_i is of order p^3 and the product of any two of them is G. Employing the Product Formula, $Y = X_1 \cap X_2$ is of order p^2 and it intersects X_3 non-trivially. Let Z be a non-trivial subgroup of $X_3 \cap Y$ of order p (note that the order of $X_3 \cap Y$ is either p or p^2). Now, X_1, X_2, X_3, Y and Z form a K_5 in the intersection graph of G, so that G is not planar. □

Lemma 3.4. *Let p be an odd prime and G be a non-cyclic group of order p^3. Then G is not planar.*

Proof. Since $p > 2$, arguing as in the proof of Lemma 3.3, we first conclude that there are at least four maximal subgroups of G, say X_1, X_2, X_3 and X_4, of order p^2. Assume that $Y = X_1 \cap X_2 \cap X_3 \cap X_4$ is non-trivial, then this group together with X_1, X_2, X_3 and X_4 form a K_5 in the intersection graph. Now let us assume that Y is trivial. In this case $\Phi(G) = 1$ where $\Phi(G)$ denotes the Frattini subgroup of G. Since G is a p-group, $G/\Phi(G)$ is elementary abelian. Thus, if $\Phi(G) = 1$ and $|G| = p^3$ then $G \cong G/\Phi(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ which is not planar, because $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ is not listed in Proposition 3.2. □

Up to isomorphism, there are exactly 5 distinct groups of order 8 and only two of them, namely D_8 (dihedral group of order 8) and Q_8 (quaternion group), are non-abelian. Both groups, whose intersection graphs are given in Figure 3.3, are planar.



(a) $\Gamma(D_8)$ (b) $\Gamma(Q_8)$

Figure 3.3 : Non-abelian planar nilpotent groups.

It is clear that if H is a proper subgroup of G and the intersection graph of H contains K_4, then G cannot be planar, because there would be a K_5 in the graph. With this simple remark we have:

Proposition 3.5. *A finite non-abelian nilpotent group is planar if and only if it is isomorphic to D_8 or Q_8.*

Proof. Suppose that G is a finite non-abelian nilpotent group which is planar. Since a nilpotent group is the direct product of its Sylow subgroups, there must be a non-abelian Sylow subgroup S of G. Let $|S| = p^\alpha$ for some prime p and natural number α. Since S is non-abelian, $\alpha \geq 3$. As S must be planar, it follows from Lemma 3.3 and Lemma 3.4 that $\alpha = 3$ and $p = 2$, which means S must be isomorphic to D_8 or Q_8. In both cases the intersection graph of S contains K_4. Therefore S cannot be a proper subgroup of G, and so $G = S$. □

Since a subgroup of a planar group is planar, the following lemma is an easy consequence of Propositions 3.2 and 3.5.

Lemma 3.6. *Let G be a finite planar group of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ where $k > 2$ and p_i are distinct prime numbers. Then $\alpha_j < 5$ for any j. Moreover, if $\alpha_j = 3$ or $\alpha_j = 4$ for some j then any Sylow p_j-subgroup of G is cyclic.*

Proof. There is only one planar group of order p^5, namely $\mathbb{Z}_{p^5}$, and only one planar group of order p^4, namely $\mathbb{Z}_{p^4}$, and four planar groups of order p^3, namely $\mathbb{Z}_{p^3}$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, D_8 and Q_8 (see Propositions 3.2 and 3.5). But, each of $\mathbb{Z}_{p^5}$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, D_8 and Q_8 contains K_4 in its intersection graph. □

A finite solvable group is a group with a composition series whose factor groups are of prime order. This means that if G is a planar solvable group of order $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$

where p_i are distinct prime numbers, then $\alpha_1+\alpha_2+\ldots+\alpha_k<6$; otherwise, there must be a chain of five proper non-trivial subgroups forming a K_5 in the intersection graph. Hence, for a finite solvable group G there are finitely many cases that must be examined, and these cases are given in Table 3.1, whose first row consist of p-groups and they are already classified.

Table 3.1 : Possible orders of a finite planar solvable group G.

$\lvert G\rvert=p^5$	$\lvert G\rvert=p^4$	$\lvert G\rvert=p^3$	$\lvert G\rvert=p^2$
$\lvert G\rvert=p^4q$	$\lvert G\rvert=p^3q$	$\lvert G\rvert=p^2q$	$\lvert G\rvert=pq$
$\lvert G\rvert=p^3q^2$	$\lvert G\rvert=p^2q^2$	$\lvert G\rvert=pqr$	
$\lvert G\rvert=p^3qr$	$\lvert G\rvert=p^2qr$		
$\lvert G\rvert=p^2q^2r$	$\lvert G\rvert=pqrt$		
$\lvert G\rvert=p^2qrt$			
$\lvert G\rvert=pqrtu$			

Note that the groups in Lemmas 3.7-3.8 and 3.11-3.13 are all solvable by the virtue of Theorems 1.7 and 1.8.

We say that non-trivial subgroups $H_1,H_2,\ldots,H_n$ of a group G are pairwise intersecting if $H_i\cap H_j\neq 1$ for any $i,j\in\{1,\ldots,n\}$.

We first eliminate groups of order p^3q and of order p^4q as non-planar groups.

Lemma 3.7. *If G is a group of order p^3q or p^4q where p and q are distinct prime numbers, then G is not planar.*

Proof. We prove the assertion for groups of order p^3q. Similar arguments apply for groups of order p^4q. Let P be a Sylow p-subgroup of G and let Q be a Sylow q-subgroup of G. By Lemma 3.6, we see that P is cyclic, otherwise G is not planar. Take a chain $A_1<A_2<P$ where $|A_1|=p$ and $|A_2|=p^2$. We have three cases to analyze: in the first case P is normal in G; in the second case Q is normal in G; and in the third case both P and Q are not normal in G.

Case I: Assume that P is normal in G. As any subgroup of a normal cyclic subgroup is also a normal subgroup, each A_i is normal in G, implying that the products A_iQ are subgroups of G. It is now clear that the five subgroups A_1, A_2, P, A_1Q, A_2Q are pairwise distinct and each of them contains A_1. Consequently, the graph of G contains K_5, and so G is not planar.

Case II: Assume that Q is normal in G. In this case the products A_iQ are subgroups of G. So, as in the first case, A_1, A_2, P, A_1Q and A_2Q form a K_5 in the graph of G, and so G is not planar.

Case III: Assume that both of P and Q are not normal in G. Let N be a minimal normal subgroup of G. Since G is solvable, N is an elementary abelian r-group for some prime r (that is, a direct product of cyclic groups $\mathbb{Z}_r$). As Q is a not normal, $r = p$. Therefore, N is an elementary abelian p-group inside the cyclic p-group P. This shows that $N \cong \mathbb{Z}_p$. Now take a subgroup T such that $N < T < P$ where $|T| = p^2$. As each Sylow p-subgroup of G contains N, we see that each Sylow p-subgroup of G intersects T. Consequently, all the Sylow p-subgroups together with the subgroups N and T are pairwise intersecting and pairwise distinct. As P is not normal, there are at least $p+1$ Sylow p-subgroups. Therefore, in the above we have at least $p+3$ pairwise distinct and pairwise intersecting subgroups. As $p+3 \geq 5$, G cannot be planar. □

Lemma 3.8. *Let G be a group of order p^3q^2 where p and q are distinct prime numbers. Then G is not planar.*

Proof. Since G is solvable, there must be a (normal) subgroup H of order either p^2q^2 or p^3q. By Lemma 3.7 we eliminate the latter case. Then H has a subgroup K of order either p^2q or pq^2. Let X be a subgroup of K of order p, and let P be a Sylow p-subgroup of G containing X. As $|X| = p$ and $|P| = p^3$, we may choose a subgroup Y of G such that $X < Y < P$ with $|Y| = p^2$. Then, H, K, X, Y, P form a K_5 in the intersection graph of G. □

Let G be a finite group and let N be a non-trivial normal subgroup of G. If G/N has at least five proper subgroups, then by the Correspondence Theorem (see Theorem 1.4) G has at least five proper subgroups all containing N and these subgroups form a K_5 in the intersection graph of G. The groups having exactly m subgroups where $m \leq 6$ are classified in [24].

Proposition 3.9 (see [24]). *A non-abelian group has at least* 6 *subgroups.*

It follows easily from the previous result that the center $Z(G)$ of any non-nilpotent planar group G is trivial. Another immediate consequence of the classification in [24] is the following.

Lemma 3.10. *Let G be a finite planar group and let N be a non-trivial normal subgroup of G. Then G/N is abelian. Moreover, letting n_s be the number of proper non-trivial subgroups of G/N, the following occur:*

1. $n_s = 0 \quad \Rightarrow \quad G/N \cong \mathbb{Z}_p$
2. $n_s = 1 \quad \Rightarrow \quad G/N \cong \mathbb{Z}_{p^2}$
3. $n_s = 2 \quad \Rightarrow \quad G/N \cong \mathbb{Z}_{p^3}$ *or* $G/N \cong \mathbb{Z}_{pq}$
4. $n_s = 3 \quad \Rightarrow \quad G/N \cong \mathbb{Z}_{p^4}$ *or* $G/N \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

for some distinct prime numbers p and q.

We use the above result to reduce the number of possible cases for the order of a finite planar solvable group. Let G be a finite planar solvable group, and let N be a minimal normal subgroup of G. Then N must be a planar elementary abelian s-group where s is a prime number. It follows from Proposition 3.2 that N is isomorphic to $\mathbb{Z}_s$ or $\mathbb{Z}_s \times \mathbb{Z}_s$. Moreover, G/N must be isomorphic to one of the groups described in Lemma 3.10. Therefore, the solvable groups of order p^3qr, p^2q^2r, $pqrt$, p^2qrt and $pqrtu$ given in Table 3.1 cannot be planar.

Lemma 3.11. *Let G be a non-nilpotent group of order p^2q where p and q are distinct prime numbers. Then, G is planar if and only if it is isomorphic to one of the following groups:*

1. $$\mathbb{Z}_q \rtimes_\alpha \mathbb{Z}_{p^2} = \langle a, b \mid a^q = b^{p^2} = 1, bab^{-1} = a^\alpha \rangle$$

   where p^2 divides $q-1$ and α is any integer not divisible by q whose order in the unit group $\mathbb{Z}_q^*$ of $\mathbb{Z}_q$ is p^2. (Moreover, such a group has exactly q subgroups of order p^2 which are all cyclic and pairwise non-intersecting, and has exactly 1 subgroup of order q, and has exactly 1 subgroup of order pq, and has exactly q subgroups of order p).

2. $$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes_\beta \mathbb{Z}_q = \langle a, b, c \mid a^p = b^p = c^q = 1, ab = ba, cac^{-1} = b, cbc^{-1} = a^{-1}b^\beta \rangle$$

where q divides $p+1$ and β is any integer such that the matrix $\theta = \begin{bmatrix} 0 & -1 \\ 1 & \beta \end{bmatrix}$ has order q in the group $GL(2,\mathbb{Z}_p)$ and such that θ has no eigenvalue in $\mathbb{Z}_p$. (Moreover, such a group has exactly 1 subgroup of order p^2 which is elementary abelian, and has exactly p^2 subgroups of order q, and has exactly $p+1$ subgroups of order p, and has no subgroup of order pq).

Proof. Let G be a non-nilpotent group of order p^2q. Let P be a Sylow p-subgroup of G and Q be a Sylow q-subgroup of G. We separate the proof into two parts. In the first case we assume that P is not normal in G and in the second case we assume that P is normal in G.

Case I: Assume that P is not normal: As P is a maximal subgroup and as it has order p^2, we see that the center of $N_G(P)$ is P, from which we conclude by applying BNCT (see Theorem 1.12) that Q is normal in G. Moreover, the Sylow Theorems (see Theorem 1.3) imply that G has q Sylow p-subgroups $P_1,..,P_q$ and $q \equiv 1 \pmod p$. As $|P| = p^2$, there are two possibilities: $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$ or $P \cong \mathbb{Z}_{p^2}$.

Case I (a): Assume that $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$: As $G = PQ$ and Q is normal in G, we see that $G/Q \cong P$. Now P has $p+1$ subgroups of order p, and hence by the Correspondence Theorem G has $p+1$ subgroups $R_1, R_2, \ldots, R_{p+1}$ of order pq (all of which contain Q). As $|P_i||R_j| = p^2pq > |G|$, we see that $P_i \cap R_j \neq 1$. Since both of q and $p+1$ are greater than or equal to 3, we see that P_1, P_2, P_3 and R_1, R_2, R_3 form a $K_{3,3}$ in the intersection graph of G. Hence, G is not planar in this case.

Case I (b): Assume that $P \cong \mathbb{Z}_{p^2}$:

Case I (b) (i): Assume that $P_i \cap P_j \neq 1$ for some distinct i and j : Let $X = P_i \cap P_j$. Then $N_G(X)$ contains both of P_i and P_j, implying that X is a normal subgroup of G of order p. Therefore, X is in every Sylow p-subgroups of G. Hence, $P_1, P_2, \ldots, P_q$ and X and QX form a K_{q+2} in the intersection graph of G. Note that as $q \equiv 1 \pmod p$, $q+2 \geq 5$. So, G is not planar in this case.

Case I (b) (ii): Assume the contrary of the previous case: That is, we assume that the intersection of any two distinct Sylow p-subgroups is trivial. As $G/Q \cong P$ and P is cyclic, G/Q has a unique subgroup of order p. From the Correspondence Theorem G has a unique subgroup of order pq. So, in this case, it is clear that G is planar, and its intersection graph is given in Figure 3.4(a). To write a presentation of G let a be a generator of Q and b be a generator of P. Then $bab^{-1} = a^{\alpha}$ for some integer α. For any natural number k, it is easy to see that $b^k ab^{-k} = a^{\alpha^k}$. This shows that $\alpha^{p^2} \equiv 1 \pmod{q}$. Moreover, $\alpha^p \not\equiv 1 \pmod{q}$, otherwise the intersection of any two Sylow p-subgroups of G is not trivial. Conversely, it is clear that any group with the given presentation has the stated subgroup structure.

Case II: Assume that P is normal in G. As G is not nilpotent, Q cannot be normal in G. We have two possibilities either there is a subgroup of G of order pq or there is no such subgroup.

Case II (a): Assume that there is a subgroup of G of order pq :

Case II (a) (i): Assume that there is a normal subgroup of G of order pq, say Y. Then Sylow q-subgroups of Y and G are the same, implying that Y and hence G has p Sylow q-subgroups $Q_1, Q_2, \ldots, Q_p$, and $p \equiv 1 \pmod{q}$. Note that the normalizers $N_G(Q_1), N_G(Q_2), \ldots, N_G(Q_p)$ must be pairwise distinct, because each $N_G(Q_i)$ has a unique Sylow q-subgroup which is Q_i. Moreover, they all have order pq. As the normalizer of a Sylow subgroup is self-normalizing (see Lemma 1.15), each $N_G(Q_i)$ is not normal in G. Therefore, we see that the $p+2$ subgroups $P, Y, N_G(Q_1), N_G(Q_2), \ldots, N_G(Q_p)$ are pairwise distinct and intersecting, forming a K_{p+2} in the intersection graph of G. Since $p+2 \geq 5$ (because $p \equiv 1 \pmod{q}$), G is not planar.

Case II (a) (ii): Assume that there is a non-normal subgroup of G of order pq, say Z. Its index p cannot be the smallest prime dividing the order of G. Hence, $p > q$. Let U be a Sylow p-subgroup of Z. Then U must be normal in Z. Note that U is contained in P (because P is normal in G) and that U is normal in P (because P is abelian). Therefore, U is normal in G. It follows from Proposition 3.9 and the explanation given before it that if the quotient group G/U is not abelian, then G is not planar. On the other hand, if G/U is abelian, then the Correspondence Theorem implies that G has a normal subgroup of order pq. We know from the previous subcase that in this case G is not planar.

Case II (b): Assume that there is no subgroup of G of order pq : In this case it is clear that G is planar. Moreover, P cannot be cyclic. Otherwise, its unique subgroup T of order p will be a normal subgroup of G, implying that the quotient group G/T will have a subgroup of order q, and hence G will have a subgroup of order pq. Therefore, $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$. As G has no subgroup of order pq, the normalizer of a Sylow q-subgroup of G must have index p^2. The subgroup Q acts by conjugation on the set of all subgroups of G of order p. As G has no subgroup of order pq, this action has no fixed point, implying that q divides $p+1$.

To write a presentation of G, let a be an element of P of order p and let c be a generator of Q. As G has no subgroup of order pq, the elements a and cac^{-1} must generate P. Letting $b := cac^{-1}$, it is enough to determine cbc^{-1} in terms of a and b. Now $cbc^{-1} = a^{\gamma}b^{\beta}$ for some integers γ and β. Conjugation by c induces an invertible linear operator f on the vector space P over the field $\mathbb{Z}_p$ and the matrix of f with respect to the basis $\{a,b\}$ of P is $\theta = \begin{bmatrix} 0 & \gamma \\ 1 & \beta \end{bmatrix}$.

