ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE ENGINEERING AND TECHNOLOGY

REGULARIZED TRACES AND SPECTRAL PROPERTIES OF DIFFERENTIAL OPERATORS

Ph.D. THESIS

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Department of Mathematics Engineering

Mathematics Engineering Programme

SEPTEMBER 2015

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Erdoğan Şen, **a Ph.D.** student of ITU **Institute of / Graduate School of Science Engineering and Technology** student ID **509102007**, successfully defended the **dissertation** entitled "**REGULARIZED TRACES AND SPECTRAL PROPERTIES OF DIFFERENTIAL OPERATORS**", which he prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

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To my parents and sister,

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REGULARIZED TRACES AND SPECTRAL PROPERTIES OF DIFFERENTIAL OPERATORS

SUMMARY

This thesis consists of five main chapters. In introduction, we give a general information about the theory of Sturm-liouville operators and previous works in the literature which is realatively close to our studies. Also establishment of the problems is given in introduction.

In the second chapter, we extend some spectral properties of regular Sturm-Liouville problems to those which consist of a Sturm-Liouville equation with discontinuous weight at two interior points together with spectral parameter-dependent boundary conditions. By modifying some techniques of [C. T. Fulton, Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions, Proc. Roy. Soc. Edinburgh Sect. A 77 (1977) 293-308; O. Sh. Mukhtarov and M. Kadakal, Some spectral properties of one Sturm-Liouville type problem with discontinuous weight, Siberian Mathematical Journal, 46 (2005) 681-694], we give an operator-theoretic formulation for the considered problem and obtain asymptotic formulas for the eigenvalues and eigenfunctions.

In the third chapter, we investigate discontinuous two-point boundary value problems with eigenparameter in the boundary conditions and with transmission conditions at the finitely many points of discontinuity. Namely we consider the discontinuous eigenvalue problem which consist of Sturm-Liouville equation

$$\tau u := -u''(x) + q(x)u(x) = \lambda u(x)$$

on $[-1, h_1) \cup (h_1, h_2) \cup ... \cup (h_m, 1]$, together with eigenparameter-dependent boundary conditions

$$\tau_1 u := \alpha_1 u(-1) + \alpha_2 u'(-1) = 0,$$
$$\tau_2 u := (\beta_1' \lambda + \beta_1) u(1) - (\beta_2' \lambda + \beta_2) u'(1) = 0$$

and transmission conditions at the points of discontinuity $x = h_i \ (i = \overline{1, m})$,

$$\tau_{2i+1} u := u(h_i - 0) - \delta_i u(h_i + 0) = 0,$$
$$\tau_{2i+2} u := u'(h_i - 0) - \delta_i u'(h_i + 0) = 0,$$

where $-1<h_1<h_2<...<h_m<1$, $q(x)$ is a given real-valued function continuous in $[-1,h_1),(h_1,h_2),...,(h_m,1]$ and has finite limits $q(h_i\pm 0)=\lim_{x\to h_i\pm 0} q(x)\ \left(i=\overline{1,m}\right)$; λ is a complex eigenvalue parameter; $\delta_i\ \left(i=\overline{1,m}\right), \alpha_j, \alpha_j^{'}, \beta_j, \beta_j^{'}\ \ (j=1,2)$ are real numbers; $|\alpha_1|+|\alpha_2|\neq 0$ and $\delta_i\neq 0\ \ \left(i=\overline{1,m}\right)$. This section organised as follows: Firstly we give operator formulation of the problem in a suitable Hilbert space (i.e., A self-adjoint linear operator A is defined in a suitable Hilbert space H such that the eigenvalues of the considered problem coincide with those of A) and then asymptotic approximate formulas of characteristic function derived for four distinct cases. Asymptotic formulas for eigenvalues and eigenfunctions of the problem is given and finally we show that the eigenfunctions of A are complete in H.

In the fourth chapter, assuming H is a separable Hilbert space, we consider the operators L_0 and L generated by the differential expressions

$$l_0(y)=-y''(x)+Ay(x)$$

and

$$l(y)=-y''(x)+Ay(x)+Q(x)y(x)$$

respectively, in the Hilbert space $H_1=L_2\left([0,1],H\right)$, with the same boundary conditions

$$y(0)=0,\ \ y'(1)+by(1)=0,\ \ b>0,$$

where A is a positive definite self-adjoint operator in H and $Q(x)$ satisfies some additional conditions.
Let the eigenvalues of the operators L_0 and L be $\mu_1\le\mu_2\le...\le\mu_n\le...$ and $\lambda_1\le\lambda_2\le...\le\lambda_n\le...$ respectively. In this section, firstly we investigate the spectrum and resolvent of the operators L_0 and L. Finally, under the conditions (1)-(3) the following formula has been found for the regularized trace of L:

$$\lim_{p\to\infty}\sum_{k=1}^{n_p}(\lambda_k-\mu_k)=\frac{1}{4}\left[trQ(1)-trQ(0)\right].$$

In the fifth chapter, we investigate the resolvent operator and completeness of eigenfunctions of a Sturm-Liouville problem with discontinuities at two points. The problem contains an eigenparameter in the one of boundary conditions. For operator-theoretic formulation of the considered problem we define an equivalent inner product in the Hilbert space $L_2[-1,1]\oplus\mathbb{C}$ and suitable self-adjoint linear operator in it.

DİFERANSİYEL OPERATÖRLERİN DÜZENLİ İZLERİ VE SPEKTRAL ÖZELLİKLERİ

ÖZET

Matematiksel fiziğin bazı problemlerinde zaman değişkenine göre kısmi türev sadece diferansiyel denklemde değil aynı zamanda sınır koşularında da ortaya çıkmaktadır. Böyle problemlere uygun olan sınır-değer problemlerinde özdeğer parametresi sadece diferansiyel denklemde değil aynı zamanda sınır koşullarında da bulunmaktadır. Süreksiz sınır-değer problemleri ise farklı fiziksel ve mekanik özellikleri bulunan cisimler arasındaki ısı ve madde iletimi veya başka geçiş süreçlerinde ortaya çıkmaktadır. Literatürde süreksiz Sturm-Liouville problemleri hakkında çalışmalar mevcuttur, ama süreksizlik noktası sayısı birden fazla olduğunda özdeğer ve özfonksiyonların asimtotik davranışlarının ve bazı spektral özelliklerinin nasıl değiştiği bu tezde incelenen konular arasındadır. Yine literatürde diferansiyel ifadede süreksiz operatör içeren Sturm-Liouville operatörlerinin düzenli izleri birkaç çalışma dışında araştırılmamıştır. Bu tezde daha genel ve farklı sınır koşullarına sahip süreksiz operatör katsayılı bir diferansiyel operatör için düzenli iz formülü elde edilmiştir.

Bu tezin esas kısmı 5 bölümden oluşmaktadır. Giriş bölümünde tezde incelenen problemler tanıtılmış, bunların uygulama alanlarından bahsedilmiş, teorik önemi belirtilmiş ve bunlarla ilgili olarak yapılan çalışmalar hakkında literatür özeti verilmiştir.

İkinci bölümde ise $\left[-1,h_1\right)\cup\left(h_1,h_2\right)\cup\left(h_2,1\right]$ aralığında tanımlı; iki noktada süreksizliğe sahip

$$\omega(x)=\begin{cases}\omega_1^2, & x\in\left[-1,h_1\right),\\ \omega_2^2, & x\in\left(h_1,h_2\right),\\ \omega_3^2, & x\in\left(h_2,1\right]\end{cases}$$

ağırlık fonksiyonuna sahip

$$-u''+q(x)u=\lambda\omega(x)u$$

diferansiyel operatörü ve

$$\cos\alpha u\left(-1\right)+\sin\alpha u'\left(-1\right)=0,$$

$$\lambda\left(\beta_1' u\left(1\right)-\beta_2' u'\left(1\right)\right)+\left(\beta_1 u\left(1\right)-\beta_2 u'\left(1\right)\right)=0,$$

şeklinde sınır koşullarının birinde özdeğer parametresinin yer aldığı

$$\gamma_1 u(h_1-0)-\delta_1 u(h_1+0)=0,$$
$$\gamma_2 u'(h_1-0)-\delta_2 u'(h_1+0)=0,$$
$$\gamma_3 u(h_2-0)-\delta_3 u(h_2+0)=0,$$
$$\gamma_4 u'(h_2-0)-\delta_4 u'(h_2+0)=0,$$

geçiş (iletim) koşullarına sahip sınır-değer probleminin özdeğerleri ve özfonksiyonları için asimtotik formül bulunmuştur. $\omega(x)\equiv 1$ ve $\gamma_i=\delta_i \ (i=\overline{1,4})$ olarak alındığında problem sürekli bir sınır-değer problemine dönüşür ve elde edilen sonuçlar [C. T. Fulton, Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions, Proc. Roy. Soc. Edinburgh Sect. A 77 (1977) 293-308] çalışmasında elde edilen sonuçlarla çakışır. Sürekisizlik noktalarının sayısını tek bir nokta olarak almamız durumunda ise sonuçlar [O. Sh. Mukhtarov and M. Kadakal, Some spectral properties of one Sturm-Liouville type problem with discontinuous weight, Siberian Mathematical Journal, 46 (2005) 681-694] çalışmasındaki sonuçlarla çakışır. Yani elde edilen sonuçlar literatürdeki sonuçların bir genelleştirilmesidir.

Üçüncü bölümde ise sınır koşulunda özdeğer parametresi olan tanım aralığında sonlu sayıda süreksiz noktaya sahip olan sınır-değer problemi; yani $[-1,h_1)\cup(h_1,h_2)\cup\ldots\cup(h_m,1]$ aralığında tanımlı

$$-u''(x)+q(x)u(x)=\lambda u(x)$$

diferansiyel operatörü;

$$\alpha_1 u(-1)+\alpha_2 u'(-1)=0,$$

$$(\beta_1'\lambda+\beta_1)u(1)-(\beta_2'\lambda+\beta_2)u'(1)=0$$

sınır koşulları ve

$$u(h_i-0)-\delta_i u(h_i+0)=0,$$
$$u'(h_i-0)-\delta_i u'(h_i+0)=0,$$

geçiş koşulları ile oluşturulan sınır-değer problemi uygun bir Hilbert uzayı ve bu uzayda kendine eş bir lineer operatör tanımlanarak problem operatör denklem olarak ifade edilmiştir. Kökleri (sıfırları) sınır-değer probleminin özdeğerleri olacak şekilde bir polinom bulunmuş ve özdeğerlerin katlılığı incelenmiştir. Daha sonra özdeğer ve özfonksiyonlar için asimtotik formüller bulunmuş, spektrumunun sadece özdeğerlerden ibaret olduğu ispatlanmış, resolvent operatörü incelenmiş, özfonksiyonlar cinsinden seri açılımı elde edilmiş ve özfonksiyonların tamlığı incelenmiştir.