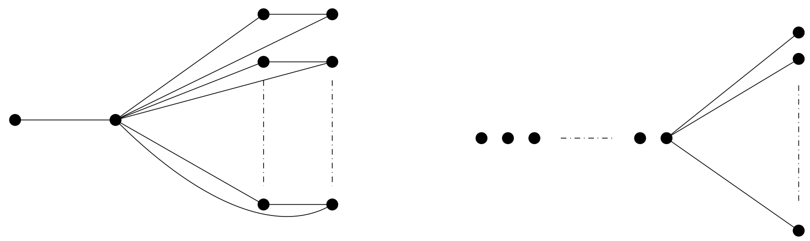
Note that G has no subgroup of order pq if and only if f has no eigenvalue in $\mathbb{Z}_p$: Indeed, if $1 \neq s \in P$ is an eigenvector of f corresponding to an eigenvalue $\lambda \in \mathbb{Z}_p$, then $f(s) = s^{\lambda}$, implying that $\langle c\rangle\langle s\rangle$ is a subgroup of G (because $\langle c\rangle\langle s\rangle = \langle s\rangle\langle c\rangle$) of order pq. Conversely, if G has a subgroup of order pq, then conjugating it by an element of G we see that there is a subgroup H of G of order pq which contains Q. Therefore, $H = \langle t\rangle\langle c\rangle$ for some $1 \neq t \in P$. As H is a subgroup, $ct = t^m c^n$ for some integers m and n. Then $t^{-m}f(t) = c^{n-1} \in P \cap Q = 1$, implying that $f(t) = t^m$.

As the order of c is prime q, the order of θ in $GL(2,\mathbb{Z}_p)$ must be q. Considering the determinants we see from the equation $\theta^q = I$ that the possibilities for the order of $-\gamma$ in $\mathbb{Z}_p^*$ is 1 or q. Suppose for a moment that the order of $-\gamma$ is q. Then q divides $p-1$, implying that $q = 2$ (because we already know that q divides $p+1$). But then $\theta^2 = I$ implies that $\theta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which has an eigenvalue in $\mathbb{Z}_p$. Therefore, the order of $-\gamma$ must be 1, implying that $\gamma \equiv -1 \pmod{p}$.

Conversely, it is clear that any group with the given presentation has the stated subgroup structure. The intersection graph of such a group is given in Figure 3.4(b). □

The prime q in the second part of the previous lemma cannot be 2. Indeed, it is easy to see that $\theta^2 \neq I$ where θ is the matrix in Lemma 3.11. Therefore, there is no planar group of order $2p^2$ where p is an odd prime.

Groups of order p^2q were classified by Hölder (see [25], [26, p. 76], [27], or [28]). The previous lemma may also be justified by analyzing the cases described in these references.



(a) $\Gamma(\mathbb{Z}_q \rtimes_\alpha \mathbb{Z}_{p^2})$ (b) $\Gamma((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes_\beta \mathbb{Z}_q)$

Figure 3.4 : Non-nilpotent planar groups of order p^2q.

Lemma 3.12. *Let G be a non-nilpotent group of order p^2q^2 where $p > q$ are distinct prime numbers. Then, G is planar if and only if it is isomorphic to*

$$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes_\beta \mathbb{Z}_{q^2} = \langle a,b,c \mid a^p = b^p = c^{q^2} = 1, ab = ba, cac^{-1} = b, cbc^{-1} = a^{-1}b^\beta \rangle$$

where q^2 divides $p+1$ and β is any integer such that the matrix $\theta = \begin{bmatrix} 0 & -1 \\ 1 & \beta \end{bmatrix}$ has order q^2 in the group $GL(2,\mathbb{Z}_p)$ and such that θ^q has no eigenvalue in $\mathbb{Z}_p$. (Moreover, such a group has exactly 1 subgroup of order p^2q, and has no subgroup of order pq^2, and has exactly 1 subgroup of order p^2 which is elementary abelian, and has exactly p^2 subgroups of order q^2 which are all cyclic and pairwise non-intersecting, and has no subgroup of order pq, and has exactly $p+1$ subgroups of order p, and has exactly p^2 subgroups of order q).

Proof. Assume that G is planar. Let P be a Sylow p-subgroup of G and Q be a Sylow q-subgroup of G. We have the following subgroup structure for G :

(I) G has no normal subgroup of prime order: This follows from Lemma 3.10.

(II) P is normal in G and Q is not normal in G : It is clear that the intersection of any two distinct Sylow p-subgroups P_1 and P_2 of G is a normal subgroup of G of order p. The normality of $P_1 \cap P_2$ may be seen easily by considering the normalizer of $P_1 \cap P_2$. Therefore, it follows from (I) that P is normal in G. As G is not nilpotent, Q is not normal in G.

(III) $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$: Using Lemma 3.10 and (I) and (II) we see that P is a minimal normal subgroup of G.

(IV) $Q \cong \mathbb{Z}_{q^2}$: It follows from Lemma 3.10 that $G/P \cong \mathbb{Z}_{q^2}$ or $G/P \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. In the case $G/P \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, there are 3 subgroups of G/P of order 2. Hence, there are 3 subgroups X_1, X_2, X_3 of G of order $2p^2$ all of which contain P. Letting Y be any subgroup of P of order p, we see that the groups X_1, X_2, X_3, P, Y form a K_5 in the intersection graph of G so that G is not planar.

(V) G has exactly one subgroup of order p^2q : As P is normal in G, any subgroup of G of order divisible by p^2 contains P. So, the number of subgroups of G of order p^2q is equal to the number of subgroups of $Q \cong G/P$ of order q. The result follows because Q is cyclic by (IV).

(VI) G has no subgroup of order pq^2 and has no subgroup of order pq : There is a unique subgroup of G of order p^2q by (V). This subgroup, say H, must be planar and P is the unique Sylow p-subgroup of H. As P is elementary abelian, H must be isomorphic to the second group found in Lemma 3.11. In particular, H has no normal subgroup of order p and has no subgroup of pq. Now, suppose for a moment that there is a subgroup U of G of order pq^2 or pq. Note that U is not in H. As P is an abelian normal subgroup of G, we see that $PN_G(U) \leq N_G(P \cap U)$. Considering the order of the subgroup $PN_G(U)$ and the uniqueness of H, we see that $H \leq PN_G(U)$. Therefore, $P \cap U$ is a normal subgroup of H of order p, which is impossible.

(VII) G has exactly p^2 subgroups of order q^2, all of which are cyclic: As Q is not normal, this follows from (IV), (VI), and the Sylow Theorems.

(VIII) The intersection of any two distinct subgroups of order q^2 is trivial: Otherwise the intersection is a subgroup of G of order q such that the order of the normalizer of the intersection is pq^2 or p^2q^2. It follows from (VI) and (I) that each of the two cases is impossible.

(IX) G has exactly p^2 subgroups of order q : This follows from (VII) and (VIII).

(X) q^2 divides $p+1$: Q acts by conjugation on the set of all subgroups of G of order p. Since by (VI) there is no subgroup of G of order pq^2 or pq, the stabilizer of any subgroup of G of order p must be the trivial subgroup of Q. Therefore, each orbit has cardinality q^2.

Conversely, it is clear that any group satisfying the above properties (I)-(X) is planar, and its intersection graph is given in Figure 3.5(a). On the left the vertices represent the subgroup of order p^2 and the $p+1$ subgroups of order p, and on the rightmost two columns the vertices represent subgroups of order q and of order q^2.

Finally, we may argue as in the proof of the second part of Lemma 3.11 to see that such a group has the given presentation. Indeed, let a be an element of P of order p and let c be a generator of Q. As G has no subgroup of order pq^2, the elements a and $b := cac^{-1}$ form a basis for the vector space P over $\mathbb{Z}_p$. Now $cbc^{-1} = a^\gamma b^\beta$ for some integers γ and β. The matrix of the conjugation on P by c is $\theta = \begin{bmatrix} 0 & \gamma \\ 1 & \beta \end{bmatrix}$. The order of θ in $GL(2,\mathbb{Z}_p)$ must be q^2 because c has order q^2 and G has no subgroup of order pq. The order of $(-\gamma)$ in $\mathbb{Z}_p^*$ is 1 or q or q^2. We see easily that the order is not q^2 (otherwise q^2 divides 2) and is not q (otherwise, $q = 2$ and $\gamma^2 = 1$ in $\mathbb{Z}_p$, and $\theta^4 = I$ implies that $\beta = 0$ in $\mathbb{Z}_p$, and so θ^2 is diagonal, implying that G has a subgroup of order pq). Therefore, $\gamma \equiv -1 \pmod p$. Moreover, as G has no subgroup of order pq we have to assume that θ^q (implying that θ) has no eigenvalue in $\mathbb{Z}_p$. □

The prime q in the previous lemma cannot be 2. Indeed, $\theta^4 = I$ implies that θ^2 is diagonal, and so G has a subgroup of order pq, where θ is the matrix in Lemma 3.12. Therefore, there is no planar group of order $4p^2$ where p is an odd prime.

Groups of order p^2q^2 were determined by Le Vavasseur in [29]. The previous result may also be proved by analyzing the cases given there.

Lemma 3.13. *Let G be a non-nilpotent group of order pqr where $p < q < r$ are distinct prime numbers. Then, G is planar if and only if it is isomorphic to*

$$\mathbb{Z}_r \rtimes_\alpha \mathbb{Z}_{pq} = \langle a,b \mid a^r = b^{pq} = 1, bab^{-1} = a^\alpha \rangle$$

where pq divides $r-1$ and α is any integer not divisible by r whose order in the unit group $\mathbb{Z}_r^*$ of $\mathbb{Z}_r$ is pq. (Moreover, such a group has exactly 1 subgroup of order pr, and has exactly 1 subgroup of order qr, and has exactly r subgroups of order pq, which are all cyclic and pairwise non-intersecting, and has exactly 1 subgroup of order r, and has exactly r subgroups of order p, and has exactly r subgroups of order q).

Proof. The Sylow Theorems imply that G has a unique Sylow r-subgroup R. Assume first that G is planar. We have the following subgroup structure for G :

(I) G has exactly 1 subgroup of order pr and exactly 1 subgroup of order qr : From Lemma 3.10, we see that $G/R \cong \mathbb{Z}_{pq}$. Thus G/R has exactly one subgroup of order p and q. Since any subgroup of G of order divisible by r contains R, the result follows.

(II) G has exactly r subgroups of order pq, which are all cyclic and pairwise non-intersecting: Let X be a Hall subgroup of G of order pq. By (I) there are unique subgroups of G of order pr and qr, say Y and Z. If X is normal in G, then $X \cap Y \cap Z = 1$ so that G is isomorphic to a subgroup of the cyclic group $\mathbb{Z}_r \times \mathbb{Z}_q \times \mathbb{Z}_p$. Therefore, X is not normal in G. As any two Hall subgroups of a finite solvable group of the same order are conjugate [15, p. 231, Theorem 4.1], there are exactly r subgroups of G of order pq. Moreover, X must be cyclic because $X \cap Y$ and $X \cap Z$ are normal subgroups of X of orders p and q whose product is X. Finally, let X_1 and X_2 be two subgroups of G of order pq such that $X_1 \cap X_2 \neq 1$. Then we see that $X_1 \cap X_2$ is a normal subgroup of G of order p or q. But then $X_1 \cap X_2$ must be contained in each of r subgroups of G of order pq. Therefore, the intersection graph of G contains K_r. As $r \geq 5$, in this case G is not planar.

(III) G has exactly r subgroups of order p and r subgroups of order q : As any subgroup of G of order p or q is contained in a subgroup of G of order pq, the result follows from (II).

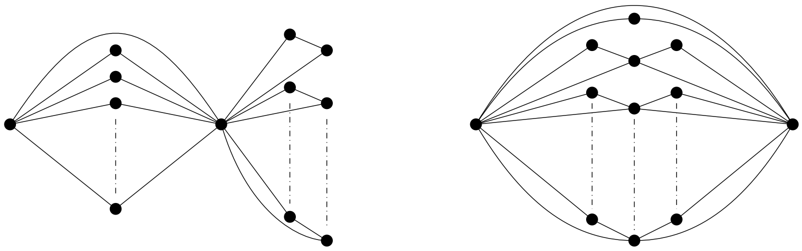
(IV) pq divides $r-1$: It follows from (III) and the Sylow Theorems that $r \equiv 1 \pmod{p}$ and $r \equiv 1 \pmod{q}$. The result follows.

Conversely, it is clear that any group satisfying the above properties (I)-(IV) is planar, and its intersection graph is given in Figure 3.5(b). The uppermost middle vertex represents the unique subgroup of G of order r, and the leftmost and the rightmost vertices represent the unique subgroups of G of order pr and qr.

Finally, we may argue as in the proof of the first part of Lemma 3.11 to see that such a group has the given presentation. □

Groups of order pqr were classified by Hölder (see [25]). One may also analyze the cases there to prove the previous result.

Lemma 3.14. *Let G be a non-nilpotent solvable group of order p^2qr where p, q and r are distinct prime numbers. Then G is not planar.*



(a) $\Gamma((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes_\beta \mathbb{Z}_{q^2})$ (b) $\Gamma(\mathbb{Z}_r \rtimes_\alpha \mathbb{Z}_{pq})$

Figure 3.5 : Non-nilpotent planar groups of orders p^2q^2 and pqr.

Proof. Assume for a moment that G is planar. Let P be a Sylow p-subgroup of G. We have the following subgroup structure for G :

(I) P is normal in G and $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and $G/P \cong \mathbb{Z}_{qr}$: As any minimal normal subgroup of a finite solvable group is elementary abelian group of prime power order, the result follows from Lemma 3.10.

(II) G has exactly 1 subgroup A of order p^2q and has exactly 1 subgroup B of order p^2r. Moreover, both A and B contain P : As P is normal, any subgroup of G of order divisible by p^2 must contain P. The result follows from (I) which implies that G/P has exactly 1 subgroup of order q and has exactly 1 subgroup of order r.

(III) The intersection graph of G contains $K_{3,3}$: It follows from (I) that P has exactly $p+1$ subgroups of order p. Take any 3 distinct subgroups of P of order p, say X_1, X_2, X_3. Then, it is clear from (II) that the intersection of any element of the set $\{X_1, X_2, X_3\}$ with any element of the set $\{A, B, P\}$ is not trivial. Thus, the intersection graph of G contains $K_{3,3}$.

Finally, we note that (III) contradicts the planarity of G. □

Finally, if G is a group of order pq where $p > q$ are prime numbers, then any proper non-trivial subgroup of G is of prime order, and so there is no edge in the intersection graph of G. Therefore, any such group is planar, and we have the following easy consequence of the Sylow Theorems.

Lemma 3.15. *Let G be a group of order pq where $p > q$ are distinct primes. Then, G is planar. If G is non-nilpotent, then q divides $p-1$ and G is isomorphic to*

$$\mathbb{Z}_p \rtimes_\alpha \mathbb{Z}_q = \langle a, b \mid a^p = b^q = 1, bab^{-1} = a^\alpha \rangle$$

where α is any integer not divisible by p whose order in the unit group $\mathbb{Z}_p^*$ of $\mathbb{Z}_p$ is q. (Moreover, such a group has exactly 1 *subgroup of order p and has exactly p subgroups of order q).*

3.3 Non-solvable Groups

In this section we show that any non-solvable finite group is not planar.

Lemma 3.16. *If G is a finite non-solvable simple group then G is not planar.*

Proof. Suppose contrarily that G is a finite non-solvable simple group which is planar. Then we have:

(I) Any Sylow subgroup of G is abelian: Let P be a Sylow p-subgroup of G for some prime p dividing $|G|$. As P is planar, it follows from Propositions 3.2 and 3.5 that P is isomorphic to one of the groups $\mathbb{Z}_{p^\alpha}$ $(\alpha \leq 5)$, $\mathbb{Z}_p \times \mathbb{Z}_p$, D_8, Q_8. However, the intersection graph of any of the groups $\mathbb{Z}_{p^5}$, D_8, Q_8 contains a K_4. Therefore, P must be isomorphic to one of the groups $\mathbb{Z}_{p^\alpha}$ $(\alpha \leq 4)$, $\mathbb{Z}_p \times \mathbb{Z}_p$.

(II) For any non-trivial Sylow subgroup P of G, its normalizer $N_G(P)$ is a non-abelian proper subgroup of G : As G is simple, the result follows from BNCT (see Theorem 1.12).