H sonsuz boyutlu ayrılabilir bir Hilbert uzayı olmak üzere $H_1=L_2(0,1;H)$ uzayında

$$l_0(y) = -y''(x) + Ay(x),$$
$$l(y) = -y''(x) + Ay(x) + Q(x)y(x)$$

diferansiyel ifadeleri ve aynı

$$y(0) = 0,\ y'(1) + b\,y(1) = 0, \quad b > 0$$

sınır koşulları ile oluşturulan operatörler sırasıyla L_0 ve L olsun. Burada A, $D(A) \subset H$ olmak üzere $D(A)$ dan H ye

$$A = A^* \geq I,\ A^{-1} \in \sigma_\infty(H)$$

koşullarını sağlayan bir operatördür ve $Q(x)$, operator fonksiyonu $[0,1]$ aralığında tanımlıdır ve aşağıdaki koşulları sağlar:

a.) Her $x \in [0,1]$ için $Q(x) : H \to H$ ikinci mertebeden zayıf türeve sahiptir. $Q''(x)$ zayıf ölçülebilirdir ve her $x \in [0,1]$ için $Q^{(i)}(x) : H \to H$ $(i = 0,1,2)$ kendine eş nükleer operatörlerdir.

b.) $\left\|Q^{(i)}(x)\right\|_{\sigma_1(H)}$ $(i = 0,1,2)$ fonksiyonları $[0,1]$ aralığında sınırlı ve ölçülebilirdir. Burada $\sigma_1(H)$, H dan H a nükleer operatörler uzayını göstermektedir.

c.) Her $f \in H$ için $\int_0^1 (Q(x)f, f)_H\, dx = 0$ dır.

L_0 operatörünün özdeğerleri $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_m \leq \ldots$ ve L operatörlerinin özdeğerleri $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m \leq \ldots$ olsun. Dördüncü bölümde L_0 ve L operatörlerinin saf ayrık spektruma sahip olduğu gösterilmiş, resolvent operatörleri için bazı eşitlikler elde edilmiş ve L operatörünün düzenli izi için

$$\lim_{p\to\infty} \sum_{k=1}^{n_p} (\lambda_k - \mu_k) = \frac{1}{4}\left[trQ(1) - trQ(0)\right]$$

şeklinde bir formül bulunmuştur. Eğer diferansiyel ifadedeki sınırsız katsayılı operatörü yani A operatörünü özdeş olarak sıfıra eşit alırsak elde edilen sonuçlar [K. Koklu, I. Albayrak, A. Bayramov, A regularized trace formula for second order differential operator equations, Mathematica Scandinavica, 107 (2010) 123-138] çalışmasındaki sonuçlar ile çakışır.

Beşinci bölümde ise [-1,1] aralığının h_1 ve h_2 gibi iki iç noktasında süreksiz olan, katsayıları sonlu

$$\ell u := \frac{1}{r(x)}\left\{-\left(\mathrm{p}(x)u'\right)' + q(x)u\right\} = \lambda u,\ x \in [-1, h_1) \cup (h_1, h_2) \cup (h_2, 1]$$

diferansiyel denkleminden,

$$u(-1)=0,$$
$$(\lambda\alpha_1+\beta_1)u(1)-(\lambda\alpha_2+\beta_2)u'(1)=0$$

sınır koşullarından ve x= h_1 , x= h_2 süreksizlik noktalarındaki

$$\gamma_1 u(h_1-0)=\delta_1 u(h_1+0),$$
$$\gamma_2 u'(h_1-0)=\delta_2 u'(h_1+0),$$
$$\gamma_3 u(h_2-0)=\delta_3 u(h_2+0),$$
$$\gamma_4 u'(h_2-0)=\delta_4 u'(h_2+0),$$

geçiş koşullarından oluşan bir Sturm-Liouville probleminin özfonksiyonlarının tamlığı incelenmiştir. Sınır-değer-geçiş problemi önce uygun Hilbert uzayında kendine eş bir operatör yardımıyla özdeğer problemi olarak ifade edilmiştir. Daha sonra bu operatörün simetrik bir operatör olduğu ispatlanmış ve özfonksiyonlar sistemine açılım teoremi ispatlanmıştır.

1. INTRODUCTION

Let L be a linear operator defined on some set of elements. An element $y \neq 0$ is called an eigenfunction of L if $Ly = \lambda y$; the number λ is called an eigenvalue of L One of the most important operators which is frequently encountered in applications is an operator of the form

$$L \equiv -\frac{d^2}{dx^2} + q(x) ,$$

where the function $q(x)$ will be assumed real and, to begin with, continuous on some interval $[a,b]$. For this operator the set of elements $y(x)$ mentioned above is determined by the obvious differentiability condition and also by certain conditions on the boundary of the interval $[a,b]$.

The most important boundary conditions for the operator L are the followings:

$$\begin{aligned} &\text{I.}\quad y(a)\cos\alpha + y'(a)\sin\alpha = 0, \\ &\qquad y(b)\cos\beta + y'(b)\sin\beta = 0, \\ &\text{II.}\ y(a) = y(b),\ \ y'(a) = y'(b). \end{aligned}$$

The boundary value problem

$$\begin{aligned} &Ly(x) \equiv -\frac{d^2 y}{dx^2} + q(x)y = \lambda y, \\ &y(a)\cos\alpha + y'(a)\sin\alpha = 0, \\ &y(b)\cos\beta + y'(b)\sin\beta = 0 \end{aligned}$$

is known in the literature as the Sturm-Liouville problem [30].

Sturmian theory is one of the most extensively developing fields in theoretical and applied mathematics [1-24. 29-46, 48, 50-52, 54-69]. Particularly, there has been an increasing interest in the spectral analysis of boundary-value problems with eigenvalue-dependent boundary conditions [2, 3, 5, 10, 11, 16, 20-23, 33-42, 50-52, 57-60, 63, 65, 66, 69]. Sturm-Liouville problems arise as a result of using the method of separation of variables to solve classical partial differential equations of physics, such as Laplace's equation, the heat equation and the wave equation. A Sturm-Liouville problem with eigenparameter contained in the boundary condition arise upon separation of variables in the one-dimensional wave and heat equations for a varied assortment of physical problems, e.g. in the diffusion of water vapour through a porous membrane and several electric circuit problems involving long cables. (for example, see [16, 41]), vibrating string problems when the string loaded additionally with point masses (for example, see [54]). Also some problems with transmission conditions arise in thermal conduction problems for a thin laminated plate (i.e., a plate composed by materials with different characteristics piled in the thickness) [32, 56]. In this class of problems, transmission conditions across the interfaces should be added since the plate is laminated. The study of the structure of the solution in the matching region of the layer with the basis solution in the plate leads to consideration of an eigenvalue problem for a second order differential operator with piecewise continuous coefficients and transmission conditions [29, 32, 45, 54, 56]. Sturm-Liouville problems with transmission conditions at one interior point have been studied by many authors [2-7, 21, 33, 35, 36, 39-43, 50, 52, 57, 59, 60, 66, 68]. In [22] and [65] Sturm-Liouville problem with transmission conditions at two interior points studied. Li et al. [31] gave the complete descriptions of self-adjoint boundary conditions of the Schrödinger operator with $\delta(x)$ and $\delta'(x)$ interaction. Adjoint and self-adjoint boundary value problems with interface conditions have been studied by Zettl [67].

In 1977, Fulton [16] considered the Sturm-Liouville eigenvalue problem

$$\begin{aligned}
&-u'' + qu = \lambda u, \\
&\cos\alpha u(a) + \sin\alpha u'(a) = 0, \\
&-\left(\beta_1 u(b) - \beta_2 u'(b)\right) = \lambda\left(\beta_1' u(b) - \beta_2' u'(b)\right)
\end{aligned}$$

and obtained asymptotic formulas for eigenvalues and eigenfunctions of this problem.

In 2004, Altınisk et al. [3] investigated the asymptotics of eigenvalues and eigenfunctions for the differential equation

$$-a(x)u''(x)+q(x)u(x)=\lambda u(x)$$

in the interval $[-1,1]$ except one inner point $x=0$ together with the eigenvalue-dependent boundary conditions

$$\alpha_1 u(-1)+\alpha_2 u'(-1)=0,$$
$$\left(\beta_1'\lambda+\beta_1\right)u(1)=\left(\beta_2'\lambda+\beta_2\right)u'(1)$$

and transmission conditions at the point of discontinuity

$$\gamma_1 u(0-)=\delta_1 u(0+),$$
$$\gamma_2 u'(0-)=\delta_2 u'(0+).$$

where $a(x)=a_1^2$ for $x\in(0,1]$ and $a(x)=a_2^2$ for $x\in[-1,0)$ and obtained asymptotic expressions for eigenvalues and eigenfunctions.

In 2005, Mukhtarov and Kadakal [39] considered the Sturm-Liouville equation

$$-u''+q(x)u=\lambda\omega(x)u$$

in the interval $[-1,0)\cup(0,1]$; where $\omega(x)$ is a discontinuous weight function such that $\omega(x)=\omega_1^2$ for $x\in[-1,0)$, and $\omega(x)=\omega_2^2$ for $x\in(0,1]$, together with the standart boundary condition at $x=-1$

$$\cos\alpha u(-1)+\sin\alpha u'(-1)=0,$$

the spectral parameter dependent boundary condition at $x=1$

$$\lambda\left(\beta_1' u(1)-\beta_2' u'(1)\right)+\left(\beta_1 u(1)-\beta_2 u'(1)\right)=0,$$

and the two transmission conditions at the point of discontinuity $x=0$

$$\gamma_1 u(-0)-\delta_1 u(+0)=0,$$
$$\gamma_2 u'(-0)-\delta_2 u'(+0)=0.$$

In the second chapter, following [39] we consider the boundary value problem for the differential equation

$$\tau u := -u'' + q(x)u = \lambda\omega(x)u \tag{1.1}$$

for $x\in[-1,h_1)\cup(h_1,h_2)\cup(h_2,1]$ (i.e., x belongs to $[-1,1]$ but the two inner points $x=h_1$ and $x=h_2$), where $q(x)$ is a real valued function, continuous in $[-1,h_1)$, (h_1,h_2) and $(h_2,1]$ with the finite limits $q(\pm h_1)=\lim_{x\to\pm h_1}$, $q(\pm h_2)=\lim_{x\to\pm h_2}$; $\omega(x)$ is a discontinuous weight function such that $\omega(x)=\omega_1^2$ for $x\in[-1,h_1)$, $\omega(x)=\omega_2^2$ for $x\in(h_1,h_2)$ and $\omega(x)=\omega_3^2$ for $x\in(h_2,1]$, $\omega>0$ together with the standart boundary condition at $x=-1$

$$L_1 u := \cos\alpha u(-1)+\sin\alpha u'(-1)=0, \tag{1.2}$$

the spectral parameter dependent boundary condition at $x=1$

$$L_2 u := \lambda\left(\beta_1' u(1)-\beta_2' u'(1)\right)+\left(\beta_1 u(1)-\beta_2 u'(1)\right)=0, \tag{1.3}$$

and the four transmission conditions at the points of discontinuity $x=h_1$ and $x=h_2$

$$L_3 u := \gamma_1 u(h_1-0)-\delta_1 u(h_1+0)=0, \tag{1.4}$$

$$L_4 u := \gamma_2 u'(h_1 - 0) - \delta_2 u'(h_1 + 0) = 0, \tag{1.5}$$

$$L_5 u := \gamma_3 u(h_2 - 0) - \delta_3 u(h_2 + 0) = 0, \tag{1.6}$$

$$L_6 u := \gamma_4 u'(h_2 - 0) - \delta_4 u'(h_2 + 0) = 0, \tag{1.7}$$

in the Hilbert space $L_2(-1, h_1) \oplus L_2(h_1, h_2) \oplus L_2(h_2, 1)$ where $\lambda \in \mathrm{C}$ is a complex spectral parameter; and all coefficients of the boundary and transmission conditions are real constants. We assume naturally that $|\alpha_1| + |\alpha_2| \neq 0$, $|\beta_1'| + |\beta_2'| \neq 0$ and $|\beta_1| + |\beta_2| \neq 0$. Moreover, we will assume that $\rho := \beta_1'\beta_2 - \beta_1\beta_2' > 0$. We find asymptotic formulas for eigenvalues and eigenfunctions of the problem (1.1)-(1.7).