(III) If P is a Sylow p-subgroup of G for some prime p dividing $|G|$, then P is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ or $\mathbb{Z}_p$. Moreover, if P is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ then $N_G(P)$ is a non-nilpotent group of order p^2q isomorphic to the group described in the second part of Lemma 3.11: Suppose that $P \cong \mathbb{Z}_{p^\alpha}$ where $\alpha \geq 2$. The unique subgroup C of the cyclic group P of order p must be normal in $N_G(P)$. Moreover, $P \neq N_G(P)$ by (I)-(II). It then follows from Lemma 3.10 that $N_G(P)/C$ is isomorphic to $\mathbb{Z}_{pr}$ and $\alpha = 2$ where r is a prime number different from p. So $N_G(P)$ is a non-abelian planar group of order p^2r having a normal cyclic subgroup of order p^2, which is impossible by the virtue of Lemma 3.11. Consequently, it follows from the proof of (I) that P is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ or $\mathbb{Z}_p$. Suppose now that P is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. If $N_G(P)/P$ has a proper non-trivial subgroup X/P, then the set $\{N_G(P), X, P\}$ and the set consisting of any distinct three subgroups of P of order p form a $K_{3,3}$ in the intersection graph of G. Hence, $N_G(P)/P$ must have prime order q so that $N_G(P)$ is a non-abelian planar group of order p^2q.

(IV) If $|G|$ is even then any Sylow 2-subgroup of G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$: Indeed, if there is a cyclic Sylow 2-subgroup S, then $S \cong \mathbb{Z}_2$ by (III). But then the N/C Lemma (see Theorem 1.16) implies that $N_G(S) = \mathbb{Z}_G(S)$ so that S has a normal complement by BNCT.

(V) G has no subgroup of order $2s$ where s is an odd prime: Suppose for a moment that G has a subgroup Y of order $2s$. Let U be a subgroup of Y of order 2, and let W be a Sylow 2-subgroup of G containing U. From (IV) we know that $U \neq W$. Let g be an element of W. Note that ${}^gY$ contains U. If Y and ${}^gY$ are distinct then the subgroups $U, W, N_G(W), Y, {}^gY$ form a K_5 in the intersection graph of G. Therefore, $Y = {}^gY$ so that $g \in N_G(Y)$. Therefore, $W \leq N_G(Y)$, implying that $WY = YW$ so that WY is a subgroup of G of order $4s$. Note that $WY \neq G$ (because Y is a normal subgroup of WY), and note that $WY \neq N_G(W)$ (because otherwise $|N_G(W)| = 2^2 s$, and it follows from (III) and Lemma 3.11 that $N_G(W)$ has no subgroup of order $2s$). Therefore, the subgroups $U, W, Y, WY, N_G(W)$ form a K_5 in the intersection graph of G. This contradicts the planarity of G.

(VI) If P is a Sylow p-subgroup of G for some prime p dividing $|G|$, then P is isomorphic to $\mathbb{Z}_p$: Assume contrarily that P is a Sylow p-subgroup of G not isomorphic to $\mathbb{Z}_p$. It follows from (III) that $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and $N_G(P)$ is a non-abelian group of order $p^2 q$ for some prime q different from p. Let Q be a Sylow q-subgroup of $N_G(P)$. It follows from (III) and Lemma 3.11 that $N_G(P) \cap N_G(Q) = Q$. Let T be a Sylow q-subgroup of G containing Q. If $T = Q$ then (II) implies that $Q \neq N_G(Q)$. If $T \neq Q$ then $N_G(Q)$ contains T by (I) so that $Q \neq N_G(Q)$. Hence, $N_G(Q)/Q$ is a non-trivial group. For any two distinct elements aQ and bQ of the quotient group $N_G(Q)/Q$, the subgroups ${}^aN_G(P)$ and ${}^bN_G(P)$ are distinct subgroups containing Q (because $N_G(P)$ is self normalizing and $N_G(P) \cap N_G(Q) = Q$.) Therefore, if $|N_G(Q)/Q| > 3$ then there are three distinct conjugates Z_1, Z_2, Z_3 of $N_G(P)$ containing Q, so that the subgroups $Z_1, Z_2, Z_3, Q, N_G(Q)$ form a K_5 in the intersection graph of G. Therefore, we must have that $|N_G(Q)/Q| = 2$, and so $|N_G(Q)| = 2q$. But then, (V) implies that $q = 2$ and so $N_G(Q)$ is a Sylow 2-subgroup of G. Now the subgroups $N_G(P), {}^zN_G(P), Q, N_G(Q), M$ form a K_5 in the intersection graph of G where zQ is any non-identity element of $N_G(Q)/Q$ and M is the normalizer in G of the Sylow 2-subgroup $N_G(Q)$ of G.

It follows from (VI) that G has square free order. But such a group is solvable by Hölder's Theorem (see Theorem 1.8). □

Corollary 3.17. *A finite non-solvable group is not planar.*

Proof. Suppose contrarily that G is a finite non-solvable group which is planar. Since solvability is closed under group extension, G must have a non-solvable simple composition factor X. It follows from Proposition 3.16 that X is not isomorphic to a subgroup of G. Thus X is isomorphic to H/N for some non-trivial subgroup H of G and for some non-trivial proper normal subgroup N of H. But then, as H is planar, Lemma 3.10 implies that X is abelian. □

4. $K_{3,3}$-FREENESS OF INTERSECTION GRAPHS

Let Γ and Λ be two graphs. We say that Γ is Λ-free if there is no subgraph of Γ which is isomorphic to Λ, i.e. Γ does not contain Λ as a subgraph. Let G be a group. For simplicity we say that G is Λ-free whenever its intersection graph is Λ-free.

In a recent work [30], Rajkumar and Devi classified finite groups whose intersection graphs does not contain one of K_5, K_4, C_5, C_4, P_4, P_3, P_2, $K_{1,3}$, $K_{2,3}$ or $K_{1,4}$ as a subgraph. Here we present the classification of finite $K_{3,3}$-free groups. Our main result in this chapter is

Theorem 4.1. *A finite* non-planar *group is $K_{3,3}$-free if and only if it is isomorphic to one of the following groups:*

1. $\mathbb{Z}_{p^6}$, $\mathbb{Z}_{p^3} \times \mathbb{Z}_q$, $\mathbb{Z}_9 \times \mathbb{Z}_3$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$, $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$, $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$, *and* D_{18}, *where* p,q *are distinct primes.*

2. *The semidirect product* $\mathbb{Z}_q \rtimes \mathbb{Z}_{p^3}$ *with* $p^3 \mid q-1$, *where* p,q *are distinct primes.*

4.1 Solvable Groups

Recall that in Proposition 3.2 we determined finite abelian groups which are $K_{3,3}$-free. Here we restate this result for convenience:

Lemma 4.2. *A finite abelian group is $K_{3,3}$-free if and only if, for some distinct primes p, q and r, it is isomorphic to one of the following groups*

$$\mathbb{Z}_{p^i}\ (0 \le i \le 6), \quad \mathbb{Z}_{p^3} \times \mathbb{Z}_q, \quad \mathbb{Z}_{p^2} \times \mathbb{Z}_q, \quad \mathbb{Z}_p \times \mathbb{Z}_q, \quad \mathbb{Z}_9 \times \mathbb{Z}_3, \quad \mathbb{Z}_4 \times \mathbb{Z}_2,$$

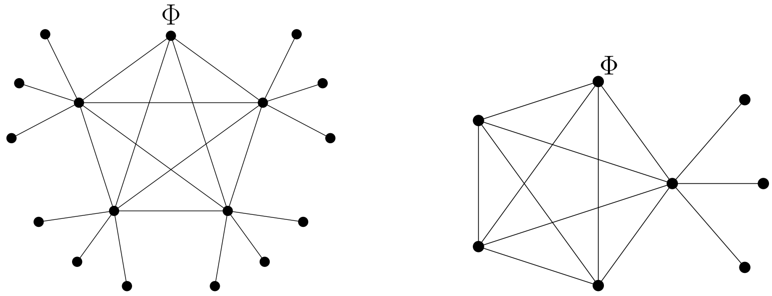
$$\mathbb{Z}_p \times \mathbb{Z}_p, \quad \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_p\ (p \neq 2).$$

Let G be a non-abelian p-group of order p^α $(\alpha > 2)$ which is $K_{3,3}$-free. Then, as the quotient of G by the Frattini subgroup $\Phi(G)$ (i.e. the intersection of all maximal subgroups) is elementary abelian, $\Phi(G)$ is a non-trivial subgroup of G. That is, there are subgroups K, L of G such that $|K| = p$, $|L| = p^2$, and K is contained in every maximal subgroup of G as well as in L. If $p > 3$, by Theorem 1.3 there are at least 6 maximal subgroups each containing a common subgroup, hence $\Gamma(G)$ contains a K_7 which is a contradiction. (Notice that for a graph being $K_{3,3}$-free is a more stringent condition than being K_6-free.) Thus G is either a 2-group or a 3-group. Also, the exponent $\alpha = 3$. To see this, suppose that $\alpha = 4$ and consider the case $p = 3$. Then there are at least four maximal subgroups of G of order p^3 and together with K and L, they form a K_6 in the intersection graph. If $\alpha = 4$, $p = 2$ and $|\Phi(G)| = p$, then $G/\Phi(G)$ is elementary abelian of rank 3 which is not listed in Lemma 4.2. Suppose that $|\Phi(G)| = p^2$. If $\Phi(G)$ is cyclic, then the subgroup $Z \leq \Phi(G)$ of order p is in the center of G. Since the intersection graph of the quaternion group of order 16 is K_9, we may further assume there are more than one minimal subgroups of G. Let K be another minimal subgroup of G, then three maximal subgroups together with Z, $\Phi(G)$, and ZK form a K_6 in $\Gamma(G)$. If $\Phi(G)$ is not cyclic, then we may take three maximal subgroups and the three subgroups of $\Phi(G)$ to form a $K_{3,3}$ in the intersection graph. Finally, $|\Phi(G)| = p^3$ implies G is cyclic which is a contradiction. There are two non-abelian groups of order 8, namely the dihedral group D_8 and the quaternion group Q_8; and also there are two non-abelian groups of order 27, namely $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$ and $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$. It can be verified that these groups are $K_{3,3}$-free (see Figure 3.3 and Figure 4.1). Thus, we almost proved that

Lemma 4.3. *A finite non-abelian nilpotent group is $K_{3,3}$-free if and only if it is isomorphic to one of the following groups*

$$D_8, \qquad Q_8, \qquad (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3, \qquad \mathbb{Z}_9 \rtimes \mathbb{Z}_3.$$

Proof. Since a nilpotent group G is the direct product of its Sylow subgroups, at least one of them must be non-abelian. However, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$ and $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$ both contains a K_5 in their intersections graphs, therefore cannot be a proper subgroup of G. Also, if G contains D_8 properly, then the three maximal subgroups together with D_8 and $\Phi(D_8)$ form a K_5 in $\Gamma(G)$. If we take another minimal subgroup K which is not a subgroup of D_8, then the subgroup $\Phi(D_8)K$ would be a sixth vertex which is connected by an edge with each vertices of K_5. Same argument is valid also for Q_8. □



(a) $\Gamma((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3)$ (b) $\Gamma(\mathbb{Z}_9 \rtimes \mathbb{Z}_3)$

Figure 4.1 : Vertices labelled with Φ represents Frattini subgroups.

Let G be a finite non-nilpotent solvable group. Following Section 3.2, we may reduce the number of cases substantially with regard to the orders of the groups. Let N be a minimal subgroup of G. By Theorem 1.5, N is an elementary abelian group and as a subgroup of G it is $K_{3,3}$-free. Moreover, N is either of rank 1 or rank 2 in virtue of Lemma 4.2. It is well-known that there is a correspondence between the subgroups of G containing N and the subgroups of G/N. Now we make a very useful observation

- the rank of N is 1 $\implies$ # subgroups of G/N is at most 6;
- the rank of N is 2 $\implies$ # subgroups of G/N is at most 3.

As a consequence of Sylow and Hall Theorems (see Theorems 1.3 and 1.6), the only possible values of $|G/N|$ are p^i $(0 \leq i \leq 5)$, pq, and p^2q, where p and q distinct prime numbers. Moreover, if N is of rank 2, then G/N is isomorphic to a cyclic group of prime or prime squared order. Therefore, the only possible cases for the order of G are

$$p^5q, \quad p^4q, \quad p^2qr; \quad p^3q, \quad p^2q, \quad p^2q^2, \quad pqr, \quad pq.$$

In [30], the $K_{2,3}$-free groups are determined as a sublist of K_5-free groups. Our preceding discussion made it apparent that if G is a $K_{2,3}$-free group and $N \lhd G$ is elementary abelian of rank 2, then either $[G:N] = 1$ or p, where p is a prime.

Returning to the possible orders of the non-nilpotent solvable $K_{3,3}$-free groups, we can still eliminate some of the cases by *ad hoc* arguments.

Lemma 4.4. *There are no finite non-nilpotent solvable group which is $K_{3,3}$-free and of order*

$$p^5q, \qquad p^4q, \qquad or \qquad p^2qr$$

where p,q,r are distinct prime numbers.

Proof. Let G be a $K_{3,3}$-free group, and let N be a minimal normal subgroup of G. First, consider the case $|G| = p^4q$. Clearly, $|N| = q$. Since the number of the subgroups of G/N is at most 6, G/N is isomorphic to $\mathbb{Z}_{p^4}$. Let $A < B < C < D$ be a chain of non-trivial p-subgroups of G. Then, one may form NA, NB and NC which are proper subgroups. As the orders of those groups are different, they form a K_7 in $\Gamma(G)$ (intersection of any two of them contains A). Therefore, G is not $K_{3,3}$-free. Similar arguments can also be applied for $|G| = p^5q$ case.

Next, suppose $|G| = p^2qr$. Clearly, N is not a p-group. Without loss of generality we may assume that $|N| = r$. Then $|G/N| = p^2q$. If G/N is not cyclic, then the number of subgroups of G/N exceeds 6. This is clear if Sylow p-subgroup of G/N is elementary abelian. And if G/N is not abelian, then there must be a non-normal subgroup (since there is no subgroup of G/N isomorphic to Q_8, it is not Hamiltonian) implying there are more than 6 subgroups. Hence, G/N must be cyclic. In this case there are subgroups of G/N of orders p^2, pq, p, and q. Then, by the Correspondence Theorem (see Theorem 1.4) there are four subgroups containing N, say A, B, C, D of orders p^2r, pqr, pr, and qr respectively. Let T be a subgroup of order p^2q. By the Product Formula T intersects A, B, and C non-trivially. That is, A,B,C,D,N,T span a subgraph in $\Gamma(G)$ containing $K_{3,3}$. □

Now we examine the other cases.

Lemma 4.5. *Let G be a non-nilpotent group of order p^3q, where p and q are distinct prime numbers. Then, G is $K_{3,3}$-free if and only if it is isomorphic to*

$$\mathbb{Z}_q \rtimes_\alpha \mathbb{Z}_{p^3} = \langle a,b \mid a^q = b^{p^3} = 1, bab^{-1} = a^\alpha \rangle$$

where p^3 divides $q-1$ and α is any integer not divisible by q whose order in the unit group $\mathbb{Z}_q^*$ of $\mathbb{Z}_q$ is p^3.

Proof. Suppose that G is $K_{3,3}$-free. Clearly, the order of the minimal normal subgroup N cannot be p^2. Therefore, we only need to consider the following two cases.

Case I: $|N| = p$. Since the number of subgroups of G/N is at most 6, we have $G/N \cong \mathbb{Z}_{p^2q}$. Let P be a Sylow p-subgroup of G containing N; and let Q be a Sylow q-subgroup. By the Correspondence Theorem P is the unique Sylow p-subgroup containing N, hence it is normal in G. As G is not a nilpotent group, Q is not a normal subgroup of G. Let H and K be the subgroups containing N with orders pq and p^2q respectively. Since NQ^g is a subgroup of order pq for any conjugate Q^g of Q which contains N and since H is the unique subgroup of order pq containing N by the Correspondence Theorem, H contains all conjugates of Q and this implies the number of Sylow q-subgroups of G is $[H:N]=p$. In particular $H \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$ and $p > q$. Then, the normalizer $N_G(Q)$ has order p^2q. Moreover, since $N_G(Q)$ is self-normalizing by Lemma 1.15, there are p conjugates of $N_G(Q)$ which are different from the normal subgroup K. By the Product Formula, P, H, K and p conjugates of $N_G(Q)$ pairwise intersect non-trivially. In other words, those subgroups form a K_{p+3} on $\Gamma(G)$. As $p \geq 3$, G cannot be $K_{3,3}$-free in this case.

Case II: $|N| = q$. Since $[G:N] = p^3$ and since the number of subgroups of G/N is at most six, G/N is isomorphic to either $\mathbb{Z}_{p^3}$ or Q_8. Notice that a group with a unique maximal subgroup is necessarily cyclic and by the Theorem 1.3 a non-cyclic p-group has at least three maximal subgroups. Therefore, G/N must have a unique minimal subgroup even if it is not cyclic.

Case II (a): $G/N \cong \mathbb{Z}_{p^3}$. Take three non-trivial p-subgroups $A < B < C$ and form NA and NB. As the orders of those groups are different, they form a K_5 in $\Gamma(G)$. Also, since G is not nilpotent, there are more than one Sylow p-subgroups of G. If A is contained in a Sylow p-subgroup D other than C, then together with D we have 6 proper non-trivial subgroups pairwise intersecting non-trivially. On the other hand, if any two Sylow p-subgroups intersect trivially, then $\Gamma(G)$ is $K_{3,3}$-free. Notice that NA is the unique subgroup of G of order pq and NB is the unique subgroup of G of order p^2q. Let $Q = \langle a \rangle$ and $P = \langle b \rangle$. We want to write a presentation for G. Since Q is normal, $bab^{-1} = a^{\alpha}$ for some integer α not divisible by q. Observe that, $b^k a b^{-k} = a^{\alpha^k}$ for any integer k. This implies $\alpha^{p^3} \equiv 1 \pmod{q}$, i.e. the order of α in the unit group $\mathbb{Z}_q^*$ divides p^3. Moreover, its order is exactly p^3, as otherwise, the intersection of some Sylow p-subgroups would be non-trivial. Conversely, the group

$$\mathbb{Z}_q \rtimes_{\alpha} \mathbb{Z}_{p^3} = \langle a, b \mid a^q = b^{p^3} = 1, bab^{-1} = a^{\alpha} \rangle$$

has the subgroup structure described above and it is $K_{3,3}$-free. See Figure 4.2.

Case II (b): $G/N \cong Q_8$. Then, there are 5 non-trivial subgroups of a Sylow p-subgroup each containing a unique minimal subgroup A. Together with NA we have 6 subgroups forming a K_6 in $\Gamma(G)$. Thus, there is no $K_{3,3}$-free group in this case. □

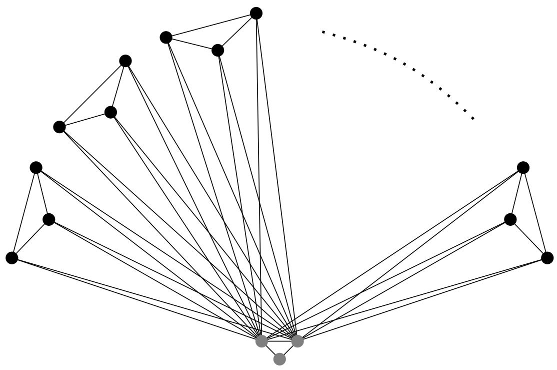


Figure 4.2 : $\Gamma(\mathbb{Z}_q \rtimes_\alpha \mathbb{Z}_{p^3})$, gray vertices represents subgroups of orders q, pq, and p^2q.

There are non-nilpotent solvable planar groups of orders

$$p^2q, \qquad p^2q^2, \qquad pqr, \qquad \text{and} \qquad pq$$

which are necessarily $K_{3,3}$-free. Previously, we proved that the groups presented at the second and third items of the following lemma are planar.

Lemma 4.6 (compare with Lemma 3.11)**.** *Let G be a non-nilpotent group of order p^2q, where p and q are distinct prime numbers. Then, G is $K_{3,3}$-free if and only if it is isomorphic to one of the following groups:*

1.

$$\mathbb{Z}_3 \rtimes \mathbb{Z}_4, \quad \textit{or} \quad D_{18},$$

2.

$$\mathbb{Z}_q \rtimes_\alpha \mathbb{Z}_{p^2} = \langle a, b \mid a^q = b^{p^2} = 1, bab^{-1} = a^\alpha \rangle$$

where p^2 divides $q-1$ and α is any integer not divisible by q whose order in the unit group $\mathbb{Z}_q^*$ of $\mathbb{Z}_q$ is p^2,

3.

$$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes_\beta \mathbb{Z}_q = \langle a, b, c \mid a^p = b^p = c^q = 1, ab = ba, cac^{-1} = b, cbc^{-1} = a^{-1}b^\beta \rangle$$

where q divides $p+1$ and β is any integer such that the matrix $\theta = \begin{bmatrix} 0 & -1 \\ 1 & \beta \end{bmatrix}$ has order q in the group $GL(2, \mathbb{Z}_p)$ and such that θ has no eigenvalue in $\mathbb{Z}_p$.

Proof. Suppose that G is $K_{3,3}$-free. There are three possible cases for the order of the minimal normal subgroup N of G.

Case I: $|N| = p$. Let P be the Sylow p-subgroup of G, and Q be a Sylow q-subgroup of G.

Case I (a): P is not a normal subgroup of G. Then N is contained in every Sylow p-subgroups as well as in some subgroups of order pq. However, there are $q = 1 + kp$ conjugates of P and since G is K_6-free, we have $p = 2$, $q = 3$ and $H := NQ$ must be a normal subgroup of G. Notice that, the three Sylow p-subgroups together with N and H form a K_5 in $\Gamma(G)$. Moreover, Q is a normal subgroup of G; otherwise, $H \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$, however $3 \nmid 2-1$. If $P \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, there would be a non-normal subgroup $K \cong \mathbb{Z}_2$, as otherwise, P would be a normal subgroup of G. Then QK is connected by an edge with two of the three Sylow p-subgroups as well as with H which is a contradiction because we assumed that G is $K_{3,3}$-free. Therefore, $P \cong \mathbb{Z}_4$ and we can easily observe that

$$\mathbb{Z}_3 \rtimes \mathbb{Z}_4 = \langle a, b \mid a^3 = b^4 = 1, bab^{-1} = a^2 \rangle$$

is $K_{3,3}$-free as it has exactly six proper non-trivial subgroups and the minimal subgroup of order 3 has degree one in the intersection graph.

Case I (b): P is the normal Sylow p-subgroup of G. As G is not a nilpotent group by assumption, Q is not a normal subgroup of G.

Suppose that there is a normal subgroup L of G of order pq containing Q. Then L contains all conjugates of Q, hence $L \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$ and in particular $q \mid p-1$. Moreover, by Lemma 1.15 any subgroup containing $N_G(Q)$ is self-normalizing and this implies $N_G(Q) \neq Q$, as $L \lhd G$ by assumption. However, $N_G(Q) \neq G$ either, thus $H := N_G(Q)$ is of order pq and it is not a normal subgroup of G. Let K be the subgroup of H of order p. Notice that since $p > q$, we have $K \lhd H$. Clearly, conjugates of H together with K and P form a K_{p+2} in $\Gamma(G)$. Therefore $p = 3$ and $q = 2$. However, any (Sylow) q-subgroup is contained in the normal subgroup L implying there is an edge between L and any conjugate of H. That is, conjugates of H together with K, P, and L span a subgraph containing $K_{3,3}$.

Now suppose that there is no normal subgroup of order pq. In particular NQ is not a normal subgroup of G. As in the previous paragraph, conjugates of NQ together with N and P form a K_{p+2} in $\Gamma(G)$. Therefore $p = 3$ and $q = 2$, as the number of Sylow

q-subgroups is $\equiv 1 \pmod q$. (Since any subgroup of index 2 must be normal, $p \neq 2$.) If $P \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, there must be a normal subgroup K of order p different from N. To see this, consider the action of Q by conjugation on the set of subgroups of order p. (Notice that there are totally four subgroups of order p.) Since N is a normal subgroup of order p and the length of an orbit of Q is either 1 or 2, there must be a subgroup K fixed by Q and different from N. However, G is generated by the elements of N, K, and Q, thus K is a normal subgroup. Then KQ is a group of order pq different from NQ and its conjugates. This is because, NQ and KQ have unique subgroups of order p which are not conjugate to each other. By the Product Formula any two subgroups of order pq intersects non-trivially. Therefore, conjugates of NQ together with the conjugates of KQ form a K_6 in $\Gamma(G)$. Finally, if $P \cong \mathbb{Z}_9$, we have the dihedral group

$$D_{18} = \langle a, b \mid a^9 = b^2 = 1, bab = a^{-1} \rangle$$

which is $K_{3,3}$-free. See Figure 4.3.

Case II: $|N| = q$. As the Sylow q-subgroup N is normal and as G is not a nilpotent group, there are at least three Sylow p-subgroups, say P_i $(1 \leq i \leq q)$.

Suppose that $G/N \cong \mathbb{Z}_p \times \mathbb{Z}_p$. By the Correspondence Theorem there are at least three subgroups H_j $(1 \leq j \leq p+1)$ of order pq each containing N. By the Product Formula, $P_i \cap H_j \neq 1$ for any $1 \leq i, j \leq 3$ and we have six vertices which span a subgraph of $\Gamma(G)$ containing a $K_{3,3}$, contradiction!

Now suppose that $G/N \cong \mathbb{Z}_{p^2}$. If $X = P_i \cap P_j$ is non-trivial for some distinct Sylow p-subgroups, then X must be a normal subgroup of G as $N_G(X)$ contains both P_i and P_j. However, this case was considered in Case I (a). If the intersection of any pair of Sylow p-subgroups is trivial, then G has a presentation

$$\mathbb{Z}_q \rtimes_\alpha \mathbb{Z}_{p^2} = \langle a, b \mid a^q = b^{p^2} = 1, bab^{-1} = a^\alpha \rangle$$

and it is planar. See Lemma 3.11 for details.

Case III: $|N| = p^2$. As the Sylow p-subgroup N is normal and as G is not a nilpotent group, any subgroup of order q is not normal in G. We want to observe that there are no subgroups of G of order pq. To see this, first suppose that there is a subgroup H of G of order pq. If H is a normal subgroup of G, obviously H contains all (Sylow) q-subgroups. Then $A = H \cap N$ is normal in H as well as in G, since N is abelian and

$\langle H, N \rangle = G$. However, this is in contradiction with the assumption that there is no normal subgroup of order p. If H is not a normal subgroup of G, since a subgroup of smallest prime index must be normal, we have $p > q$. Then, again A is a normal subgroup of H and of G and we have the same contradiction. Therefore, there is no subgroup of G of order pq. In that case, G has a presentation

$$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes_\beta \mathbb{Z}_q = \langle a, b, c \mid a^p = b^p = c^q = 1, ab = ba, cac^{-1} = b, cbc^{-1} = a^{-1}b^\beta \rangle$$

and it is planar. See Lemma 3.11 for details. □

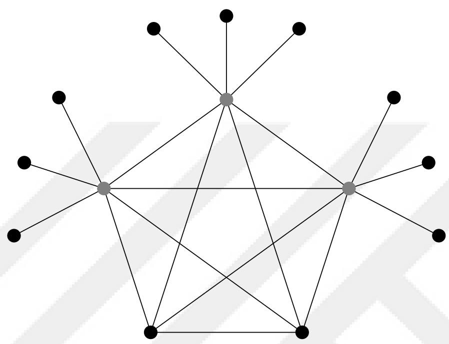


Figure 4.3 : $\Gamma(D_{18})$, gray colored vertices represents subgroups of order pq.

Lemma 4.7 (compare with Lemma 3.12)**.** *Let G be a non-nilpotent group of order p^2q^2, where $p > q$ are distinct prime numbers. Then, G is $K_{3,3}$-free if and only if it is isomorphic to*

$$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes_\beta \mathbb{Z}_{q^2} = \langle a, b, c \mid a^p = b^p = c^{q^2} = 1, ab = ba, cac^{-1} = b, cbc^{-1} = a^{-1}b^\beta \rangle$$

where q^2 divides $p+1$ and β is any integer such that the matrix $\theta = \begin{bmatrix} 0 & -1 \\ 1 & \beta \end{bmatrix}$ has order q^2 in the group $GL(2, \mathbb{Z}_p)$ and such that θ^q has no eigenvalue in $\mathbb{Z}_p$.

Proof. Suppose that G is $K_{3,3}$-free. First we shall observe that the minimal normal subgroup N of G must be a Sylow subgroup. To this end, let us assume $|N| = p$. Then, $G/N \cong \mathbb{Z}_{pq^2}$ and there exists a unique Sylow p-subgroup P containing N. Since Sylow p-subgroups are conjugate and since N is a normal subgroup, P is also a normal subgroup of G. Let Q be a Sylow q-subgroup. By assumption G is not nilpotent, hence Q is not a normal subgroup of G. (Notice that assuming $|N| = q$, one may deduce in a similar fashion that the unique Sylow q-subgroup is a normal subgroup of G. However, this is not possible for $p > q$.)

Let A, B, C be the subgroups of respective orders pq, pq^2, p^2q containing N and Q_i $(1 \leq i \leq 3)$ be three Sylow q-subgroups. Let X be a group of order q. Since N is a normal subgroup, NX is a group of order pq containing X. This implies A contains any group of order q, as it is the unique subgroup of order pq containing N. That is $A \cap Q_i$ is non-trivial for $1 \leq i \leq 3$. This is also true for B and C, and so A, B, C together with Q_i span a subgraph containing a $K_{3,3}$ in $\Gamma(G)$.

By the preceding discussion we conclude that N is the normal Sylow p-subgroup of G and since it is minimal, $N \cong \mathbb{Z}_p \times \mathbb{Z}_p$ by Theorem 1.5. We know that if the rank of the minimal normal subgroup N is two, then G/N has at most three subgroups, hence it is a cyclic group of prime or prime squared order. Since the order of G is p^2q^2, we conclude $Q \cong \mathbb{Z}_{q^2}$, where Q is a Sylow q-subgroup. Let K be the unique subgroup of G of order p^2q containing N. Since any subgroup of order p^2q contains N, we see that K is the unique subgroup of G of order p^2q. Moreover, since $NX = K$ for any subgroup X of order q, K contains all subgroups of G of order q. Also, since any subgroup of a $K_{3,3}$-free group is also $K_{3,3}$-free, K is isomorphic to the third group stated in the previous Lemma 4.6. In particular, there is no subgroup of K of order pq and in turn this implies there are no subgroups of G of order pq^2 or pq. To see this observe that if $H < G$ is of order pq, then $H \cap K$ contains a subgroup of order p by the Product Formula. However, K contains every subgroup of order q which implies $H < K$, contradiction! Similar argument works when $|H| = pq^2$. Hence G is a group with a normal Sylow p-subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ and a non-normal Sylow q-subgroup isomorphic to $\mathbb{Z}_{q^2}$ and there are no subgroups of G of order pq or of order pq^2. Such a group has a presentation

$$(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes_\beta \mathbb{Z}_{q^2} = \langle a, b, c \mid a^p = b^p = c^{q^2} = 1, ab = ba, cac^{-1} = b, cbc^{-1} = a^{-1}b^\beta \rangle$$

and it is planar. See Lemma 3.12 for details. □

Lemma 4.8 (compare with Lemma 3.13)**.** *Let G be a non-nilpotent group of order pqr, where $p < q < r$ are distinct prime numbers. Then, G is $K_{3,3}$-free if and only if it is isomorphic to*

$$\mathbb{Z}_r \rtimes_\alpha \mathbb{Z}_{pq} = \langle a, b \mid a^r = b^{pq} = 1, bab^{-1} = a^\alpha \rangle$$

where pq divides $r-1$ and α is any integer not divisible by r whose order in the unit group $\mathbb{Z}_r^*$ of $\mathbb{Z}_r$ is pq.

Proof. Let R be a Sylow r-subgroup. Applying Sylow Theorems it can be easily observed that $R \lhd G$. Since $|G/R| = pq$ and $p < q$, we see that either $G/R \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p$ or $G/R \cong \mathbb{Z}_p \times \mathbb{Z}_q$. Observe that in the first case the number of subgroups of G/R is $q+3$ implying $G/R \cong S_3$ as this number is at most six.

Suppose that $G/R \cong S_3$. By the Correspondence Theorem, there is a unique subgroup N of order $3r$ and three subgroups L_i $(1 \leq i \leq 3)$ of order $2r$ containing R. Since N is a Hall $\{3,r\}$-subgroup and R is normal, N is a normal subgroup of G as well (see Theorem 1.6). Let Q be the Sylow 3-subgroup of G contained in N. Then, by Lemma 1.15, Q is a normal subgroup of G. Let H be a Hall $\{2,3\}$-subgroup of G. If H is not a normal subgroup of G, then the number of its conjugates is $[G:H] = r$ and those subgroups together with Q form a K_{r+1} in $\Gamma(G)$. Since $r \geq 5$, the intersection graph cannot be $K_{3,3}$-free in this case. Also, if $H \lhd G$ then it is easy to observe that the subgroups H, N, R together with L_i $(1 \leq i \leq 3)$ form a subgraph containing $K_{3,3}$.