In the third chapter, we examine eigenvalues and eigenfunctions of one discontinuous eigenvalue problem which consist of Sturm-Liouville equation

$$\tau u := -u''(x) + q(x)u(x) = \lambda u(x) \tag{1.8}$$

on $[-1, h_1) \cup (h_1, h_2) \cup \ldots \cup (h_m, 1]$, together with eigenparameter-dependent boundary conditions

$$\tau_1 u := \alpha_1 u(-1) + \alpha_2 u'(-1) = 0, \tag{1.9}$$

$$\tau_2 u := (\beta_1'\lambda + \beta_1) u(1) - (\beta_2'\lambda + \beta_2) u'(1) = 0 \tag{1.10}$$

and transmission conditions at the points of discontinuity $x = h_i$ $(i = \overline{1, m})$,

$$\tau_{2i+1} u := u(h_i - 0) - \delta_i u(h_i + 0) = 0, \tag{1.11}$$

$$\tau_{2i+2} u := u'(h_i - 0) - \delta_i u'(h_i + 0) = 0, \tag{1.12}$$

where $-1<h_1<h_2<...<h_m<1$, $q(x)$ is a given real-valued function continuous in $[-1,h_1),(h_1,h_2),...,(h_m,1]$ and has finite limits $q(h_i \pm 0)=\lim_{x\to h_i\pm 0} q(x)$ $\left(i=\overline{1,m}\right)$; λ is a complex eigenvalue parameter; δ_i $\left(i=\overline{1,m}\right)$, α_j, $\alpha_j^{'}$, β_j, $\beta_j^{'}$ $(j=1,2)$ are real numbers; $|\alpha_1|+|\alpha_2|\neq 0$ and $\delta_i\neq 0$ $\left(i=\overline{1,m}\right)$. As following [16] we assume everywhere that $\rho=\beta_1^{'}\beta_2-\beta_1\beta_2^{'}>0$.

Third chapter organised as follows: following the operator formulation of the problem (1)-(5) in a suitable Hilbert space in Sect. 3.2, asymptotic approximate formulas of characteristic function derived for four distinct cases in Sect. 3.3, asymptotic formulas for eigenvalues and eigenfunctions of the problem (1)-(5) is given in Sect. 3.4, and in the last section we examined the completeness of eigenfunctions of the problem (1.8)-(1.12).

Let H be a separable Hilbert space. We denote the inner product in H by $(\cdot,\cdot)$ and the norm in H by $\|\cdot\|$. Let f be a strongly measurable function defined on $[0,1]$ with values in H such that

1.) The scalar function $(f(x),g)$ is Lebesgue measurable for every $g\in H$ in the interval $[0,1]$;

2.) $\int_0^1 \|f(x)\|^2 dx<\infty.$

The set of all functions f satisfying the above conditions is denoted by $H_1=L_2(0,1;H)$. If the inner product of two arbitrary elements f_1 and f_2 of the space H_1 is defined as

$$(f_1,f_2)_{H_1}=\int_0^1 (f_1(x),f_2(x))dx \quad (f_1,f_2\in H_1),$$

then H_1 becomes a separable Hilbert space [24]. The norm in the space H_1 is denoted by $\|\cdot\|_{H_1}$. $\sigma_\infty(H)$ denotes the set of compact operators from H to H. If $A\in\sigma_\infty(H)$ then A^*A is a nonnegative self-adjoint operator and $\left(A^*A\right)^{1/2}\in\sigma_\infty(H)$ [13]. Let the nonzero eigenvalues of the operator $\left(A^*A\right)^{1/2}$ be $s_1\geq s_2\geq\ldots\geq s_k$ $(0\leq k\leq\infty)$. Here each eigenvalue is repeated according to its multiplicity. The numbers $s_1,s_2,\ldots,s_k$ are called s-numbers of the operator A. If $k<\infty$, then $s_j=0$ where $j=k+1,k+2,\ldots$. The s-numbers of the operator A is also denoted by $s_k(A)$ $(k=1,2,\ldots)$. If A is a normal operator, that is $A^*A=AA^*$ then, $s_k(A)=\left|\lambda_k(A)\right|$ $(k=1,2,\ldots)$ [13]. Here, $\lambda_1(A),\lambda_2(A),\ldots,\lambda_k(A)$ are the non-zero eigenvalues of the operator A. We will denote the set of all operators $A\in\sigma_\infty(H)$ such that the s-numbers of which satisfy the condition $\sum_{k=1}^{\infty}s_k^p(A)<\infty$ by σ_p or $\sigma_p(H)$. The set σ_p is a separable Banach space with respect to the norm

$$\|A\|_{\sigma_p(H)}=\left[\sum_{k=1}^{\infty}s_k^p(A)\right]^{\frac{1}{p}}.$$

(see [13]).

For $p=1$ the space $\sigma_1(H)$ is called the space of kernel operators. Thus an operator in $\sigma_1(H)$ is called a kernel operator. If $A\in\sigma_1(H)$ then for any linear bounded operator $B:H\to H$ we have $AB,BA\in\sigma_1(H)$ and

$$\|BA\|_{\sigma_1(H)}\leq\|B\|\|A\|_{\sigma_1(H)},$$
$$\|AB\|_{\sigma_1(H)}\leq\|B\|\|A\|_{\sigma_1(H)}.$$

(see [13]). If $A\in\sigma_1(H)$ and $\{e_j\}_{j=1}^{\infty}\subset H$ is any orthonormal basis then the series

$$\sum_{j=1}^{\infty}\left(Ae_j,e_j\right)$$

is convergent and the sum of the series $\sum_{j=1}^{\infty}\left(Ae_j, e_j\right)$ does not depend on the choice of the basis $\left\{e_j\right\}_{j=1}^{\infty}$. The sum of the series $\sum_{j=1}^{\infty}\left(Ae_j, e_j\right)$ is said to be matrix trace of A and is denoted by trA (see [13]). We have

$$trA = \sum_{k=1}^{\upsilon(A)} \lambda_k\left(A\right).$$

Here each eigenvalue is counted according to its own algebraic multiplicity number and $\upsilon(A)$ denotes the sum of algebraic multiplicity of non-zero eigenvalues of A (see [13]). A self-adjoint operator is said to have purely-discrete spectrum if its spectrum consist of eigenvalues $\left\{\lambda_j\right\}_{j=1}^{\infty}$ of finite multiplicity and $\lim_{j\to\infty}\left|\lambda_j\right| = \infty$.

The first work about the theory of regularized traces of differential operators belongs to Gelfand and Levitan [17]. They considered the Sturm-Liouville operator

$$-y'' + \left[q(x) - \lambda\right] y = 0,$$

with boundary conditions

$$y'(0) = y'\left(\pi\right) = 0,$$

where $q(x) \in C^1\left[0,\pi\right]$. Under the condition $\int_0^{\pi} q\left(x\right) dx = 0$ they obtained the formula

$$\sum_{n=0}^{\infty}\left(\mu_n - \lambda_n\right) = \frac{1}{4}\left(q\left(0\right) + q\left(\pi\right)\right).$$

Here, μ_n are the eigenvalues of the operator $-y'' + q(x)\, y = \lambda y$ and $\lambda_n = n^2$ are the eigenvalues of the same operator with $q(x) = 0$. The limit given in the form

$$\lim_{n\to\infty} \sum_{k=1}^{n}\left(\lambda_k - \mu_k\right)$$

is called the regularized trace of operator L.

First, the trace formulas for the Sturm-Liouville operator were obtained in [15, 17]. Afterwards these works have been followed by numerous studies. The bibliography on the subject is very extensive and we refer to the list of the works in [30, 48]. The trace formulas related to the Sturm-Liouville problem with bounded self-adjoint operator were considered in [1, 8, 9, 14, 25].

Köklü et al. [25] under some additional conditions obtained the formula

$$\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\left(\lambda_{mn}-\left(m-\frac{1}{2}\right)\right)-\int_{0}^{1} trQ(x)\,dx\right]=\frac{1}{4}\left[trQ(1)-trQ(0)\right]$$

for the operator L generated by the differential expression

$$-y''(x)+Q(x)y(x)$$

with the boundary conditions

$$y(0)=0, \quad y(1)+ay(1)=0, \quad a>0$$

in the Hilbert space $H_1=L_2\left(\left[0,1\right],H\right)$ where $Q(x):\mathrm{H}\rightarrow\mathrm{H}$ self-adjoint nuclear operator and H is a separable Hilbert space.

Let L_0 and L be operators which are formed by the differential expressions

$$l_0(\mathrm{y})=-y''(x)$$

and

$$l(y)=-y''(x)+Q(x)y(x)$$

respectively, in the space $L_2(0,\pi;H)$ with the same boundary conditions

$$y(0) = -y(\pi), \quad y'(0) = -y'(\pi)$$

where H is a separable Hilbert space. Bayramov et al. [9] obtained the formula

$$\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\left(\lambda_{mn}^2 - 2(2m-1)^4\right) - \frac{4(2m-1)^2}{\pi}\int_0^{\pi} trQ(x)\,dx \right.$$
$$\left. -\frac{1}{\pi}\int_0^{\pi} trQ^2(x)\,dx - \frac{1}{\pi^2} tr\left(\int_0^{\pi} Q(x)\,dx\right)^2\right] = 0$$

for the regularized trace of operator L.

Let H be a separable Hilbert space. In the Hilbert space $H_1 = L_2([0,1],H)$ we consider the self-adjoint operator L generated by the expression

$$l(y) = -y''(x) + Ay(x) + Q(x)y(x)$$

with the boundary conditions

$$y(0) = 0, \quad y'(1) + by(1) = 0, \quad b > 0, \tag{1.13}$$

where A is a positive definite self-adjoint operator in H, which is the inverse to a compact operator; we may assume that $A \geq I$ where I is the identity operator and suppose that the operator function $Q(x)$ satisfies the following conditions:

(1) $Q(x)$ has a weak derivative of second order in interval [0,1]. The operator function $Q''(x)$ is weakly measurable and for every $x \in [0,1]$, $Q^{(i)}(x): H \to H$ $(i = 0,1,2)$ are self-adjoint nuclear operators.

(2) The functions $\left\| Q^{(i)}(x) \right\|_{\sigma_1(H)}$ $(i = 0,1,2)$ are bounded and measurable in the interval $[0,1]$. Here $\sigma_1(H)$ denotes the space of the nuclear operators from H to H.

(3) $\int_0^1 \left(Q(x) f, f \right)_H dx = 0$ for every $f \in H$.

Let L_0 be the operator generated by the differential expression $l_0(y) = -y''(x) + Ay(x)$ and the boundary conditions (1.13).

In the fourth chapter, we obtain a formula for the operators L_0 and $L = L_0 + Q$. This formula is said to be regularized trace formula.

In this work, the problem that we consider is different from [25] by appearence of unbounded operator coefficient.

The trace formulas can be used for approximate calculation of the first eigenvalues of an operator [48], and in order to establish necessary and sufficient conditions for a set of complex numbers to be spectrum of an operator [49].

The operators L_0 and L have purely-discrete spectrum [19]. Let the eigenvalues of the operators L_0 and L be $\mu_1 \le \mu_2 \le ... \le \mu_n \le ...$ and $\lambda_1 \le \lambda_2 \le ... \le \lambda_n \le ...$ respectively.

Let $\gamma_1 \le \gamma_2 \le ... \le \gamma_n \le ...$ be the eigenvalues of the operator A and $\phi_1, \phi_2, ..., \phi_n, ...$ be the orthonormal eigenfunctions corresponding to these eigenvalues. It is known [19] that, if

$$\gamma_j \sim a j^{\alpha} \text{ as } j \to \infty \quad (a > 0, \alpha > 2)$$

then

$$\lambda_n, \mu_n \sim d n^{\frac{2\alpha}{2+\alpha}} \text{ as } n \to \infty, \tag{1.14}$$

here $d>0.$ By using this asymptotic formula, it is easily seen that the sequence $\{\mu_n\}$ has a subsequence $\mu_{n_1}<\mu_{n_2}<...<\mu_{n_p}<...$ such that

$$\mu_k-\mu_{n_p}>d_1\left(k^{\frac{2\alpha}{2+\alpha}}-n_p^{\frac{2\alpha}{2+\alpha}}\right)\ \left(k=n_p+1,n_p+2,...\right), \quad \textbf{(1.15)}$$

here $d_1>0.$

In this work, under the conditions (1)-(3) the following formula has been found for the regularized trace of L:

$$\lim_{p\to\infty}\sum_{k=1}^{n_p}(\lambda_k-\mu_k)=\frac{1}{4}\left[trQ(1)-trQ(0)\right].$$

In 2010, Wang et al. [60] studied completeness of eigenfunctions of the following Sturm-Liouville problem with eigenvalue-dependent boundary conditions and transmission conditions at one interior point:

$$\begin{aligned}
&-\left(a(x)u'(x)\right)'+q(x)u(x)=\lambda u(x),\\
&\alpha_1 u(-1)+\alpha_2 u'(-1)=0,\\
&\lambda\left(\beta_1' u(1)-\beta_2' u'(1)\right)+\beta_1 u(1)-\beta_2 u'(1)=0,\\
&u(0+)-\alpha_3 u(0-)-\beta_3 u'(0-)=0,\\
&u(0+)-\alpha_4 u(0-)-\beta_4 u'(0-)=0.
\end{aligned}$$

In 2014, Aydemir and Mukhtarov [6] investigated the completeness of eigenfunctions of the following boundary-value problem:

$$\begin{aligned}
&-p(x)y''(x)+q(x)y(x)=\lambda y(x),\\
&\cos\alpha y(-\pi)+\sin\alpha y'(-\pi)=0,\\
&\cos\beta y(\pi)+\sin\beta y'(\pi)=0
\end{aligned}$$

where singularity of the solution $y=y(x,\lambda)$ prescribed by transmission conditions

$$\beta_{11}^{-}y'(0-)+\beta_{10}^{-}y(0-)+\beta_{11}^{+}y'(0+)+\beta_{10}^{+}y(0-)=0,$$
$$\beta_{21}^{-}y'(0-)+\beta_{20}^{-}y(0-)+\beta_{21}^{+}y'(0+)+\beta_{20}^{+}y(0-)=0.$$

In chapter 5, we shall investigate the Sturm-Liouville equation

$$\ell u := \frac{1}{r(x)}\left\{-\left(\mathrm{p}(x)u'\right)' + q(x)u\right\} = \lambda u, \tag{1.16}$$

on three disjoint intervals $[-1,h_1)$, (h_1,h_2) and $(h_2,1]$ with the eigenparameter dependent boundary condition

$$u(-1)=0\,, \tag{1.17}$$

$$(\lambda\alpha_1+\beta_1)u(1)-(\lambda\alpha_2+\beta_2)u'(1)=0\ , \tag{1.18}$$

and the transmission conditions

$$\delta_1 u(h_1+0)=\gamma_1 u(h_1-0), \tag{1.19}$$

$$\delta_2 u'(h_1+0)=\gamma_2 u'(h_1-0), \tag{1.20}$$

$$\delta_3 u(h_2+0)=\gamma_3 u(h_2-0), \tag{1.21}$$

$$\delta_4 u'(h_2+0)=\gamma_4 u'(h_2-0). \tag{1.22}$$

Here $p(x)$, $q(x), r(x)$ are continuous functions on $I=[-1,h_1)\cup(h_1,h_2)\cup(h_2,1]$; and have finite limits $p(h_i\pm 0)=\lim_{h_i\to\pm 0}\mathrm{p(x)}$, $q(h_i\pm 0)=\lim_{h_i\to\pm 0} q(\mathrm{x})$, $r(h_i\pm 0)=\lim_{h_i\to\pm 0} r(\mathrm{x})$ $(\mathrm{i}=1,2)$; $\lambda\in\mathbb{C}$ is eigenparameter; α_i,β_i $(i=1,2)$, δ_j,γ_j (j $=1,2,3,4$) are real numbers and $\delta_j\gamma_j\neq 0$ ($j=1,2,3,4$). Also throughout this paper, we assume that $\rho := \alpha_1\beta_2-\alpha_2\beta_1>0$, $p(\mathrm{x})>0$ and $r(x)>0$.

We investigate the resolvent operator and completeness of eigenfunctions of the problem (1.16)-(1.22). For operator-theoretic formulation of the considered problem we define an equivalent inner product in the Hilbert space $L_2[-1,1]\oplus\mathbb{C}$ and suitable self-adjoint linear operator in it. We obtain the resolvent operator and prove compactness of it. Finally we prove the main theorem about expansion in series of eigenfunctions. In the special case that $p(x)=q(x)\equiv 1$ and the transmission coefficients $\delta_i=\gamma_i$ $(i=\overline{1,4})$ in the results obtained in this work coincide with corresponding results in the classical continuous Sturm-Liouville operator.

2. ASYMPTOTIC PROPERTIES OF EIGENVALUES AND EIGENFUNCTIONS OF A STURM-LIOUVILLE PROBLEM WITH DISCONTINUOUS WEIGHT FUNCTION

The results of this chapter were the object of the article "Asymptotic properties of eigenvalues and eigenfunctions of a Sturm-Liouville problem with discontinuous weight function, Miskolc Mathematical Notes, Vol. 15, No. 1, pp. 197-209, 2014".

2.1 Operator-Theoretic Formulation of the Problem

In this section, we introduce a special inner product in the Hilbert space $\left(L_2\left(-1,h_1\right)\oplus L_2\left(h_1,h_2\right)\oplus L_2\left(h_2,1\right)\right)\oplus \mathbb{C}$ and define a linear operator A in it so that the problem (1.1)-(1.7) can be interpreted as the eigenvalue problem for A. To this end, we define a new Hilbert space inner product on $H := \left(L_2\left(-1,h_1\right)\oplus L_2\left(h_1,h_2\right)\oplus L_2\left(h_2,1\right)\right)\oplus \mathbb{C}$ by

$$\begin{aligned}\left\langle F,G\right\rangle_H &= \omega_1^2\int_{-1}^{h_1} f(x)\overline{g(x)}dx+\omega_2^2\,\frac{\delta_1\delta_2}{\gamma_1\gamma_2}\int_{h_1}^{h_2} f(x)\overline{g(x)}dx \\ &+\omega_3^2\,\frac{\delta_1\delta_2\delta_3\delta_4}{\gamma_1\gamma_2\gamma_3\gamma_4}\int_{h_2}^{1} f(x)\overline{g(x)}dx+\frac{\delta_1\delta_2\delta_3\delta_4}{\rho\gamma_1\gamma_2\gamma_3\gamma_4}\,f_1\overline{g_1}\end{aligned}$$

for $F=\begin{pmatrix} f(x)\\ f_1\end{pmatrix}$ and $G=\begin{pmatrix} g(x)\\ g_1\end{pmatrix}\in H$. For convenience we will use the notations

$$R_1\left(u\right) := \beta_1 u(1)-\beta_2 u'(1),\quad R_1'\left(u\right) := \beta_1' u(1)-\beta_2' u'(1).$$

In this Hilbert space we construct the operator $A: H\to H$ with domain

$$D(A)=\left\{F=\begin{pmatrix} f(x) \\ f_1 \end{pmatrix} \mid f(x),\, f'(x) \text{ are absolutely continuous in } [1,h_1]\cup[h_1,h_2]\right.$$
$$\cup[h_2,1];\text{ and has finite limits } f(h_1\pm 0),\ f(h_2\pm 0),\ f'(h_1\pm 0),\ f'(h_2\pm 0);$$
$$\tau f\in L_2(-1,h_1)\oplus L_2(h_1,h_2)\oplus L_2(h_2,1);\ L_1 f=L_3 f=L_4 f=L_5 f=L_6 f=0,$$
$$\left. f_1=R_1'(f)\right\} \tag{2.1}$$

which acts by the rule

$$AF=\begin{pmatrix} \frac{1}{\omega(x)}\left[-f''+q(x)f\right] \\ -R_1(f) \end{pmatrix} \text{ with } F=\begin{pmatrix} f(x) \\ R_1'(f) \end{pmatrix}\in D(A). \tag{2.2}$$

Thus we can pose the boundary-value-transmission problem (1.1)-(1.7) in H as

$$AU=\lambda U,\quad U:=\begin{pmatrix} u(x) \\ R_1'(u) \end{pmatrix}\in D(A). \tag{2.3}$$

It is readily verified that the eigenvalues of A coincide with those of the problem (1.1)-(1.7).

Theorem 2.1. The operator A is symmetric.