Suppose that $G/R \cong \mathbb{Z}_{pq}$. By the Correspondence Theorem, there are unique subgroups N of order pr and M of order qr. As in the preceding paragraph both M and N are normal subgroups. Let K be a subgroup of order pq. Clearly, K is not a normal subgroup of G and in particular it has r conjugates. Now assume that there exist two distinct conjugates K_1 and K_2 of K such that their intersection $X = K_1 \cap K_2$ is non-trivial. Then, as $|X|$ is either p or q, we have $X \lhd G$; and this implies X is contained in all conjugates of K. That is, conjugates of K together with X form a K_{r+1} in the intersection graph which is a contradiction as $r \geq 5$. Therefore, any two distinct subgroup of order pq intersects trivially. Such a group has a presentation

$$\mathbb{Z}_r \rtimes_\alpha \mathbb{Z}_{pq} = \langle a, b \mid a^r = b^{pq} = 1, bab^{-1} = a^\alpha \rangle$$

and it is planar. See Lemma 3.13 for details. □

Finally, intersection graph of any group of order pq consists of isolated vertices and so $K_{3,3}$-free. For further references we state it as a lemma.

Lemma 4.9. *Let G be a non-nilpotent group of order pq, where p,q are prime numbers and $p > q$. Then, $q \mid p-1$ and*

$$G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$$

is $K_{3,3}$-free.

4.2 Non-solvable Groups

First, we shall show that there is no finite non-abelian simple group which is $K_{3,3}$-free. To this end, we need the following result.

Theorem 4.10 (see [31, Theorem 1])**.** *If the finite group G contains a maximal subgroup M which is nilpotent of class less than* 3*, then G is solvable.*

As a consequence of Theorem 4.10, if a Sylow p-subgroup P of G is maximal and $|P| = p^3$, then G is solvable.

Proposition 4.11. *If G is a finite non-abelian simple group, then* $\Gamma(G)$ *contains a* $K_{3,3}$ *as a subgraph.*

Proof. Consider a finite simple group G which is $K_{3,3}$-free. Then there exists a minimal finite simple group U which is isomorphic to a non-abelian composition factor of some subgroup of G. Thus, U must be $K_{3,3}$-free.

Minimal simple groups are known (see [32, Corollary 1]). Thus, U is isomorphic to one of the following groups: $PSL_2(q)$, $Sz(q)$, $PSL_3(3)$.

In view of Feit-Thompson Theorem, 2 divides $|U|$. Let S be a Sylow 2-subgroup of U. Then S is a 2-group from Lemmas 4.2 or 4.3. Thus, either $S \cong \mathbb{Z}_{2^i}$, where $1 \leq i \leq 6$, or $S \in \{\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4 \times \mathbb{Z}_2, D_8, Q_8\}$.

Since the intersection graphs of each of the 2-groups $\mathbb{Z}_4 \times \mathbb{Z}_2$, D_8 and Q_8 contains $K_{1,3}$, those groups must be maximal in U. By Theorem 4.10, U is solvable. A contradiction.

Suppose that S is cyclic. Then, by Theorem 1.13, S has a normal complement in U which contradicts with the assumption that U is simple.

Thus, $S \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. By BNCT (see Theorem 1.12), the normalizer $N_U(S)$ properly contains S. Clearly, $N_U(S)$ is a proper subgroup of U, as otherwise, S would be a normal subgroup.

Normalizers of Sylow 2-subgroups of finite simple groups are known (see [33, Corollary]). Thus, U is isomorphic to either $PSL_2(q)$, where $q \cong \pm 3 \pmod 8$ (in this case $N_U(S) \cong A_4$), or to $PSL_3(3)$. But $PSL_3(3)$ properly contains S_4 (see [34]), therefore is not $K_{3,3}$-free. If $U \cong PSL_2(q)$, where $q \cong \pm 3 \pmod 8$, then there is a subgroup $H \cong D_{q\pm 1}$ of U which is a subgroup of odd index (see [35, Table 8.7]). Take

S to be a Sylow 2-subgroup of H. Then H, $N_U(S)$, S and three proper subgroups of S form a graph which contains $K_{3,3}$. A contradiction. □

Corollary 4.12. *A finite non-solvable group is not $K_{3,3}$-free.*

Proof. Let G be a finite non-solvable group. Since G has a non-abelian simple composition factor which is not $K_{3,3}$-free by Proposition 4.11, G is not $K_{3,3}$-free as well. □

Our main result follows from Lemmas 4.2, 4.3, 4.5, 4.6, 4.7, 4.8, 4.9 and Corollary 4.12.

5. CONNECTIVITY OF INTERSECTION GRAPHS

In this chapter, we classify finite solvable groups whose intersection graphs are not 2-connected and finite nilpotent groups whose intersection graphs are not 3-connected.

Let Γ be a simple graph with vertex set $V(\Gamma)$. A sequence $\gamma = (v_0, v_1, \ldots, v_k)$ of vertices is a *path* of length k between v_0 and v_k, if each consecutive pair of vertices are adjacent in Γ. We call two or more paths with the same end points *internally independent* provided that none of them have a common inner vertex with another. (For brevity, we usually omit 'internally' and say simply 'independent paths'.) A graph is *connected* if any two of its vertices are linked by a path. Let $\mathscr{C}$ be a subset of $V(\Gamma)$ such that the induced subgraph by $\mathscr{C}$ is connected. If $\mathscr{C}$ is a maximal subset of $V(\Gamma)$ with this property, then we say $\mathscr{C}$ is a *component* of Γ. Alternatively, we may define an equivalence relation $\sim$ on $V(\Gamma)$ by using the adjacency of vertices inductively: If $x \sim y$ and $\{y, z\}$ is an edge, then $x \sim z$. (Of course, we also insert $x \sim x$ for every $x \in V(\Gamma)$.) Then a subset $\mathscr{C}$ of $V(\Gamma)$ is said to be a component of Γ, if $\mathscr{C}$ is an equivalence class. Clearly, Γ is connected if and only if $V(\Gamma)$ is the single component.

Let G be a group. It is not difficult to determine finite non-simple groups having a disconnected intersection graph:

Theorem 5.1. *Let G be a finite non-simple group. Then $\Gamma(G)$ is* not *connected if and only if for some prime numbers p and q one of the following holds.*

1. *$G \cong \mathbb{Z}_p \times \mathbb{Z}_p$, or $G \cong \mathbb{Z}_p \times \mathbb{Z}_q$.*
2. *$G \cong N \rtimes A$ where $N \cong \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$, $A \cong \mathbb{Z}_q$, $N_G(A) = A$, and N is a minimal normal subgroup of G.*

In [8], Shen proved this result and also showed that intersection graphs of (non-abelian) simple groups are connected, thereby completed the classification for all finite groups. Here we shall give a different proof for Theorem 5.1 which is due to I. M. Isaacs. In an earlier work [13], Lucido classified finite groups whose poset of proper non-trivial subgroups are connected. Obviously, $\Gamma(G)$ is connected if and only if the poset of proper non-trivial subgroups of G is connected.

The aim of the present chapter is to give a more detailed account of the "connectivity" of intersection graphs. For a connected graph Γ, a subset $\mathscr{S}$ of the vertex set $V(\Gamma)$ is said to be a *separating set*, if removal of the vertices in $\mathscr{S}$ yields more than one components. We say Γ is *k-connected* if $|V(\Gamma)| > k$ and there is no separating set of cardinality $< k$. We define the *connectivity* $\kappa(\Gamma)$ of Γ as the greatest value of k such that Γ is k-connected. By convention, the connectivity of the complete graph K_n on n vertices is $n-1$. Hence, 1-connected graphs form the class of connected graphs with at least two vertices. Clearly, Γ is not connected if and only if $\kappa(\Gamma) = 0$. By abuse of notation, we denote the connectivity of the intersection graph of G by $\kappa(G)$. For solvable groups we proved the following theorem.

Theorem 5.2. *Let G be a finite solvable group. Then $\kappa(G) < 2$ if and only if for some prime numbers p and q one of the following holds.*

1. *$|G| = p^{\alpha}$ with $0 \leq \alpha \leq 2$.*
2. *$|G| = p^3$ and neither $G \cong Q_8$ nor $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.*
3. *$|G| = p^2 q$ with a Sylow p-group P such that either*
   - *(a) $P \cong \mathbb{Z}_{p^2}$, or*
   - *(b) $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and there exists a non-normal subgroup of G of order p.*
4. *$G = PQ$ is a group of order $p^{\alpha} q$ ($\alpha \geq 3$) with P being the normal Sylow p-group of G such that either*
   - *(a) P is elementary abelian, Q acts on P irreducibly, and the order of $N_G(Q)$ is at most pq, or*
   - *(b) $N := \Phi(P)$ is elementary abelian, Q acts on both N and P/N irreducibly, and either $N_G(Q) = Q$ or $N_G(Q) = NQ \cong \mathbb{Z}_p \times \mathbb{Z}_q$.*

In particular, any solvable group whose order is divisible by at least three distinct primes is 2-connected.

Intuitively, intersection graphs should be highly connected graphs and if there are some examples of such graphs with 'low' connectivity, they must be exceptional. By Menger's Theorem (see [36, Theorem 3.3.6]), a graph is k-connected if and only if it contains k independent paths between any two vertices. Hence, if $\Gamma(G)$ is 3-connected, there must exist sufficiently many vertices in the intersection graph forming at least three independent paths between any pair of vertices. However, claiming the existence of those subgroups and also verifying that they intersect non-trivially sufficiently many times seems to be a fairly complicated problem for the class of solvable groups. For nilpotent groups we obtain the following theorem.

Theorem 5.3. *Let G be a finite nilpotent group. Then $\kappa(G) < 3$ if and only if for some prime numbers p, q, and r one of the following holds.*

1. $|G| = p^{\alpha}$ $(0 \leq \alpha \leq 3)$ *and neither* $G \cong Q_8$ *nor* $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.
2. *G is a group of order p^4 such that*
   (a) $G \cong \mathbb{Z}_{p^4}$, *or*
   (b) $\Phi(G) \cong \mathbb{Z}_{p^2}$ *and* $G \not\cong Q_{16}$, *or*
   (c) $\Phi(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $Z(G) < \Phi(G)$ *and*
   $G \not\cong \langle a, b, c \mid a^9 = b^3 = 1, ab = ba, a^3 = c^3, bcb^{-1} = c^4, aca^{-1} = cb^{-1} \rangle$.
3. $G \cong \mathbb{Z}_{p^3q}$, $G \cong \mathbb{Z}_{p^2q}$, $G \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \times \mathbb{Z}_q$, *or* $G \cong \mathbb{Z}_{pqr}$.

Moreover, any solvable group whose order is divisible by at least four distinct primes is 3-connected.

5.1 Preliminaries

Let $V(G)$ be the set of proper non-trivial subgroups of G. This vertex set $V(G)$ (of $\Gamma(G)$) naturally carries a poset structure under set inclusion and its minimal elements are the minimal subgroups of G. A subset $\mathscr{S}$ of $V(G)$ is *upward closed* if whenever $H \in \mathscr{S}$ and $H \leq K$, then also $K \in \mathscr{S}$.

Proposition 5.4. *For a finite group G with $|V(G)| > k$ the following statements are equivalent:*

(i) $\Gamma(G)$ *is k-connected.*

(ii) There is no "upward closed" separating set $\mathscr{S}$ *of* $\Gamma(G)$ *with* $|\mathscr{S}| < k$*.*

(iii) There are at least k independent paths in $\Gamma(G)$ *between any pair of "minimal" subgroups.*

Proof. (i) $\Longleftrightarrow$ (ii): By definition a graph is k-connected if and only if there is no separating set of cardinality $< k$. Thus, all we need to do is to show that any minimal separating set for $\Gamma(G)$ is upward closed (except, if $\Gamma(G)$ is a complete graph). Take a vertex $S \in \mathscr{S}$ where $\mathscr{S}$ is a minimal separating set. By the minimality of $\mathscr{S}$, for any two vertices $H, K \in V(G) \setminus \mathscr{S}$ there is a path $\gamma = (H, \dots, K)$ traversing only the points in $(V(G) \setminus \mathscr{S}) \cup \{S\}$. Suppose that H and K belong to the different components (obtained after removing all the vertices in $\mathscr{S}$). So γ necessarily visits S, i.e. $\gamma = (H, \dots, S, \dots, K)$. If $\bar{S} \in V(G)$ and $S < \bar{S}$, then $\bar{\gamma} = (H, \dots, \bar{S}, \dots, K)$ is also a path from H to K and therefore $\bar{S} \in \mathscr{S}$. Since S was chosen arbitrarily, $\mathscr{S}$ is upward closed.

(i) $\Longleftrightarrow$ (iii): Menger's Theorem states that a graph is k-connected if and only if it contains k independent paths between any two vertices. Therefore, it is enough to show that existence of k independent paths between any pair of minimal subgroups implies the existence of k independent paths between any pair of subgroups in $V(G)$. If there exists a unique minimal subgroup of G, then $\Gamma(G)$ is a complete graph on more than k vertices, thus it is k-connected. Suppose that there are more than one minimal subgroups of G. Let $X, Y \in V(G)$ be two distinct vertices and A, B be two minimal subgroups with $\gamma_i = (A, A_i, \dots, B_i, B)$, $1 \le i \le k$, are independent paths between them. Suppose that neither X nor Y are minimal subgroups. There are two cases that may occur:

Case I: X and Y contains a common minimal subgroup, say A. Then $\bar{\gamma}_i := (X, A_i, Y)$ are independent paths provided that no coincidence occurs. If X coincides with, say A_1, then replace $\bar{\gamma}_1$ with (X, Y). If, in addition, Y coincides with, say A_2, then substitute $(X, \dots, B_1, B_2, \dots, Y)$ for $\bar{\gamma}_2$.

Case II: X and Y contains distinct minimal subgroups, say A and B respectively. In this case, we may simply take $\bar{\gamma}_i = (X, A_i, B_i, Y)$ as independent paths between X and Y. If X or Y coincides with some inner vertex, we may simply shorten the path accordingly.

Finally, it is easy to see that above arguments can still be applied with minor changes if one of X and Y is a minimal subgroup. □

Obviously, if a graph is k-connected, then the degree (valency) of any vertex is at least k. In view of Proposition 5.4 (iii) we make the following convention: For a finite group G, we say

"G satisfies the k-valency condition"

provided that any minimal subgroup of G is contained strictly by at least k proper subgroups.

A vertex v of a connected graph Γ is called a *cut-vertex*, if removing v from Γ renders a disconnected graph, i.e. if $\{v\}$ is a separating set for Γ. For the complete graph K_2, we shall regard any of its two vertices as a cut-vertex. (This is not a standard convention.) Hence, $\kappa(\Gamma) = 1$ if and only if there exists a cut-vertex of Γ.

Lemma 5.5. *Let G be a finite nilpotent group. Then there exists a cut-vertex of $\Gamma(G)$ if and only if G is isomorphic to one of the following groups*

$$\mathbb{Z}_{p^3}, \qquad \mathbb{Z}_{p^2} \times \mathbb{Z}_p, \qquad \mathbb{Z}_{p^2} \times \mathbb{Z}_q$$

for some prime numbers p and q.

Proof. Let G be a finite nilpotent group such that there is a cut-vertex M in $\Gamma(G)$. By Proposition 5.4 (ii), M can be taken as a maximal subgroup of G. Actually, M must be a maximal subgroup unless $\Gamma(G)$ is a complete graph. Suppose that $\Gamma(G) \cong K_n$. Obviously it must be the complete graph on two vertices. In other words, G has a unique maximal subgroup and a unique minimal subgroup different from the maximal subgroup. This is possible only if $G \cong \mathbb{Z}_{p^3}$ for some prime number p. (Observe that a finite group has a unique maximal subgroup if and only if it is isomorphic to a cyclic group of prime power order.)

Next, suppose that $\Gamma(G)$ is not complete. Clearly, there are more than one minimal subgroups. Let N be a minimal normal subgroup of G. Since G is the direct product of its Sylow subgroups and since a p-group has a normal subgroup of prime order, N is a minimal subgroup of G. As any subgroup contains a minimal subgroup, any component of the graph obtained by removing the vertex M and all the incident edges to M from $\Gamma(G)$ contains at least one minimal subgroup. Let A be a minimal subgroup which is not in the same component with N. Since (A, NA, N) is a path between them, $M = NA$. It can be easily seen that NA is a maximal subgroup of the nilpotent group G if and only if G is isomorphic to either $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$, or $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$ for some prime numbers p and q. □

As can be observed from the proof of the Lemma 5.5, it is important to know when two minimal subgroups generate a preferably small proper subgroup. Accordingly, it is easier to describe the connectivity of groups with *many* normal subgroups such as p-groups. On the other hand, it is known that any simple group can be generated by two elements. Let us recapitulate some basic group theoretical facts that are essential for our arguments.