Proof. Let $F=\begin{pmatrix} f(x) \\ R_1'(f) \end{pmatrix}$ and $G=\begin{pmatrix} g(x) \\ R_1'(g) \end{pmatrix}$ be arbitrary elements of $D(A)$. Twice integrating by parts we find

$$\begin{aligned}\langle AF,G\rangle_H-\langle F,AG\rangle_H &= W\left(f,\overline{g};h_1-0\right)-W\left(f,\overline{g};-1\right)\\ &+\frac{\delta_1\delta_2}{\gamma_1\gamma_2}\left(W\left(f,\overline{g};h_2-0\right)-W\left(f,\overline{g};h_1+0\right)\right)\\ &+\frac{\delta_1\delta_2\delta_3\delta_4}{\gamma_1\gamma_2\gamma_3\gamma_4}\left(W\left(f,\overline{g};1\right)-W\left(f,\overline{g};h_2+0\right)\right)\\ &+\frac{\delta_1\delta_2\delta_3\delta_4}{\rho\gamma_1\gamma_2\gamma_3\gamma_4}\left(R_1'(f)R_1(\overline{g})-R_1(f)R_1'(\overline{g})\right)\end{aligned} \tag{2.4}$$

where, as usual, $W(f,g;x)$ denotes the Wronskian of f and g; i.e.,

$$W(f,g;x) := f(x)g'(x) - f'(x)g(x).$$

Since $F,G \in D(A)$, the first components of these elements, i.e. f and g satisfy the boundary condition (1.2). From this fact we easily see that

$$W\left(f,\overline{g};-1\right) = 0, \tag{2.5}$$

since $\cos\alpha$ and $\sin\alpha$ are real. Further, as f and g also satisfy both transmission conditions, we obtain

$$W\left(f,\overline{g};h_1-0\right) = \frac{\delta_1\delta_2}{\gamma_1\gamma_2} W\left(f,\overline{g};h_1+0\right), \tag{2.6}$$

$$W\left(f,\overline{g};h_2-0\right) = \frac{\delta_1\delta_2\delta_3\delta_4}{\gamma_1\gamma_2\gamma_3\gamma_4} W\left(f,\overline{g};h_2+0\right). \tag{2.7}$$

Moreover, the direct calculations give

$$R_1'(f)R_1(\overline{g}) - R_1(f)R_1'(\overline{g}) = -\rho W\left(f,\overline{g};1\right). \tag{2.8}$$

Now, inserting (2.5)-(2.8) in (2.4), we have

$$\langle AF,G\rangle_H = \langle F,AG\rangle_H \qquad (F,G \in D(A))$$

and so A is symmetric.

Recalling that the eigenvalues of (1.1)-(1.7) coincide with the eigenvalues of A, we have the next corollary:

Corollary 2.1. All eigenvalues of (1.1)-(1.7) are real.

Since all eigenvalues are real it is enough to study only the real-valued eigenfunctions. Therefore we can now assume that all eigenfunctions of (1.1)-(1.7) are real-valued.

2.2. Asymptotic Formulas for Eigenvalues and Fundamental Solutions

Let us define fundamental solutions

$$\varphi(x,\lambda)=\begin{cases}\varphi_1(x,\lambda), & x\in[-1,h_1),\\ \varphi_2(x,\lambda), & x\in(h_1,h_2),\\ \varphi_3(x,\lambda), & x\in(h_2,1]\end{cases} \text{ and } \chi(x,\lambda)=\begin{cases}\chi_1(x,\lambda), & x\in[-1,h_1),\\ \chi_2(x,\lambda), & x\in(h_1,h_2),\\ \chi_3(x,\lambda), & x\in(h_2,1]\end{cases}$$

of **(1.1)** by the following procedure. We first consider the next initial-value problem:

$$-u''+q(x)u=\lambda\omega_1^2u,\ x\in[-1,h_1] \tag{2.9}$$

$$u(-1)=\sin\alpha, \tag{2.10}$$

$$u'(-1)=-\cos\alpha. \tag{2.11}$$

By virtue of ([55], Theorem 1.5) the problem (2.9)-(2.11) has a unique solution $u=\varphi_1(x,\lambda)$ which is an entire function of $\lambda\in C$ for each fixed $x\in[-1,h_1]$. Similarly,

$$-u''+q(x)u=\lambda\omega_2^2u,\ \ x\in[h_1,h_2] \tag{2.12}$$

$$u(h_1)=\frac{\gamma_1}{\delta_1}\varphi_1(h_1,\lambda), \tag{2.13}$$

$$u'(h_1)=\frac{\gamma_2}{\delta_2}\varphi_1'(h_1,\lambda), \tag{2.14}$$

has a unique solution $u=\varphi_2(x,\lambda)$ which is an entire function of $\lambda\in C$ for each fixed $x\in[h_1,h_2]$. Continuing in this manner

$$-u''+q(x)u=\lambda\omega_3^2u,\quad x\in[h_2,1] \tag{2.15}$$

$$u(h_2)=\frac{\gamma_3}{\delta_3}\varphi_2(h_2,\lambda), \tag{2.16}$$

$$u'(h_2)=\frac{\gamma_4}{\delta_4}\varphi_2'(h_2,\lambda), \tag{2.17}$$

has a unique solution $u=\varphi_3(x,\lambda)$ which is an entire function of $\lambda\in C$ for each fixed $x\in[h_2,1]$. Slightly modifying the method of ([55], Theorem 1.5) we can prove that the initial-value problem

$$-u''+q(x)u=\lambda\omega_3^2u,\ x\in[h_2,1] \tag{2.18}$$

$$u(1)=\beta_2'\lambda+\beta_2, \tag{2.19}$$

$$u'(1)=\beta_1'\lambda+\beta_1 \tag{2.20}$$

(2.18)-(2.20) has a unique solution $u=\chi_3(x,\lambda)$ which is an entire function of spectral parameter $\lambda\in C$ for each fixed $x\in[h_2,1]$. Similarly,

$$-u''+q(x)u=\lambda\omega_2^2u,\quad x\in[h_1,h_2] \tag{2.21}$$

$$u(h_2)=\frac{\delta_3}{\gamma_3}\chi_3(h_2,\lambda), \tag{2.22}$$

$$u'(h_2)=\frac{\delta_4}{\gamma_4}\chi_3'(h_2,\lambda), \tag{2.23}$$

has a unique solution $u=\chi_2(x,\lambda)$ which is an entire function of $\lambda\in C$ for each fixed $x\in[h_1,h_2]$. Continuing in this manner

$$-u''+q(x)u=\lambda\omega_3^2 u,\quad x\in[-1,h_1] \tag{2.24}$$

$$u(h_1)=\frac{\delta_1}{\gamma_1}\chi_2(h_1,\lambda), \tag{2.25}$$

$$u'(h_1)=\frac{\delta_2}{\gamma_2}\chi_2'(h_1,\lambda), \tag{2.26}$$

has a unique solution $u=\chi_1(x,\lambda)$ which is an entire function of $\lambda\in C$ for each fixed $x\in[-1,h_1]$.

By virtue of (2.10) and (2.11) the solution $\varphi(x,\lambda)$ satisfies the first boundary condition (1.2). Moreover, by (2.13), (2.14), (2.16) and (2.17), $\varphi(x,\lambda)$ satisfies also transmission conditions (1.4)-(1.7). Similarly, by (2.19), (2.20), (2.22), (2.23), (2.25) and (2.26) the other solution $\chi(x,\lambda)$ satisfies the second boundary condition (1.3) and transmission conditions (1.4)-(1.7). It is well-known from the theory of ordinary differential equations that each of the Wronskians $\Delta_1(\lambda)=W(\varphi_1(x,\lambda),\chi_1(x,\lambda))$, $\Delta_2(\lambda)=W(\varphi_2(x,\lambda),\chi_2(x,\lambda))$ and $\Delta_3(\lambda)=W(\varphi_3(x,\lambda),\chi_3(x,\lambda))$ are independent of x in $[-1,h_1]$, $[h_1,h_2]$ and $[h_2,1]$ respectively.

Lemma 2.1. The equality $\Delta_1(\lambda)=\frac{\delta_1\delta_2}{\gamma_1\gamma_2}\Delta_2(\lambda)=\frac{\delta_1\delta_2\delta_3\delta_4}{\gamma_1\gamma_2\gamma_3\gamma_4}\Delta_3(\lambda)$ holds for each $\lambda\in C$.

Proof. Since the above Wronskians are independent of x, using (2.16), (2.17), (2.19), (2.20), (2.22), (2.23), (2.25) and (2.26) we find

$$\begin{aligned}\Delta_1(\lambda) &= \varphi_1(h_1,\lambda)\chi_1'(h_1,\lambda)-\varphi_1'(h_1,\lambda)\chi_1(h_1,\lambda)\\ &= \left(\frac{\delta_1}{\gamma_1}\varphi_2(h_1,\lambda)\right)\left(\frac{\delta_2}{\gamma_2}\chi_2'(h_1,\lambda)\right)-\left(\frac{\delta_2}{\gamma_2}\varphi_2'(h_1,\lambda)\right)\left(\frac{\delta_1}{\gamma_1}\chi_2(h_1,\lambda)\right)\\ &= \frac{\delta_1\delta_2}{\gamma_1\gamma_2}\Delta_2(\lambda)=\left(\frac{\delta_1\delta_3}{\gamma_1\gamma_3}\varphi_3(h_2,\lambda)\right)\left(\frac{\delta_2\delta_4}{\gamma_2\gamma_4}\chi_3'(h_2,\lambda)\right)\\ &\qquad -\left(\frac{\delta_2\delta_4}{\gamma_2\gamma_4}\varphi_3'(h_2,\lambda)\right)\left(\frac{\delta_1\delta_3}{\gamma_1\gamma_3}\chi_3(h_2,\lambda)\right)=\frac{\delta_1\delta_2\delta_3\delta_4}{\gamma_1\gamma_2\gamma_3\gamma_4}\Delta_3(\lambda).\end{aligned}$$

Corollary 2.2. The zeros of $\Delta_1(\lambda)$, $\Delta_2(\lambda)$ and $\Delta_3(\lambda)$ coincide.

In view of Lemma 2.1 we denote $\Delta_1(\lambda)$, $\frac{\delta_1\delta_2}{\gamma_1\gamma_2}\Delta_2(\lambda)$ and $\frac{\delta_1\delta_2\delta_3\delta_4}{\gamma_1\gamma_2\gamma_3\gamma_4}\Delta_3(\lambda)$ by $\Delta(\lambda)$. Recalling the definitions of $\varphi_i(x,\lambda)$ and $\chi_i(x,\lambda)$, we can state the next corollary.

Corollary 2.3. The function $\Delta(\lambda)$ is an entire function.

Theorem 2.2. The eigenvalues of (1.1)-(1.7) are the roots of $\Delta(\lambda)=0$.

Proof. Let $\Delta(\lambda_0)=0.$ Then $W(\varphi_1(x,\lambda_0),\chi_1(x,\lambda_0))=0$ for all $x\in[-1,h_1]$. Consequently, the functions $\varphi_1(x,\lambda_0)$ and $\chi_1(x,\lambda_0)$ are linearly dependent, i.e., $\chi_1(x,\lambda_0)=k\varphi_1(x,\lambda_0)$, $x\in[-1,h_1]$, for some $k\neq 0.$ By (2.10) and (2.11), from this equality, we have

$$\begin{aligned}&\cos\alpha\chi(-1,\lambda_0)+\sin\alpha\chi'(-1,\lambda_0)=\cos\alpha\chi_1(-1,\lambda_0)+\sin\alpha\chi_1'(-1,\lambda_0)\\ &=k\left(\cos\alpha\varphi_1(-1,\lambda_0)+\sin\alpha\varphi_1'(-1,\lambda_0)\right)=k\left(\cos\alpha\sin\alpha+\sin\alpha(-\cos\alpha)\right)=0,\end{aligned}$$

and so $\chi(x,\lambda_0)$ satisfies the first boundary condition (1.2). Recalling that the solution $\chi(x,\lambda_0)$ also satisfies the other boundary condition (1.3) and transmission conditions (1.4)-(1.7). We conclude that $\chi(x,\lambda_0)$ is an eigenfunction of (1.1)-(1.7); i.e., λ_0 is an eigenvalue. Thus, each zero of $\Delta(\lambda)$ is an eigenvalue. Now let λ_0 be an eigenvalue and let $u_0(x)$ be an eigenfunction with this eigenvalue. Suppose that

$\Delta(\lambda_0) \neq 0$. Whence $W(\varphi_1(x,\lambda_0),\chi_1(x,\lambda_0)) \neq 0$, $W(\varphi_2(x,\lambda_0),\chi_2(x,\lambda_0)) \neq 0$ and $W(\varphi_3(x,\lambda_0),\chi_3(x,\lambda_0)) \neq 0$. From this, by virtue of the well-known properties of Wronskians, it follows that each of the pairs $\varphi_1(x,\lambda_0)$, $\chi_1(x,\lambda_0)$; $\varphi_2(x,\lambda_0)$, $\chi_2(x,\lambda_0)$ and $\varphi_3(x,\lambda_0)$, $\chi_3(x,\lambda_0)$ is linearly independent. Therefore, the solution $u_0(x)$ of (1.1) may be represented as

$$u_0(x) = \begin{cases} c_1\varphi_1(x,\lambda_0) + c_2\chi_1(x,\lambda_0), & x \in [-1,h_1), \\ c_3\varphi_2(x,\lambda_0) + c_4\chi_2(x,\lambda_0), & x \in (h_1,h_2), \\ c_5\varphi_3(x,\lambda_0) + c_6\chi_3(x,\lambda_0), & x \in (h_2,1], \end{cases}$$

where at least one of the coefficients c_i $\left(i = \overline{1,6}\right)$ is not zero. Considering the true equalities

$$L_\upsilon(u_0(x)) = 0, \quad \upsilon = \overline{1,6}, \qquad \textbf{(2.27)}$$

as the homogenous system of linear equations in the variables c_i $\left(i = \overline{1,6}\right)$ and taking (2.13), (2.14), (2.16), (2.17), (2.22), (2.23), (2.25) and (2.26) into account, we see that the determinant of this system is equal to $-\frac{(\delta_1\delta_2\delta_3\delta_4)^2}{\gamma_1\gamma_2\gamma_3\gamma_4}\Delta^4(\lambda_0)$ and so it does not vanish by assumption. Consequently the system (2.27) has the only trivial solution $c_i = 0$ $\left(i = \overline{1,6}\right)$. This is a contradiction. And the proof is complete.

Theorem 2.3. Let $\lambda = \mu^2$ and $\operatorname{Im}\mu = t$. Then the following asymptotic equalities hold as $|\lambda| \to \infty$:

(1) In case $\sin\alpha \neq 0$

$$\varphi_1^{(k)}(x,\lambda) = \sin\alpha \frac{d^k}{dx^k}\cos\left[\mu\omega_1(x+1)\right] + O\left(\frac{1}{|\mu|^{1-k}}\exp\left(|t|\,\omega_1(x+1)\right)\right), \qquad \textbf{(2.28)}$$

$$\varphi_2^{(k)}(x,\lambda)=\frac{\gamma_1}{\delta_1}\sin\alpha\frac{d^k}{dx^k}\cos\left[\mu\left(\omega_2 x+\omega_1 h_1+\omega_1\right)\right]$$
$$+O\left(\frac{1}{|\mu|^{1-k}}\exp\left(|t|\left(\omega_2 x+\omega_1 h_1+\omega_1\right)\right)\right), \quad \textbf{(2.29)}$$

$$\varphi_3^{(k)}(x,\lambda)=\frac{\gamma_1\gamma_3}{\delta_1\delta_3}\sin\alpha\frac{d^k}{dx^k}\cos\left[\mu\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\right]$$
$$+O\left(\frac{1}{|\mu|^{1-k}}\exp\left(|t|\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\right)\right). \quad \textbf{(2.30)}$$

(2) In case $\sin\alpha=0$

$$\varphi_1^{(k)}(x,\lambda)=\frac{-1}{\mu\omega_1}\cos\alpha\frac{d^k}{dx^k}\sin\left[\mu\omega_1(x+1)\right]+O\left(\frac{1}{|\mu|^{2-k}}\exp\left(|t|\,\omega_1(x+1)\right)\right), \quad \textbf{(2.31)}$$

$$\varphi_2^{(k)}(x,\lambda)=-\frac{\gamma_1}{\mu\delta_1}\cos\alpha\frac{d^k}{dx^k}\sin\left[\mu\left(\omega_2 x+\omega_1 h_1+\omega_1\right)\right]$$
$$+O\left(\frac{1}{|\mu|^{2-k}}\exp\left(|t|\left(\omega_2 x+\omega_1 h_1+\omega_1\right)\right)\right), \quad \textbf{(2.32)}$$

$$\varphi_3^{(k)}(x,\lambda)=-\frac{\gamma_1\gamma_3}{\mu\delta_1\delta_3}\cos\alpha\frac{d^k}{dx^k}\sin\left[\mu\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\right]$$
$$+O\left(\frac{1}{|\mu|^{2-k}}\exp\left(|t|\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\right)\right). \quad \textbf{(2.33)}$$

for $k=0$ and $k=1$. Moreover, each of these asymptotic equalities holds uniformly for x.

Proof. Asymptotic formulas for $\varphi_1(x,\lambda)$ and $\varphi_2(x,\lambda)$ are found in ([55], Lemma 1.7) and ([39], Theorem 3.2) respectively. But the formulas for $\varphi_3(x,\lambda)$ need individual considerations, since this solution is defined by the initial condition with some special nonstandart form. The initial-value problem (2.15)-(2.17) can be transformed into the equivalent integral equation

$$
\begin{aligned}
u(x) = {} & \frac{\gamma_3}{\delta_3}\varphi_2(h_2,\lambda)\cos\mu\omega_3 x + \frac{\gamma_4}{\mu\omega_3\delta_4}\varphi_2'(h_2,\lambda)\sin\mu\omega_3 x \\
& + \frac{\omega_3}{\mu}\int_{h_2}^{x}\sin\left[\mu\omega_3(x-y)\right]q(y)u(y)\,dy.
\end{aligned} \qquad \textbf{(2.34)}
$$

Let $\sin\alpha \neq 0$. Inserting (2.29) in (2.34) we have

$$
\begin{aligned}
\varphi_3(x,\lambda) = {} & \frac{\gamma_1\gamma_3}{\delta_1\delta_3}\sin\alpha\cos\left[\mu\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\right] \\
& + \frac{\omega_3}{\mu}\int_{h_2}^{x}\sin\left[\mu\omega_3(x-y)\right]q(y)\varphi_3(y,\lambda)\,dy \\
& + O\left(\frac{1}{|\mu|}\exp\left(|t|\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\right)\right).
\end{aligned} \qquad \textbf{(2.35)}
$$

Multiplying this by $\exp\left(-|t|\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\right)$ and denoting

$$
F(x,\lambda) = \exp\left(-|t|\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\right)\varphi_3(x,\lambda),
$$

we have the following integral equation

$$
\begin{aligned}
F(x,\lambda) = {} & \frac{\gamma_1\gamma_3}{\delta_1\delta_3}\sin\alpha\exp\left(-|t|\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\right)\cos\left[\mu\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\right] \\
& + \frac{\omega_3}{\mu}\int_{h_2}^{x}\sin\left[\mu\omega_3(x-y)\right]\exp\left(-|t|\,\omega_3(x-y)\right)q(y)F(y,\lambda)dy + O\left(\frac{1}{\mu}\right).
\end{aligned}
$$

Putting $M(\lambda) = \max_{x\in[h_2,1]}\left|F(x,\lambda)\right|$, from the last equation we derive that

$$
M(\lambda) \le M_0\left(\left|\frac{\gamma_1\gamma_3}{\delta_1\delta_3}\right| + \frac{1}{\mu}\right)
$$

for some $M_0 > 0$. Consequently, $M(\lambda) = O(1)$ as $|\lambda| \to \infty$, and so $\varphi_3(x,\lambda) = O\left(\exp\left(|t|\left(\omega_3 x + \omega_2 h_2 + \omega_1\right)\right)\right)$ as $|\lambda| \to \infty$. Inserting the integral term of (2.35) yields (2.30) for $k = 0$. The case $k = 1$ of (2.30) follows at once on differentiating (2.29) and making the same procedure as in the case $k = 0$. The proof of (2.33) is similar to that of (2.30).