Recall that the *Frattini subgroup* $\Phi(G)$ of a group G is the intersection of all maximal subgroups of G. It is well-known that the quotient of a finite p-group by its Frattini subgroup is elementary abelian. Moreover, $\Phi(G)$ is the minimal subgroup with this property. Therefore, $\Phi(G) = 1$ if and only if G is elementary abelian (see Theorem 1.9). Notice that $\Phi(G)$ is a normal (even characteristic) subgroup of G.

The p-core $O_p(G)$ of a finite group G is the intersection of all Sylow p-subgroups of G. Like $\Phi(G)$ it is a characteristic subgroup; actually, it is the unique largest normal p-subgroup of G. In a finite solvable group G, the factors of every chief series are elementary abelian of prime power order. In particular, every minimal normal subgroup of G is elementary abelian (see Theorem 1.5). Hence, for a non-trivial solvable group G, there exists a prime $p \mid |G|$ such that $O_p(G)$ is non-trivial.

A finite group G is called *supersolvable* if it posseses a normal series with each factor group is cyclic of prime order. If a finite group is supersolvable, then every maximal subgroup is of prime index (see [37, Problem 3B.7(b)]); and therefore, any maximal chain of subgroups have the same length. Let G be a group of order $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ where p_i $(1 \leq i \leq k)$ are distinct prime numbers. We define the *order length* of G as $\ell(G) := \sum_{i=1}^{k} \alpha_i$. Clearly, for a supersolvable group G, the order length $\ell(G)$ is equal to the length of a maximal chain. Supersolvable groups form a class between the class of nilpotent groups and the class of solvable groups.

We close this section by presenting another structural result. Observe that the intersection graph of the trivial group 1 as well as the intersection graph of $\mathbb{Z}_p$ (p is a prime) are empty graphs. However, we set $|V(1)| = -1$ and $|V(\mathbb{Z}_p)| = 0$ to make the statement of the following Proposition easier. Moreover, we adopt the following convention

$$\kappa(1) = -2, \qquad \kappa(\mathbb{Z}_p) = -1, \qquad \kappa(\mathbb{Z}_{p^2}) = \kappa(K_1) = 0.$$

Notice that this is in conformity with the our previous convention that $\kappa(K_n) = n-1$.

Proposition 5.6. *Let G be a finite group and N be a normal subgroup of G. If G/N is k-connected, then G is $(k+x-1)$-connected where x is the length of the series*

$$1 < N_1 < N_2 < \cdots < N_x = N$$

such that $N_i \lhd G$ for each $1 \leq i \leq x$. In particular, $\kappa(G/N) \leq \kappa(G)$.

Proof. Let G and N be as in the hypothesis of the Proposition. Let A and B be two minimal subgroups of G. If $\kappa(G/N) = -2$, then there is a normal series

$$1 < N_1 < N_2 < \cdots < N_x = G,$$

and we may easily form $x-2$ independent paths $\gamma_i = (A, N_iA, N_iB, B)$, $1 \leq i \leq x-2$, between A and B. (In case of a possible coincidence of the vertices we can safely shorten the paths.) A similar argument shows that we may construct $x-1$ independent paths if $\kappa(G/N) = -1$.

Next suppose that $\kappa(G/N) \geq 0$, i.e. $|V(G/N)| \geq 1$. By the Correspondence Theorem there is a bijection between the subgroups of the quotient group G/N and the subgroups of G that are containing N. Observe that NA and NB correspond to some subgroups of G/N that are either trivial or minimal. Then, as G/N is k-connected by the assumption, we may construct at least k additional independent paths $\gamma_j = (A, \ldots, B)$, $x \leq j \leq k+x-1$, such that the inner vertices represents some proper subgroups of G containing N. □

Corollary 5.7. *Let G be a supersolvable group with $\ell := \ell(G)$. Then $\kappa(G) \geq \ell - 3$. In particular, all p-groups of order $> p^\alpha$ are $(\alpha-2)$-connected.* □

5.2 Non-simple Groups

Proof of Theorem 5.1 (Isaacs). Let G be a finite non-simple group and N be a minimal normal subgroup of G. Suppose that $\Gamma(G)$ is not connected. Let A be a subgroup of G which does not lie in the component of N in $\Gamma(G)$. Then $NA = G$, as otherwise, (N, NA, A) would be a path between N and A. Also $N \cap A = 1$, as otherwise, N and A would be linked via the subgroup $N \cap A$. Therefore $[G : N] = |A|$. Since this equality holds for every subgroup that does not lie in the component containing N, it holds also for any non-trivial subgroup of A. As a consequence $[G : N] = |A| = q$ is a prime number. Moreover, A is a maximal subgroup of G. To see this, suppose that there exists a proper subgroup B containing A. Since B does not lie in the same component with N, we have $|B| = q$, i.e. B coincides with A.

Let Q be a Sylow q-subgroup of G containing A. Since A is a maximal subgroup, either $G = Q$ or $A = Q$. In the first case since N is a minimal normal subgroup and G is a q-group, the order of N is q. As N and A are distinct subgroups of same order q and as $G = NA$, we see that $G \cong \mathbb{Z}_q \times \mathbb{Z}_q$. Clearly, $\Gamma(\mathbb{Z}_q \times \mathbb{Z}_q)$ is not connected.

In the latter case since G is not a q-group and since $G = NA$, there must be a prime p dividing $|N|$ and different from q. We want to show that N is a p-group. Suppose contrarily that N is not a p-group. Let P be a Sylow p-subgroup of N and $T = N_G(P)$. (Notice that $G \neq T$, as N is a minimal normal subgroup.) By the Frattini Argument (see Theorem 1.14) $G = NT$ which, in turn, implies that $q \mid |T|$. Since A is a Sylow q-subgroup, some conjugate of T contains A. However, this contradicts with the maximality of A. Therefore, N is a p-subgroup. Further, N must be an elementary abelian subgroup since it is a minimal normal subgroup.

Consider the normalizer $N_G(A)$. Since A is a maximal subgroup, there are two possibilities. If A is a normal subgroup of G, then A centralizes N; hence, $|N| = p$ and $G \cong \mathbb{Z}_p \times \mathbb{Z}_q$. Clearly, $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ is not connected. And if A is self-normalizing, G is a group described as in the second part of Theorem 5.1. To conclude the proof it is enough to show that $\Gamma(G)$ is not connected in such a case.

Let H be a proper non-trivial subgroup of G. We want to show that H is either a subgroup of the unique (normal) Sylow p-subgroup N of G or it is a Sylow q-subgroup. Obviously, $\Gamma(G)$ is not connected if this is the case. Suppose contrarily, H is neither a p-subgroup nor a q-subgroup. Then $q \mid |H|$ as $|G| = |NA| = p^{\alpha}q$ for some integer $\alpha \geq 1$. Hence, H contains a conjugate of A and we may suppose that H contains A by replacing H with some conjugate of it if necessary. Then $NH = G$ and it follows that $N \cap H \triangleleft G$. (Notice that $N \cap H$ is normalized by N as N is an abelian subgroup and $N \cap H$ is normalized by H as N is a normal subgroup.) Since N is a minimal normal subgroup, either $N \cap H = 1$ or $N \cap H = N$ yielding either $|H| = q$ or $H = G$. However, this contradicts with the assumption that H is a proper subgroup which is not a q-subgroup.

□

Notice that for a finite non-simple group G, the connectivity of G is 1 if and only if G satisfies the 1-valency condition.

5.3 Solvable Groups

Lemma 5.8. *Let G be a finite solvable group. Then $\kappa(G) = 2$ if and only if G satisfies the 2-valency condition.*

Proof. Sufficiency is obvious. Let G be a finite solvable group satisfying the 2-valency condition. We want to show that there exist at least two independent paths between any pair of minimal subgroups A_1 and A_2. If $\langle A_1, A_2 \rangle$ is a second maximal subgroup, then clearly there are two independent paths between them. Thus, for the rest we assume $\langle A_1, A_2 \rangle$ is either G or a maximal subgroup. Let M be a maximal subgroup of prime index and N be a minimal normal subgroup. Notice that since G is solvable, there exist a subgroup of prime index and minimal normal subgroups are elementary abelian. Further, let $A_1 < H_1, K_1$ and $A_2 < H_2, K_2$ such that $NA_1 \neq H_1$ and $NA_2 \neq H_2$.

Case I: Suppose that N is of prime index in G and take $M = N$.

Case I (a): $A_1, A_2 < M = N$. Obviously (A_1, M, A_2) is a path and $\langle A_1, A_2 \rangle = M \cong \mathbb{Z}_p \times \mathbb{Z}_p$. And the order of G is either p^3 or p^2q. By the Product formula, (A_1, H_1, H_2, A_2) is also a path and independent from the first.

Case I (b): M and A_1 are distinct p-groups. Then G is also a p-group and in turn $|G| = p^2$ since M must be a cyclic group of prime order. However, intersection graph of a group of order p^2 or pq consists of isolated vertices and G does not satisfy 2-valency condition in that case.

Case I (c): M is an elementary abelian p-group of rank $n \geq 2$ and $A_1 \cong \mathbb{Z}_q$. In particular $|G| = p^n q$. Observe that $A_1 \ntrianglelefteq G$, as otherwise, G would be an abelian group contradicting with the fact that M is a minimal normal subgroup. Moreover, $O_p(H_1)$ and $O_p(K_1)$ are trivial (again this is because M is a minimal normal subgroup) and this in turn implies $H_1, K_1 \trianglelefteq N_G(A_1) < G$. (Notice that this implies $n \geq 3$). Hence, we may assume $A_1 < H_1 < K_1 = N_G(A_1)$. If $A_2 < M$, then we have the two independent paths (A_1, H_1, M, A_2) and (A_1, K_1, T, A_2) where T is a subgroup of order p^{n-1} containing A_2. And if A_2 is a conjugate of A_1, then (A_1, H_1, M, H_2, A_2) and (A_1, K_1, T, K_2, A_2) are two independent paths between A_1 and A_2 where $H_2 < K_2$.

Case II: Suppose that $[G : N]$ is not prime. Then $NA_1 \neq G$, $NA_2 \neq G$. If one of NA_1 and NA_2 coincides with M, say NA_1, then we may take (A_1, H_1, NA_2, A_2) and (A_1, K_1, M, H_2, A_2) as independent paths. If both NA_1 and NA_2 coincides with M, then we may take (A_1, M, A_2) and (A_1, H_2, N, H_2, A_2). Finally, if $NA_1 \neq M$ and $NA_2 \neq M$, then (A_1, NA_1, NA_2, A_2) and (A_1, H_1, M, H_2, A_2) are two independent paths between A_1 and A_2. □

Lemma 5.9. *Let G be a finite p-group. Then $\kappa(G) < 2$ if and only if*

1. $|G| = p^\alpha$, $0 \leq \alpha \leq 2$,

2. $|G| = p^3$ *and* $G \ncong Q_8$ *or* $G \ncong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$

In particular, all p-groups of order $> p^3$ are 2-connected.

Proof. By Lemma 5.8, all we need to do is to determine p-groups which does not satisfy the 2-valency condition. Clearly, intersection graph of a group of order p^α, $0 \leq \alpha \leq 2$, is either empty graph or consists of isolated vertices. Hence 2-valency condition does not hold for those groups.

Suppose that $|G| = p^3$. If G has a unique maximal subgroup, then $G \cong \mathbb{Z}_{p^3}$ and $\Gamma(G) \cong K_2$. So it is not 2-connected in this case. If G has more than one maximal subgroup and $\Phi := \Phi(G)$ is non-trivial, then either G has a unique minimal subgroup (which is Φ) or there are minimal subgroups different from Φ. In the first case, $G \cong Q_8$ and $\Gamma(Q_8) \cong K_4$. That is, G is 3-connected. In the latter case, ΦA is a maximal subgroup of G and it is the unique subgroup of order p^2 containing A, as all the maximal subgroups contain Φ. If Φ is trivial, then G is elementary abelian and by the Correspondence Theorem any minimal (normal) subgroup is contained in $p+1$ maximal subgroups. Therefore, G is 2-connected in this case, as the 2-valency condition holds. .

Suppose that $|G| = p^{\alpha}$, $\alpha > 3$. Then any minimal subgroup of G is contained in a subgroup of order p^2 and by a subgroup of order p^3. Hence G satisfies 2-valency condition. □

Lemma 5.10. *Let G be a group of order p^2q with a Sylow p-subgroup P. Then $\kappa(G) < 2$ if and only if one of the following holds.*

1. *$P \cong \mathbb{Z}_{p^2}$.*

2. *$P \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and there exists a non-normal subgroup of G of order p.*

Proof. Let G be a group of order p^2q with a Sylow p-subgroup P and a Sylow q-subgroup Q.

Case I: $P \cong \mathbb{Z}_{p^2}$. If $P \lhd G$, then G has a unique subgroup of order p. However, this implies any q-subgroup is contained in one and only one subgroup (of order pq). Assume that $P \ntriangleleft G$. Since P is a cyclic group, P and any conjugate of it contains a unique minimal subgroup, hence either $O_p(G) \cong \mathbb{Z}_p$ is the unique subgroup of order p or any pair of Sylow p-subgroups intersects trivially. Clearly, in the first case there exists a unique subgroup containing Q. In the latter case, $O_q(G) = Q$ is a normal subgroup of G and there exists a normal subgroup $M \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p$ containing all subgroups of order p. Those two facts imply that M is the unique subgroup containing Q.

Case II: $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

Case II (a): $Q \lhd G$. Clearly, any subgroup of order p is contained in a subgroup of order p^2 and by a subgroup of order pq. We shall observe that Q is also contained in at least two subgroups of order pq. Let $U, V < P$ be two distinct subgroups of order p. Clearly, QU and QV are of order pq. Suppose that $QU = QV$. As $\langle U, V\rangle = P$, this gives a contradiction. Hence, QU and QV are distinct subgroups containing Q and 2-valency condition holds. Notice that any subgroup of order p is normal in G in this case.

Case II (b): $Q \ntriangleleft G$.

Case II (b)(i): $P \triangleleft G$. First, we shall observe that either there is no normal subgroup of G of order p or there are more than one. As $Q \ntriangleleft G$, the index of $N_G(Q)$ is either p or p^2 and this implies $q \mid p-1$ or $q \mid p+1$. Consider the action of Q on the subgroups of P by conjugation. Since the length of an orbit is either 1 or q, the number of fixed points (the number of normal subgroups of order p) may be 0, 2, or a multiple of q. Next, we determine the groups in which Q is contained in at most one subgroup. If Q acts on P irreducibly (without fixed points) and Q is contained in subgroup M of order pq, then $M \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p$ and it normalizes Q. Moreover, it is the unique subgroup containing Q as $Q \ntriangleleft G$. If there are distinct normal subgroups U and V of order p, then clearly UQ and UV are two distinct subgroups containing Q. Finally, we determine the groups in which a (non-normal) subgroup T of order p is not contained in a subgroup of order pq. As we have seen that groups in which Q acts on P irreducibly does not satisfy 2-valency condition, we further assume that there exist two normal subgroups U and V of order p. Suppose that T is contained in a subgroup of order pq. Then as $T \ntriangleleft G$, we have $M \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p$. On the other hand, both UQ and VQ cannot be isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_q$, as otherwise, $Q < Z(G)$ implying G is abelian. That is, one of UQ and VQ is isomorphic to $\mathbb{Z}_p \rtimes \mathbb{Z}_q$ which is impossible. Therefore, T is not contained in any subgroup of order pq.

Case II (b)(ii): $P \ntriangleleft G$. We show that there is no such group. Suppose that it exists. Since G is solvable, there exists a normal subgroup M of order pq and $U := O_p(G)$ is non-trivial.

(I) U is the only normal subgroup of order p. Suppose contrarily $V \triangleleft G$ be a normal subgroup of order p different from U. Then $\langle U, V \rangle$ be a normal Sylow p-subgroup which is a contradiction.

(II) $M = UQ$ and contains all q-subgroups. As M is a normal subgroup, it contains all (Sylow) q-subgroups. Therefore $M \cong \mathbb{Z}_p \times \mathbb{Z}_q$ and $q \mid p-1$. In particular, there are

$[M:Q]=p$ subgroups of order q. From (I), we know that U is the unique subgroup of order p normalized by Q, hence $M=UQ$.

(III) $N_G(Q)=QZ$ for a subgroup Z of order p and $QZ\cong\mathbb{Z}_p\times\mathbb{Z}_q$. By (II), $[G:N_G(Q)]=p$ and since $q\mid p-1$, we have $N_G(Q)\cong\mathbb{Z}_p\times\mathbb{Z}_q$.