Theorem 2.4. Let $\lambda = \mu^2$, $\mu = \sigma + it$. Then the following asymptotic formulas hold for the eigenvalues of the boundary-value-transmission problem (1.1)-(1.7):

Case 1: $\beta_2' \neq 0$, $\sin\alpha \neq 0$

$$\mu_n = \frac{\pi(n-1)}{\omega_3 + \omega_2 h_2 + \omega_1} + O\left(\frac{1}{n}\right), \tag{2.36}$$

Case 2: $\beta_2' \neq 0$, $\sin\alpha = 0$

$$\mu_n = \frac{\pi\left(n - \frac{1}{2}\right)}{\omega_3 + \omega_2 h_2 + \omega_1} + O\left(\frac{1}{n}\right), \tag{2.37}$$

Case 3: $\beta_2' = 0$, $\sin\alpha \neq 0$

$$\mu_n = \frac{\pi\left(n - \frac{1}{2}\right)}{\omega_3 + \omega_2 h_2 + \omega_1} + O\left(\frac{1}{n}\right), \tag{2.38}$$

Case 4: $\beta_2' = 0$, $\sin\alpha = 0$

$$\mu_n = \frac{\pi n}{\omega_3 + \omega_2 h_2 + \omega_1} + O\left(\frac{1}{n}\right), \tag{2.39}$$

Proof. Let us consider only the case 1. Putting $x = 1$ in

$$\Delta_3(\lambda) = \varphi_3(x,\lambda)\chi_3'(x,\lambda) - \varphi_3'(x,\lambda)\chi_3(x,\lambda)$$

and inserting $\chi_3(1,\lambda)=\beta_2'\lambda+\beta_2,\ \chi_3'(1,\lambda)=\beta_1'\lambda+\beta_1$ we have the following representation for $\Delta_3(\lambda)$:

$$\Delta_3(\lambda)=\left(\beta_1'\lambda+\beta_1\right)\varphi_3(1,\lambda)-\left(\beta_2'\lambda+\beta_2\right)\varphi_3'(1,\lambda). \tag{2.40}$$

Putting $x=1$ in (2.30) and inserting the result in (2.40), we derive now that

$$\begin{aligned}\Delta_3(\lambda)&=\frac{\delta_2\delta_4}{\gamma_2\gamma_4}\omega_3\beta_2'(\sin\alpha)\mu^3\sin\left[\mu\left(\omega_3+\omega_2h_2+\omega_1\right)\right]\\&+O\left(|\mu|^2\exp\left(2|t|\left(\omega+\omega_2h_2+\omega_1\right)\right)\right).\end{aligned} \tag{2.41}$$

By applying the Rouché Theorem, it follows that $\Delta_3(\lambda)$ has the same number of zeros inside the contour as the leading term in (2.41). Hence, if $\lambda_0<\lambda_1<\lambda_2...$ are the zeros of $\Delta_3(\lambda)$ and $\mu_n^2=\lambda_n,$ we have

$$\frac{\pi(n-1)}{\omega_3+\omega_2h_2+\omega_1}+\delta_n \tag{2.42}$$

for sufficiently large $n,$ where $\left|\delta_n\right|<\frac{\pi}{4(\omega_3+\omega_2h_2+\omega_1)}$ for sufficiently large $n.$ By putting in (2.41) we have $\delta_n=O\left(\frac{1}{n}\right)$, and the proof is completed in Case 1. The proofs for the other cases are similar.

Theorem 2.5. The following asymptotic formulas hold for the eigenfunctions

$$\varphi_{\lambda_n}(x)=\begin{cases}\varphi_1(x,\lambda_n), & x\in[-1,h_1),\\ \varphi_2(x,\lambda_n), & x\in(h_1,h_2),\\ \varphi_3(x,\lambda_n), & x\in(h_2,1]\end{cases}$$

of (1.1)-(1.7):

Case 1: $\beta_2'\neq 0$, $\sin\alpha\neq 0$

$$\varphi_{\lambda_n}(x)=\begin{cases}\sin\alpha\cos\left[\frac{\omega_1\pi(n-1)(x+1)}{\omega_2+\omega_1}\right]+O\left(\frac{1}{n}\right),\ x\in[-1,h_1),\\ \frac{\gamma_1}{\delta_1}\sin\alpha\cos\left[\frac{(\omega_2x+\omega_1h_1+\omega_1)\pi(n-1)}{\omega_2+\omega_1h_1+\omega_1}\right]+O\left(\frac{1}{n}\right),\ x\in(h_1,h_2),\\ \frac{\gamma_1\gamma_3}{\delta_1\delta_3}\sin\alpha\cos\left[\frac{(\omega_3x+\omega_2h_2+\omega_1)\pi(n-1)}{\omega_3+\omega_2h_2+\omega_1}\right]+O\left(\frac{1}{n}\right),\ x\in(h_2,1].\end{cases}$$

Case 2: $\beta_2'\neq 0$, $\sin\alpha=0$

$$\varphi_{\lambda_n}(x)=\begin{cases}-\frac{\omega_1+\omega_2}{\omega_1}\frac{\cos\alpha}{\pi\left(n-\frac{1}{2}\right)}\sin\left[\frac{\omega_1\pi\left(n-\frac{1}{2}\right)(x+1)}{\omega_2+\omega_1}\right]+O\left(\frac{1}{n^2}\right),\ x\in[-1,h_1),\\ \frac{-\gamma_1}{\delta_1}\frac{\omega_1+\omega_2}{\omega_1}\frac{\cos\alpha}{\pi\left(n-\frac{1}{2}\right)}\sin\left[\frac{(\omega_2x+\omega_1h_1+\omega_1)\pi\left(n-\frac{1}{2}\right)}{\omega_2+\omega_1h_1+\omega_1}\right]+O\left(\frac{1}{n^2}\right),\ x\in(h_1,h_2),\\ \frac{-\gamma_1\gamma_3}{\delta_1\delta_3}\frac{\omega_1+\omega_2}{\omega_1}\frac{\cos\alpha}{\pi\left(n-\frac{1}{2}\right)}\sin\left[\frac{(\omega_3x+\omega_2h_2+\omega_1)\pi\left(n-\frac{1}{2}\right)}{\omega_3+\omega_2h_2+\omega_1}\right]+O\left(\frac{1}{n^2}\right),\ x\in(h_2,1].\end{cases}$$

Case 3: $\beta_2'=0$, $\sin\alpha\neq 0$

$$\varphi_{\lambda_n}(x)=\begin{cases}\sin\alpha\cos\left[\frac{\omega_1\pi\left(n-\frac{1}{2}\right)(x+1)}{\omega_2+\omega_1}\right]+O\left(\frac{1}{n}\right),\ x\in[-1,h_1),\\ \frac{\gamma_1}{\delta_1}\sin\alpha\cos\left[\frac{(\omega_2x+\omega_1h_1+\omega_1)\pi\left(n-\frac{1}{2}\right)}{\omega_2+\omega_1h_1+\omega_1}\right]+O\left(\frac{1}{n}\right),\ x\in(h_1,h_2),\\ \frac{\gamma_1\gamma_3}{\delta_1\delta_3}\sin\alpha\cos\left[\frac{(\omega_3x+\omega_2h_2+\omega_1)\pi\left(n-\frac{1}{2}\right)}{\omega_3+\omega_2h_2+\omega_1}\right]+O\left(\frac{1}{n}\right),\ x\in(h_2,1].\end{cases}$$

Case 4: $\beta_2'=0$, $\sin\alpha=0$

$$\varphi_{\lambda_n}(x)=\begin{cases}-\frac{\omega_1+\omega_2}{\omega_1}\frac{\cos\alpha}{\pi n}\sin\left[\frac{\omega_1\pi n(x+1)}{\omega_2+\omega_1}\right]+O\left(\frac{1}{n^2}\right),\ x\in[-1,h_1),\\ \frac{-\gamma_1}{\delta_1}\frac{\omega_1+\omega_2}{\omega_1}\frac{\cos\alpha}{\pi n}\sin\left[\frac{(\omega_2x+\omega_1h_1+\omega_1)\pi n}{\omega_2+\omega_1h_1+\omega_1}\right]+O\left(\frac{1}{n^2}\right),\ x\in(h_1,h_2),\\ \frac{-\gamma_1\gamma_3}{\delta_1\delta_3}\frac{\omega_1+\omega_2}{\omega_1}\frac{\cos\alpha}{\pi n}\sin\left[\frac{(\omega_3x+\omega_2h_2+\omega_1)\pi n}{\omega_3+\omega_2h_2+\omega_1}\right]+O\left(\frac{1}{n^2}\right),\ x\in(h_2,1].\end{cases}$$

All these asymptotic formulas hold uniformly for x.

Proof. Let us consider only the Case 1. Inserting (2.30) in the integral term of (2.35), we easily see that

$$\int_{h_2}^{x} \sin\left[\mu\omega_3\left(x-y\right)\right]q\left(y\right)\varphi_3\left(y,\lambda\right)dy = O\left(\exp\left(\left|t\right|\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\right)\right).$$

Inserting in (2.28) yields

$$\begin{aligned}\varphi_3\left(x,\lambda\right) = \frac{\gamma_1\gamma_3}{\delta_1\delta_3}\sin\alpha\cos\left[\mu\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\right] \\ +O\left(\frac{1}{|\mu|}\exp\left|t\right|\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\right).\end{aligned} \quad \textbf{(2.43)}$$

We already know that all eigenvalues are real. Furthermore, putting $\lambda=-H,\quad H>0$ in (2.41) we infer that $\omega\left(-H\right)\to\infty$ as $H\to+\infty$, and so $\omega\left(-H\right)\neq 0$ for sufficiently large $R>0$. Consequently, the set of eigenvalues is bounded below. Letting $\sqrt{\lambda_n}=\mu_n$ in (2.43) we now obtain

$$\varphi_3\left(x,\lambda_n\right) = \frac{\gamma_1\gamma_3}{\delta_1\delta_3}\sin\alpha\cos\left[\mu_n\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\right]+O\left(\frac{1}{\mu_n}\right)$$

since $t_n = lm\mu_n$ for sufficiently large n. After some calculation, we easily see that

$$\cos\left[\mu_n\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\right] = \cos\left[\frac{\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\pi\left(n-1\right)}{\omega_3+\omega_2 h_2+\omega_1}\right]+O\left(\frac{1}{n}\right).$$

Consequently,

$$\varphi_3\left(x,\lambda_n\right) = \frac{\gamma_1\gamma_3}{\delta_1\delta_3}\sin\alpha\cos\left[\frac{\left(\omega_3 x+\omega_2 h_2+\omega_1\right)\pi\left(n-1\right)}{\omega_3+\omega_2 h_2+\omega_1}\right]+O\left(\frac{1}{n}\right).$$

In a similar method, we can deduce that

$$\varphi_2\left(x,\lambda_n\right) = \frac{\gamma_1}{\delta_1}\sin\alpha\cos\left[\frac{\left(\omega_2 x+\omega_1 h_1+\omega_1\right)\pi\left(n-1\right)}{\omega_2+\omega_1 h_1+\omega_1}\right]+O\left(\frac{1}{n}\right),$$

and

$$\varphi_1(x,\lambda_n)=\sin\alpha\cos\left[\frac{\omega_1\pi(n-1)(x+1)}{\omega_2+\omega_1}\right]+O\left(\frac{1}{n}\right).$$

Thus the proof of the theorem completed in Case 1. The proofs for the other cases are similar.

3. SPECTRAL PROPERTIES OF DISCONTINUOUS STURM-LIOUVILLE PROBLEMS WITH A FINITE NUMBER OF TRANSMISSION CONDITIONS

The results of this chapter are gathered in the article "Spectral properties of discontinuous Sturm–Liouville problems with a finite number of transmission conditions, Mediterranean Journal of Mathematics, DOI 10.1007/s00009-014-0487-x, in press (with O. Sh. Mukhtarov)".