(IV) $Z\lhd G$ which contradicts with (I). □

As is seen by Lemma 5.10, many of the groups of order p^2q does not satisfy 2-valency condition. Compare it with the following result.

Lemma 5.11. *Let G be a group of order p^2q. Then G is 3-connected if and only if*

$$G\cong\langle a,b,c\mid a^p=b^p=c^q=1, ab=ba, cac^{-1}=a^{\lambda}, cbc^{-1}=b^{\lambda}\rangle$$

where $q\mid p-1$ and $\lambda>1$ is any integer such that $\lambda^q\equiv 1\pmod{p}$.

Proof. Let G be a 3-connected group of order p^2q and let Q be a q-subgroup of G. Take a minimal p-subgroup U of G and let P be a Sylow p-subgroup containing U.

Case I: $Q\lhd G$. Clearly, QU is the unique subgroup of order pq containing U. Moreover, there exist at least two distinct Sylow p-subgroups containing U, as otherwise, 3-valency condition does not hold. This, in turn, implies that $N_G(U)=G$. Suppose that Sylow p-subgroups are cyclic. Then, U is the unique subgroup of order p and QU is the unique subgroup containing Q. Again 3-valency condition cannot be satisfied. Now, suppose that Sylow p-subgroups are elementary abelian. Since U is a normal subgroup of G and since this must be the case for any minimal p-subgroup, P is also a normal subgroup of G which is a contradiction.

Case II: $Q\not\lhd G$. Since G is a solvable group, the p-core $O_p(G)$ is a non-trivial normal subgroup of G. Thus, we shall consider following two sub-cases.

Case II (a): $P\not\lhd G, U\lhd G$. Suppose that Sylow p-subgroups are cyclic. As in Case I, PQ is the unique subgroup containing Q and this case can be discarded. Now, suppose that Sylow p-subgroups are elementary abelian. However, by the proof of Lemma 5.10 we know that no such group exists.

Case II (b): $P\lhd G$. As in previous cases, P cannot be a cyclic subgroup. Thus $P\cong\mathbb{Z}_p\times\mathbb{Z}_p$. We claim that $N_G(Q)=Q$. Assume contrarily that $N_G(Q)$ is a group of order pq.

Then $N_G(Q)$ is self-normalizing and by the Product Formula any two distinct conjugate of it intersect at Q; however, Q is normal in both of them which is a contradiction. Therefore, any subgroup of order pq must be isomorphic to $\mathbb{Z}_p \rtimes \mathbb{Z}_q$ and since G is 3-connected there must exist at least three such subgroups containing Q. To write a presentation for G, let a,b,c be three elements generating G such that a,b are of order p and c is of order q. Moreover, we may suppose that c normalizes $\langle a\rangle$, $\langle b\rangle$, and $\langle ab^k\rangle$ where $k \geq 1$ is an integer. (Notice that any subgroup of order p is generated by some element ab^k for some integer k.) In other words, we have the relations $cac^{-1} = a^{\lambda_1}$, $cbc^{-1} = b^{\lambda_2}$, and $cab^kc^{-1} = (ab^k)^t = a^t b^{tk}$ for some integers λ_1, λ_2, t. On the other hand, $cab^kc^{-1} = a^{\lambda_1}b^{\lambda_2 k}$ implying $\lambda_1 \equiv \lambda_2 \pmod{p}$ and hence we may take $\lambda := \lambda_1 = \lambda_2$. As a consequence all p-subgroups are normal in G. Notice that $\lambda = 1$ implies $Q \lhd G$, hence $\lambda > 1$. Moreover, since $a = c^qac^{-q} = a^{\lambda^q}$, we have $\lambda^q \equiv 1 \pmod{p}$. Conversely, it can be verified that a group with this presentation is of order p^2q.

Let G a the group with the given presentation. To conclude the proof, we shall show that G is 3-connected. We claim G satisfies 3-valency condition. From the previous arguments, $\langle c\rangle$ is contained in at least three subgroups of order pq and any element of order q acts on P in the same way as c does. Moreover, all p-subgroups are normal and there are clearly more than three proper subgroups containing any subgroup of order p. Finally, since the maximal subgroups of G form a complete graph in $\Gamma(G)$ by the Product Formula, we deduce that G is 3-connected. □

Proof of Theorem 5.2. Let G be a finite solvable group which is not a p-group. (Finite p-groups that are not 2-connected are presented in Lemma 5.9.) Since G is a solvable group by assumption, there exists a maximal subgroup M of G of prime index. By Lemma 5.8, it is enough to determine groups for which the 2-valency condition does not hold. Suppose that G does not satisfy 2-valency condition. Let A be a minimal subgroup of order q such that A is (strictly) contained in at most one proper subgroup of G.

First, suppose that $q^2 \mid |G|$ and let Q be the Sylow q-subgroup containing A. Then either $M = Q$ and $|G| = pq^2$ where $[G:M] = p$, or $M \neq Q$ and $[G:M] = q$ as Q is the unique subgroup containing A. The first case was considered in Lemma 5.10. In the latter case, if $Q \lhd G$ then $|G| = pq^2$ as Q is the only proper subgroup containing

A; hence, we again refer to Lemma 5.10. And if $Q \ntriangleleft G$, then we may further assume that M is a normal subgroup. (Notice that since Sylow q-subgroup Q is maximal, there must be a normal subgroup of G of index q in that case.) Moreover, M must be a minimal normal subgroup as Q is the unique proper subgroup containing A. Since G is solvable $M \cong \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ for some prime p different from q. However, this is impossible as $q \mid |M|$.

Next, suppose that A is a Sylow q-subgroup. If $|G| = pq$, clearly 2-valency condition does not hold and the case $|G| = p^2q$ was already considered in Lemma 5.10. Suppose that $p, r \mid |G|$ where p and r distinct prime numbers different from q. Since G is solvable, there exist a Hall $\{p,q\}$-subgroup and a Hall $\{q,r\}$-subgroup containing A. Hence, we may assume $|G| = p^{\alpha}q$, $\alpha \geq 3$. If $A \triangleleft G$, then it is contained in more than one proper subgroup. Thus $A \ntriangleleft G$. Furthermore, $P \triangleleft G$ in this case, where P is the Sylow p-subgroup of G. Suppose to the contrary that $P \ntriangleleft G$. Since G is solvable, there exists a normal subgroup M of index p. Hence, M contains all Sylow q-subgroups implying $[M : N_M(A)] = [G : N_G(A)]$. On the other hand, since M is the unique proper subgroup of G containing A, either $N_M(A) = A$ or $N_M(A) = M$. In the first case $N_G(A)$ would be a subgroup order pq contradicting with the assumption that A is contained in at most one subgroup. And in the latter case A would be a normal subgroup of G which is again a contradiction. Therefore $P \triangleleft G$. For the rest of the proof, we take $Q := A$. Let N be a minimal normal subgroup of G. Since G is solvable N is elementary abelian. Now we examine two cases:

Case I: $P = N$. Thus, P is elementary abelian. We claim that G does *not* satisfy 2-valency condition if and only if Q acts on P irreducibly and the order of $N_G(Q)$ is at most pq. Sufficiency is obvious. For the necessity, observe that if K is a minimal subgroup properly containing Q, then either Q is a normal subgroup of K (hence $K \leq N_G(Q)$) or $O_p(K)$ is a non-trivial normal subgroup of K (hence $O_p(K) \triangleleft G$ and the action of Q is not irreducible).

Case II: $P \neq N$.

Case II (a): P is elementary abelian. A Theorem of Gaschütz (see Theorem 1.11) states that an abelian normal p-subgroup has a complement in G if and only if it has a complement in a Sylow p-subgroup. Clearly, N is complemented in the elementary abelian p-subgroup P. Let K be the complement of N in G. Then NQ and K are two distinct subgroups containing A.

Case II (b): P is *not* elementary abelian. As NQ is a proper subgroup of G containing A, we conclude that P and N are the only proper non-trivial normal subgroups of G. Moreover, N coincides with $\Phi(P)$, since $\Phi(P)$ is a non-trivial characteristic subgroup of P. Notice that a characteristic subgroup of a normal subgroup is normal in the whole group (see [14, Lemma 5.20]). That is, P is a p-group such that its Frattini subgroup N is elementary abelian. Moreover, since Q is contained in at most one subgroup, either $N_G(Q) = Q$ or $N_G(Q) = NQ$. Notice that in the latter case we have $NQ \cong \mathbb{Z}_p \times \mathbb{Z}_q$. Consider the action of Q on the set of subgroups of P by conjugation. It is easy to see that the fixed points of this action must be precisely P, N, and the trivial subgroup. Clearly, Q acts on N irreducibly and by the Correspondence Theorem the induced action of Q on P/N is also irreducible. Conversely, if the action of Q on N and P/N are irreducible, then N is the only proper non-trivial subgroup of P fixed by Q. To see this, take an element $a \in Q$ and consider its action. If $1 < X < N$, then $X^a \neq X$ by assumption. Let $N < NX \neq P$ and $X^a = Y$. We want to show that $X \neq Y$. By assumption $(NX/N)^a \neq NX/N$. However, $(NX/N)^a = (NX)^a/N = NY/N$ implying $X \neq Y$. □

Remark 5.1. The "smallest" non-solvable group is the alternating group A_5 on five letters and its order is divisible by three distinct primes. However, it does not satisfy the 2-valency condition. To be more precise, if H is a subgroup of order 5, then there is exactly one proper subgroup, say K, of A_5 containing H. To see this, first observe that any maximal subgroup M of A_5 has index ≥ 5, as otherwise, there would be a homomorphism $\phi\colon A_5 \to S_{G/M}$ with a non-trivial kernel which is impossible. Hence the only possibility for the order of K is 10. Since H is not a normal subgroup of A_5 and since H is normalized by the maximal subgroup K, we see that K is the unique subgroup containing H.

5.4 Nilpotent Groups

As it was mentioned at the beginning of this chapter, to show that the intersection graph of a given solvable group is 3-connected we must claim the existence of "sufficiently" many vertices to construct at least three independent paths for any pair of minimal subgroups which seems to be not an easy task. (Or, conversely, we must claim the non-existence of vertices to verify that the graph is not 3-connected.) Of course, Hall Theorems enables us to claim that 3-valency condition is satisfied if there are at least four distinct prime divisors of the order of the group. Also, it is not difficult to show that such groups are indeed 3-connected (compare with Corollory 5.13 below). However, if there are less than four prime divisors things are more complicated. Therefore, in this section we restrict our attention to nilpotent groups.

Lemma 5.12. *Let G be a finite supersolvable group. Then $\kappa(G) = 3$ if and only if G satisfies the 3-valency condition.*

Proof. Sufficiency is obvious. Let G be a finite supersolvable group satisfying the 3-valency condition. We want to show that there are at least three independent paths between any pair of minimal subgroups A and B. Clearly, we may suppose that $\ell(G) \geq 3$, since groups of order p^2 and pq does not satisfy even 1-valency condition. Notice that as G is supersolvable, any maximal subgroup is of prime index; and thus, if X is a non-trivial subgroup of G which is not minimal, then X intersects any maximal subgroup non-trivially.

Case I: G has exactly one maximal subgroup. Then G is a cyclic group of prime power order p^{α} and it satisfies 3-valency condition if and only if $\alpha \geq 5$ which is the case if and only if $\kappa(G) \geq 3$.

Case II: G has exactly two maximal subgroups. If G is a p-group, then the number of maximal subgroups $\equiv 1 \pmod{p}$ (see Theorem 1.3). Also, if $|G|$ is divisible by three distinct prime divisors, then there would be at least three maximal subgroups (containing the corresponding Hall subgroups). Hence $|G| = p^a q^b$. Obviously, maximal subgroups must be normal and hence $G \cong P \times Q$ is nilpotent group where P and Q are Sylow p- and Sylow q- subgroups respectively. Observe that if $H \trianglelefteq P$ and $K \trianglelefteq Q$, then $HK \trianglelefteq G$ as G is the direct product of P and Q. However, since any maximal subgroup of a p-group is normal, P and Q have exactly one maximal subgroups meaning both are cyclic groups of prime power order and in turn G is also a cyclic group. It can be easily observed that 3-valency condition is equivalent to the 3-connectedness for such groups.

Case III: G has at least three maximal subgroups. Let M_i be maximal subgroups, X_i be subgroups containing A, and Y_i be subgroups containing B for $1 \leq i \leq 3$. Then (A, X_i, M_i, Y_i, B), $1 \leq i \leq 3$, are three independent paths between A and B. Of course, in case of a coincidence the corresponding paths can be shortened accordingly. □

Corollary 5.13. *Let G be a finite supersolvable group with $\kappa(G) < 3$. Then the number of prime divisors of $|G|$ is at most three. Moreover, if there are exactly three distinct prime divisors, then $|G|$ is square-free.*

Proof. Obviously, if there are more than three distinct prime divisors of $|G|$, then G satisfies 3-valency condition, hence is 3-connected as well. Let $|G| = p^\alpha q^\beta r^\gamma$ where p, q, r distinct prime numbers and $\alpha \geq 2$. Let A be a minimal subgroup. If A is a p-subgroup, then A is properly contained in a Sylow p-subgroup, by a Hall $\{p, q\}$-subgroup, and by a Hall $\{p, r\}$-subgroup. If A is a q-subgroup, then A is contained in Hall $\{p, q\}$-subgroup, by a Hall $\{q, r\}$-subgroup and by a maximal subgroup containing the corresponding Hall $\{q, r\}$-subgroup. Similarly, there are at least three proper subgroups containing A whenever A is a r-subgroup. □

By Corollory 5.7, we know that if G is a supersolvable group such that $\kappa(G) < 3$, then $\ell(G)$ is at most 5. Moreover, by using Corollory 5.13 (and ignoring the $\ell(G) \leq 2$ cases), we may reduce the possible cases for the order of G into the following list

Table 5.1 : Possible orders of a finite supersolvable group G with $\kappa(G) < 3$.

$\lvert G\rvert = p^5$,	$\lvert G\rvert = p^4$,	$\lvert G\rvert = p^3$,
$\lvert G\rvert = p^4 q$,	$\lvert G\rvert = p^3 q$,	$\lvert G\rvert = p^2 q$,
$\lvert G\rvert = p^3 q^2$,	$\lvert G\rvert = p^2 q^2$,	$\lvert G\rvert = pqr$.

Actually, we may still eliminate some further cases.

Lemma 5.14. *Let G be a finite supersolvable group with $\kappa(G) < 3$. Then the order of G must equal to one of the following*

$$p^\alpha \ (0 \leq \alpha \leq 4), \quad p^3 q, \quad p^2 q^2, \quad p^2 q, \quad pqr, \quad pq$$

where p, q, and r are distinct prime numbers. Moreover, if G is nilpotent, then $|G| \neq p^2 q^2$; and if G is nilpotent and of order $p^3 q$, then G is cyclic.

Proof. By Lemma 5.12 we know that G is 3-connected, if 3-valency condition holds. Since G is supersolvable group, any minimal subgroup is contained in at least $\ell(G)-2$ proper subgroups. Thus, $\ell(G) < 5$. This eliminates the first column of Table 5.1.

Let G be a nilpotent group of order p^2q^2 and let A be a minimal subgroup of G. Obviously, G is an abelian group and the normal subgroup A of G is contained in a subgroup of order pq, by a subgroup of order p^2q and by the Sylow subgroup. Hence, G satisfies 3-valency condition.

Let G be a nilpotent group of order p^3q. Then the Sylow q-subgroup Q of G is a normal subgroup and any minimal subgroup of order p is contained in a subgroup of order p^2, by a subgroup of order p^3, and by a subgroup of order pq. Suppose that Q is not contained in more than two proper subgroups. However, this is possible only if the Sylow p-subgroup P is normal and $P \cong \mathbb{Z}_{p^3}$. Thus, G is cyclic as well. □

Lemma 5.15. *Let G be a finite p-group. Then $\kappa(G) < 3$ if and only if one of the following holds.*

1. $|G| = p^\alpha$ $(0 \leq \alpha \leq 3)$ *and neither* $G \cong Q_8$ *nor* $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.
2. *G is a group of order p^4 such that*
   - (a) $G \cong \mathbb{Z}_{p^4}$, *or*
   - (b) $\Phi(G) \cong \mathbb{Z}_{p^2}$ *and* $G \ncong Q_{16}$, *or*
   - (c) $\Phi(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$, $Z(G) < \Phi(G)$ *and*
     $G \ncong \langle a,b,c \mid a^9 = b^3 = 1, ab = ba, a^3 = c^3, bcb^{-1} = c^4, aca^{-1} = cb^{-1} \rangle$.

In particular, all p-groups of order $> p^4$ are 3-connected.

Proof. Obviously, $|G| = p^2$ implies $\Gamma(G)$ consists of isolated vertices, hence it cannot be connected. So, let's assume $|G| > p^2$.

Case I: $|G| = p^3$. First, suppose that $\Phi(G) \neq 1$. If all maximal subgroups of G are cyclic, then G has a unique minimal subgroup and its intersection graph is complete. However, $\Gamma(\mathbb{Z}_{p^3})$ has two vertices whereas $\Gamma(Q_8)$ has four, thus only the latter is 3-connected among them. If there exists a maximal subgroup $M \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then any minimal subgroup X of M which is different from $\Phi(G)$ is not contained in any maximal subgroup other than M, as $\langle X, \Phi(G) \rangle$ uniquely determines M. That is, G does not satisfy 3-valency condition in such case. Now, suppose that $\Phi(G)$ is trivial, i.e. G is isomorphic to the elementary abelian group of rank 3. By the Correspondence Theorem, any minimal subgroup is contained in $p+1$ maximal subgroups. Also, since any two maximal subgroups of a p-group intersects non-trivially (by the Product Formula), maximal subgroups form a complete subgraph in $\Gamma(G)$. Therefore, G is 3-connected in this case.

Case II: $|G| = p^4$. Recall that the *rank* of a p-group is the dimension of $G/\Phi(G)$ as a vector space over the field of p-elements. If the rank of G is four or three, i.e. $\Phi(G) \cong 1$ or $\mathbb{Z}_p$, then for any minimal subgroup X we may form $\Phi(G)X$ which is contained in at least $p+1$ maximal subgroups of G. Clearly, G is 3-connected in this case. On the other hand, if the rank of G is one, i.e. $G \cong \mathbb{Z}_{p^4}$, then $\Gamma(G)$ has exactly three vertices and hence cannot be 3-connected. Now we shall confine ourselves to the case G is of rank two.

Suppose that $\Phi(G) \cong \mathbb{Z}_{p^2}$. If G has a unique minimal subgroup then it is isomorphic to the quaternion group Q_{16} and its intersection graph is complete, hence 3-connected as well. Let us assume there exists a minimal subgroup X of G which is different from the minimal subgroup P of $\Phi(G)$. Notice that P is a necessarily normal subgroup of G. Then the only maximal subgroup containing X is $M := \Phi(G)X$ as any maximal subgroup contains $\Phi(G)$. This in turn implies that PX is the only subgroup of order p^2 containing X, since P is the Frattini subgroup of M. Therefore G does not satisfy 3-valency condition in such a case, hence it is not 3-connected.

Suppose that $\Phi(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$. If G is abelian, then it is isomorphic to $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$ and any minimal subgroup is contained in the Frattini subgroup, hence it is 3-connected. If G is not abelian, then either $Z(G) = \Phi(G)$ or $Z(G) < \Phi(G)$. This is because, any cyclic extension of a central subgroup is abelian and $Z(G)$ intersects any normal subgroup non-trivially whenever G is a p-group. Let $Z(G) = \Phi(G)$. Then a minimal subgroup P is normal in G if and only if P is a subgroup of $\Phi(G)$. We show that G is 3-connected in such a case. Let P_i, $1 \leq i \leq p+1$ be minimal subgroups of $\Phi(G)$ and let X, Y be two arbitrary minimal subgroups that are not contained in $\Phi(G)$. We show that there are at least three independent paths between any pair of minimal subgroups. Clearly, this holds if the endpoints are P_i and P_j for any $i \neq j$. Let $A_i := P_iX$ for $1 \leq i \leq p+1$.

Since $X \not\leq \Phi(G)$ and $|A_i| = p^2$, $A_i \cap A_j = X$ for $i \neq j$. Then, we may form three internally independent paths (X, A_1, M, P_i), (X, A_2, N, P_i), and (X, A_3, T, P_i) between X and P_i where M, N, and T are mutually distinct maximal subgroups. Let $B_i := P_i Y$ for $1 \leq i \leq p+1$. Clearly, (X, A_i, B_i, Y), $1 \leq i \leq p+1$ are independent paths between X and Y.

Let $Z(G) < \Phi(G)$. Obviously, $Z := Z(G)$ is the unique minimal normal subgroup of G. Observe that any subgroup Y of order p or p^2 which is not a subgroup of $\Phi(G)$ is contained in exactly one maximal subgroup M. Otherwise, there exist two distinct maximal subgroups such that their intersection strictly contains $\Phi(G)$ which is impossible. Moreover, the Frattini subgroup $Y \cap \Phi(G)$ of M must be a normal subgroup of G, as it is fixed by the inner automorphisms of G. Hence $Z = Y \cap \Phi(G)$. Also, if $Y \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then there exists a subgroup X of order p which is not contained in $\Phi(G)$. Clearly, $Y = ZX$ is the only subgroup of order p^2 containing X. Therefore, G does not satisfy 3-valency condition in such case. Let us assume any subgroup Y of order p^2 different from $\Phi(G)$ is cyclic. By the above argument, the unique minimal subgroup of Y is Z and there are no minimal subgroups of G which is not contained in $\Phi(G)$. Clearly, G is 3-connected in this case. Now we show that under these conditions G is unique up to isomorphism.

(I) There exists a maximal subgroup A which is abelian, moreover $A \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$. By the N/C Lemma (see Theorem 1.16), $N_G(\Phi(G))/C_G(\Phi(G)) = G/C_G(\Phi(G))$ can be embedded into $\mathrm{Aut}(\Phi(G)) \cong \mathbb{Z}_p \times \mathbb{Z}_p$ which is of order $(p^2-1)(p^2-p)$. Then $C_G(\Phi(G)) = \Phi(G)$ implies $p^2 \mid |\mathrm{Aut}(\Phi(G))|$ and this is impossible. Also, since the center of G is a proper subgroup of $\Phi(G)$, then $C_G(\Phi(G))$ is not the whole group G either. Thus, $A := C_G(\Phi(G))$ is an abelian subgroup of order p^3; and since any maximal subgroup Y of A different from $\Phi(G)$ is cyclic, $A \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$.

(II) $A := C_G(\Phi(G))$ is the unique abelian group of order p^3, moreover $M \cong \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$ for any maximal subgroup M different from A. Suppose that there exists an abelian subgroup $B \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$ different from A. Then, as $\langle A, B \rangle = G$ and $A \cap B = \Phi(G)$, the center Z of G contains $\Phi(G)$ which is a contradiction. Therefore, any maximal subgroup M other than A is isomorphic to $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$, since any non-Frattini subgroup of order p^2 is cyclic.

(III) G has a presentation

$$\langle a,b,c \mid a^{p^2} = b^p = 1, ab = ba, a^p = c^{kp}, bcb^{-1} = c^{1+p}, aca^{-1} = c^{1+mp}b^n \rangle$$

for some suitable values of the prime p and integers k,m,n. Let $a,b \in A$ and $c \in M$ such that a,c are of order p^2 and b is of order p. Clearly, those elements generate G and we have $ab = ba$, $a^p, c^p \in Z$, and $bcb^{-1} = c^{1+p}$ as $\langle b,c \rangle \cong \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$. Moreover, any conjugate of c can be written as $c^\gamma b^\beta$ for some integers γ, β; and since $ac^pa^{-1} = c^p = c^{rp}$ where $\gamma \equiv r \pmod p$, we have $r = 1$.

(IV) $p = 3$. We want to show that for $p > 3$, there is an element g of order p such that $g \notin \Phi(G)$. Then the subgroup generated by this element is a minimal one and it is not contained in the Frattini subgroup contrary to our assumption. Thus, we shall deduce $p = 3$. Using the above relations, we may obtain $b^\beta c^\gamma = c^\gamma b^\beta c^{\beta\gamma p}$ and $ac^x = c^x ab^{xn}u$ where $u = c^{(\frac{1}{2}x(x-1)n + xm)p}$. Clearly $ac^x \notin \Phi(G)$ for $p \nmid x$. By some further computation

$$(ac^x)^p = c^{xp\{1+\frac{1}{6}(p-1)p(p+1)xn\}} a^p b^{\frac{1}{2}p(p-1)xn} u^{\frac{1}{2}p(p-1)} = c^{xp\{1+\frac{1}{6}(p-1)p(p+1)xn\}} a^p.$$

However, this formula implies that $(ac^{-k})^p = 1$ for $p \neq 3$.

(V) Without loss of generality we may take $k = 1, m = 0, n = -1$. Suppose that $p = 3$. Clearly, $k \in \{-1,1\}$ and $m,n \in \{-1,0,1\}$. However, $n = 0$ implies that $C_G(\langle c \rangle)$ is an abelian group of order p^3 (compare with (I)). As $C_G(\langle c \rangle)$ is different from A, this contradicts with (II). Moreover, using the relation presented in (IV), we see that $(ac^k)^3 = c^{3(n-k)}$. Therefore, n and k have opposite parity, as otherwise, $\langle ac^k \rangle$ would be a minimal subgroup which is not contained in $\Phi(G)$. Thus, there are totally six distinct triples (k,m,n) that we shall consider. If triples (k_1,m_1,n_1) and (k_2,m_2,n_2) yields isomorphic groups, we simply write $(k_1,m_1,n_1) \sim (k_2,m_2,n_2)$. Now substituting a^{-1} for a yields an automorphism of G showing that $(1,0,-1) \sim (-1,0,1)$, $(1,-1,-1) \sim (-1,1,1)$, and $(1,1,-1) \sim (-1,-1,1)$. Also, it can be verified that the automorphisms $\varphi\colon a \mapsto a, b \mapsto b, c \mapsto c^2$ and $\psi\colon a \mapsto ab, b \mapsto b, c \mapsto b^{-1}cb$ yields $(1,0,-1) \sim (-1,1,1)$ and $(1,0,-1) \sim (1,1,-1)$ respectively. Hence, we have

$$(1,0,-1) \sim (1,-1,-1) \sim (1,1,-1) \sim (-1,0,1) \sim (-1,1,1) \sim (-1,-1,1)$$

Conversely, it can be verified that a group with this presentation is of order 81 and all minimal subgroups are contained in $\Phi(G)$. For the classification of groups of order p^4, the reader may refer to [26, p. 140].

□

Proof of Theorem 5.3. It can be easily verified that nilpotent groups of order p^2q do not satisfy the 3-valency condition. (This is also a consequence of Lemma 5.11.) Also, a nilpotent group of order pqr is cyclic and does not satisfy the 3-valency condition. Then the first part of the Theorem follows from Lemmas 5.12, 5.15, and 5.14.

For the second part we argue as follows. Let A and B be two distinct minimal subgroups of a finite solvable group G such that there are at least four distinct prime divisors of $|G|$. Suppose that A and B are of same order, say p. Let A_q, A_r, and A_s be some maximal Hall subgroups of G containing A such that their indexes is a power of prime numbers q, r, and s respectively. Also, let B_q, B_r, and B_s be some maximal Hall subgroups containing B. (Of course, $[G : B_q] = q^\alpha$ for some integer α, and so on.) By the Product Formula (A, A_q, B_r, B), (A, A_r, B_s, B), and (A, A_s, B_q, B) are three independent paths between A and B. Similar arguments can be applied if $|A| \neq |B|$. □

6. CONCLUSIONS AND RECOMMENDATIONS

In this theses we confine the study of intersection graphs to the class of finite groups. However, the definition applies for any abstract group and we may still employ combinatorial arguments for large classes of infinite groups albeit their nature is quite different. As a first step toward this direction we might consider the finiteness conditions in infinite groups and especially (subgroup) growth phenomenon in groups (see [38]).

Due to the simplicity of its definition intersection graphs can be related with many other notions in mathematics. Actually, one of the motivations for us to study those objects is to gain a new perspective into looking old contents. In this chapter we present two such headings with potential problems.

6.1 Word Problem

In [39] Dehn introduced the identity [word] problem together with the transformation [conjugacy] problem and the isomorphism problem. For a finitely generated group G, the *word problem* is the problem of finding an algorithmic procedure that can decide whether two given words on the same generators are identical. This question can be related with the construction of the intersection graph.

Let $G = \langle g_1, g_2, \ldots, g_n \rangle$ (possibly with some relators which we omit to write) and let $H_1, H_2 \leq G$ be two 'known' subgroups. Here by the word 'known' we mean that we know a generator set for H_1 and for H_2. Elements of each subgroup can be expressed in terms of their generators which in turn are some words on $g_1, g_2, \ldots, g_n$. Therefore, there are words $w_1 \in H_1$ and $w_2 \in H_2$ such that $w_1 \ncong 1 \ncong w_2$ and $w_1 w_2^{-1} \cong 1$ if and only if $\{H_1, H_2\}$ is an edge in $\Gamma(G)$. Besides the word problem we propose the following

Subgroup intersection problem: The problem of finding an algorithmic procedure that can decide whether two given subgroups intersect non-trivially.

Following questions are natural in this context.

Question 1. Is there any finitely generated group in which the subgroup intersection problem is solvable whereas the word problem is not?

Question 2. Is there any finitely generated group in which the word problem is solvable whereas the subgroup intersection problem is not?

6.2 Graphs of (Sub)groups

Intersection graphs can be seen as particular instances of graphs of groups introduced by Serre [40]. Let us begin with the somewhat less standard definition of graphs which is again due to Serre: A *graph* is an ordered quadruple $\Gamma = (E,V;\iota,\lambda)$ where E is a set of edges, V is a nonempty set of vertices disjoint from E, ι is a mapping of E onto V, called the incidence function, and λ is an involutory permutation of E, called the dart-reversing involution. Note that in this setting edges have an orientation. A *morphism of graphs* $f\colon (E,V;\iota,\lambda) \to (E',V';\iota',\lambda')$ is a function $f\colon E \sqcup V \to E' \sqcup V'$ such that it takes edges to edges and vertices to vertices, i.e. $fE \subseteq E'$, $fV \subseteq V'$, and it is incidence preserving in the sense $f\iota = \iota' f$ and $f\lambda = \lambda' f$. Defining the composition of morphisms in the obvious way yields the category **Grph**.

A *graph of groups* over a graph $\Gamma = (V,E;\iota,\lambda)$ is an assignment of a vertex group G_v to each vertex $v \in V$ and an edge group G_e to each edge $e \in E$ with injective homomorphisms $\varphi_{e,0}$ and $\varphi_{e,1}$ from G_e to the $G_{\iota(e)}$ and $G_{\iota(\lambda e)}$, respectively. Here of course $G_e = G_{\lambda e}$ for every $e \in E$. An intersection graph $\Gamma(G)$ of a group G is a particular instance for a graph of groups. This can be readily seen by observing in $\Gamma(G)$ every edge can be identified with the intersection group of subgroups that its end points represents. And canonical inclusions can serve as boundary monomorphisms. In this setting $\Gamma(G)$ might be called *graph of subgroups* of G.

Fundamental group of a graph of groups can be defined as the fundamental group of the union of "vertex" spaces and "edge" spaces having vertex groups and edge groups as fundamental groups respectively and gluing maps induces monomorphisms of the edge groups into vertex groups. Let $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$ be two groups and $H_1 < G_1$ and $H_2 < G_2$ be two subgroups along with an isomorphism $\varphi\colon H_1 \to H_2$. Then the *amalgamated free product* of G_1 and G_2 along φ is the group given by

$$G_1 *_\varphi G_2 = G_1 *_H G_2 = \langle S_1 \sqcup S_2 \mid R_1 \sqcup R_2 \sqcup \{h_1 \varphi^{-1}(h_1) \mid h_1 \in H_1\} \rangle$$

where H is an abstract group isomorphic to H_j, $j = 1,2$. Another similar construction is this: Let $G = \langle S \mid R \rangle$ be a group and K_1, K_2 be two subgroups together with an isomorphism $\psi \colon K_1 \to K_2$. Then the *HNN extension* of G relative to ψ is the group given by

$$G*_{\psi} = G*_K = \langle S, t \mid R \sqcup \{tk_1t^{-1}\psi^{-1}(k_1) \mid k \in K_1\} \rangle$$

where t is a new symbol (called the stable letter) and K is an abstract group isomorphic to K_j, $j = 1,2$. Fundamental groups of graphs of groups can be constructed as the iterations of the amalgamated free products and HNN extensions.

Martin R. Bridson formulated the following question.

Question 3. Let $\pi(\Gamma(G))$ be the fundamental group of the graph of subgroups of $\Gamma(G)$, where G is a finite group. By a result of Karass, Pietrowski, and Solitar [41], $\pi(\Gamma)$ is virtually free, i.e. it has a free subgroup of finite index. Now, consider G as a graph of groups with a single vertex and no edge. Then the canonical inclusions from the vertex and edge groups of $\Gamma(G)$ to G extends to a homomorphism from $\pi(\Gamma(G))$ to G. Is the kernel of this homomorphism the lowest index free subgroup of $\pi(\Gamma(G))$?

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