3.1. Operator Formulation

By using the method introduced in [40] we shall define direct sum of Hilbert spaces but with the usual inner product replaced by appropriate multiples. Namely, in the Hilbert space $H = L_2(-1,1) \oplus \mathbb{C}$ we define an inner product by

$$\langle F,G\rangle := \sum_{j=0}^{m}\left(\prod_{i=0}^{j}\delta_i^2\right)\int_{h_j}^{h_{j+1}} f(x)\overline{g(x)}dx + \frac{\prod_{i=1}^{m}\delta_i^2}{\rho} f_1\overline{g_1},$$

where $h_0 = -1,\quad h_{m+1} = 1,\quad \delta_0 = 1$, for

$$F := \begin{pmatrix} f(x) \\ f_1 \end{pmatrix},\ G := \begin{pmatrix} g(x) \\ g_1 \end{pmatrix} \in H.$$

For convenience we put

$$R_1(u) := \beta_1 u(1) - \beta_2 u'(1),$$
$$R_1'(u) := \beta_1' u(1) - \beta_2' u'(1).$$

The function $f(x)$ is defined on $[-1,h_1)\cup(h_1,h_2)\cup...\cup(h_m,1]$ and has finite limits $f(h_i\pm 0):=\lim_{x\to h_i\pm 0} f(x)$ $\left(i=\overline{1,m}\right)$. By $f_i(x)$ $\left(i=\overline{1,m+1}\right)$ we denote the functions

$$f_1(x):=\begin{cases} f(x), & x\in[-1,h_1), \\ \lim_{x\to h_1-0} f(x), & x=h_1, \end{cases} \quad f_2(x):=\begin{cases} \lim_{x\to h_1+0} f(x), & x=h_1, \\ f(x), & x\in(h_1,h_2),... \\ \lim_{x\to h_2-0} f(x), & x=h_2, \end{cases}$$

$$f_m(x):=\begin{cases} \lim_{x\to h_{m-1}+0} f(x), & x=h_{m-1}, \\ f(x), & x\in(h_{m-1},h_m), \\ \lim_{x\to h_m-0} f(x), & x=h_m, \end{cases} \quad f_{m+1}(x):=\begin{cases} \lim_{x\to h_m+0} f(x), & x=h_m, \\ f(x), & x\in(h_m,1] \end{cases}$$

which are defined on $\Omega_1:=[-1,h_1]$, $\Omega_2:=[h_1,h_2],...,\Omega_m:=[h_{m-1},h_m]$, $\Omega_{m+1}:=[h_m,1]$ respectively.

In the Hilbert space H we introduce a linear operator A on the domain

$$D(A):=\left\{F\in H \,\middle|\, f_i(x),\ f_i'(x) \text{ are absolutely continuous in } \Omega_i \left(i=\overline{1,m+1}\right),\ \tau f\in L^2[-1,1],\ \tau_{2i+1}u:=u(h_i-0)-\delta_i u(h_i+0)=0,\ \tau_{2i+2}u:=u'(h_i-0)-\delta_i u'(h_i+0)=0 \left(i=\overline{1,m+1}\right) \text{ and } f_1=R_1'(f)\right\}$$

by action low

$$AF=\begin{pmatrix} \tau f \\ -R_1(f) \end{pmatrix}.$$

Then we can rewrite the considered problem (1.8)-(1.12) in the operator formulation as

$$AF=\lambda F$$

where

$$F := \begin{pmatrix} f(x) \\ -R_1'(f) \end{pmatrix} \in D(A).$$

Consequently, the problem (1.8)-(1.12) can be considered as the eigenvalue problem for the operator A. Obviously, we have

Lemma 3.1. The eigenvalues of the boundary value problem (1.8)-(1.12) coincide with those of A, and its eigenfunctions are the first components of the corresponding eigenelements of A.

Lemma 3.2. The domain $D(A)$ is dense in H.

Proof. Let $F = \begin{pmatrix} f(x) \\ f_1 \end{pmatrix} \in H$, $F \perp D(A)$ and let $\widetilde{C}_0^\infty$ be the set of all functions

$$\phi(x) = \begin{cases} \phi_1(x), x \in [-1, h_1), \\ \phi_2(x), x \in (h_1, h_2), \\ \quad\vdots \\ \phi_{m+1}(x), x \in (h_m, 1] \end{cases}$$

for $\varphi_1(x) \in \widehat{C}_0^\infty[-1, h_1)$, $\varphi_2(x) \in \widehat{C}_0^\infty(h_1, h_2), \ldots, \varphi_{m+1}(x) \in \widehat{C}_0^\infty(h_m, 1]$. Since $\widehat{C}_0^\infty \oplus 0 \subset D(A)$ $(0 \in \mathbb{C})$ and $U = \begin{pmatrix} u(x) \\ 0 \end{pmatrix} \in \widehat{C}_0^\infty \oplus 0$ is orthogonal to F, we have

$$\langle F, U \rangle = \sum_{j=0}^{m} \left(\prod_{i=0}^{j} \delta_i^2 \right) \int_{h_j}^{h_{j+1}} f(x) \overline{u(x)} dx.$$

We can learn that $f(x)$ is orthogonal to $\widehat{C}_0^\infty$ in $L^2[-1,1]$, this implies $f(x) = 0$. So for all $G = \begin{pmatrix} g(x) \\ g_1 \end{pmatrix} \in D(A)$, $\langle F, G \rangle = \frac{\prod_{i=1}^{m} \delta_i^2}{\rho} f_1 \overline{g_1} = 0$. Thus $f_1 = 0$ since $g_1 = R_1'(g)$ can be chosen arbitrarily. So $F = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which proves the assertation.

Theorem 3.1. The linear operator A is symmetric in H.

Proof. Let $F,\ G\in D(A)$. By two partial integrations, we get

$$\begin{aligned}\langle AF,G\rangle &=\langle F,AG\rangle+\left(W\left(f,\overline{g};h_1-0\right)-W\left(f,\overline{g};-1\right)\right)\\ &+\delta_1^2\left(W\left(f,\overline{g};h_2-0\right)-W\left(f,\overline{g};h_1+0\right)\right)\\ &+\delta_1^2\delta_2^2\left(W\left(f,\overline{g};h_3-0\right)-W\left(f,\overline{g};h_2+0\right)\right)+\ldots \qquad \textbf{(3.1)}\\ &+\prod_{i=1}^{m}\delta_i^2\left(W\left(f,\overline{g};1\right)-W\left(f,\overline{g};h_m+0\right)\right)\\ &+\frac{\prod_{i=1}^{m}\delta_i^2}{\rho}\left(R_1'(f)R_1\left(\overline{g}\right)-R_1(f)R_1'\left(\overline{g}\right)\right)\end{aligned}$$

where

$$W(f,\overline{g};x)=f(x)\overline{g}'(x)-f'(x)\overline{g}(x)$$

denotes the Wronskian of the functions f and $\overline{g}$. Since f and $\overline{g}$ satisfy the boundary condition (1.9), it follows that

$$W\left(f,\overline{g};-1\right)=0. \qquad \textbf{(3.2)}$$

From the transmission conditions (1.11)-(1.12) we get

$$W\left(f,g;h_i-0\right)=\delta_i^2W\left(f,g;h_i+0\right),\ i=\overline{1,m}. \qquad \textbf{(3.3)}$$

Furthermore,

$$
\begin{aligned}
&R_1'(f)R_1(\overline{g})-R_1(f)R_1'(\overline{g})\\
&=\left(\beta_1' f(1)-\beta_2' f'(1)\right)\left(\beta_1\overline{g}(1)-\beta_2\overline{g}'(1)\right)-\left(\beta_1 f(1)-\beta_2 f'(1)\right)\left(\beta_1'\overline{g}(1)-\beta_2'\overline{g}(1)\right)\\
&=\left(\beta_2\beta_1'-\beta_2'\beta_1\right)f'(1)\overline{g}(1)+\left(\beta_2'\beta_1-\beta_2\beta_1'\right)f(1)\overline{g}'(1)\\
&=\rho\left(f'(1)\overline{g}(1)-f(1)\overline{g}'(1)\right)=-\rho W(f,g;1).
\end{aligned}
\tag{3.4}
$$

Finally, substituting (3.2)-(3.4) in (3.1) then we get

$$
\langle AF,G\rangle=\langle F,AG\rangle\quad\left(F,G\in D(A)\right). \tag{3.5}
$$

Now we can write the following theorem with the helps of Theorem 3.1, Naimark's Patching Lemma [44] and using the similar way as in [40].

Theorem 3.2. The linear operator A is self-adjoint in H.

Corollary 3.1. All eigenvalues of the problem (1.8)-(1.12) are real.

We can now assume that all eigenfunctions are real-valued.

Corollary 3.2. If λ_1 and λ_2 are two different eigenvalues of the problem (1.8)-(1.12), then the corresponding eigenfunctions u_1 and u_2 of this problem are orthogonal in the sense of the following equality:

$$
\sum_{j=0}^{m}\left(\prod_{i=0}^{j}\delta_i^2\right)\int_{h_j}^{h_{j+1}}u_1(x)u_2(x)\,dx+\frac{\prod_{i=1}^{m}\delta_i^2}{\rho}R_1'(u_1)R_1'(u_2)=0.
$$

We need the following lemma, which can be proved by the same technique as in [57].

Lemma 3.3. Let the real-valued function $q(x)$ be continuous in $[-1,1]$ and $f(\lambda),\ g(\lambda)$ are given entire functions. Then for any $\lambda \in C$ initial value problem

$$-u'' + q(x)u = \lambda u, \quad x \in [-1,1],$$

$$u(-1) = f(\lambda), u'(-1) = g(\lambda) \ \big(\text{or } u(1) = f(\lambda), u'(1) = g(\lambda)\big)$$

has a unique solution $u = u(x,\lambda)$ which is an entire function of λ for each fixed $x \in [-1,1]$.

We shall define two solutions

$$\varphi_\lambda(x) = \begin{cases} \varphi_{1\lambda}(x),\ x \in [-1,h_1), \\ \varphi_{2\lambda}(x),\ x \in (h_1,h_2), \\ \vdots \\ \varphi_{(m+1)\lambda}(x),\ x \in (h_m,1], \end{cases} \quad \text{and } \chi_\lambda(x) = \begin{cases} \chi_{1\lambda}(x),\ x \in [-1,h_1), \\ \chi_{2\lambda}(x),\ x \in (h_1,h_2), \\ \vdots \\ \chi_{(m+1)\lambda}(x),\ x \in (h_m,1], \end{cases}$$

of the equation (1.8) as follows: Let $\varphi_{1\lambda}(x) := \varphi_1(x,\lambda)$ be the solution of equation (1.8) on $[-1,h_1]$, which satisfies the initial conditions

$$u(-1) = \alpha_2,\ u'(-1) = -\alpha_1. \tag{3.6}$$

By virtue of Lemma 3.1, after defining this solution, we may define the solution $\varphi_2(x,\lambda) := \varphi_{2\lambda}(x)$ of equation (1.8) on $[h_1,h_2]$ by means of the solution $\varphi_1(x,\lambda)$ by the initial conditions

$$u(h_1) = \delta_1^{-1}\varphi_1(h_1,\lambda),\ u'(h_1) = \delta_1^{-1}\varphi_1'(h_1,\lambda). \tag{3.7}$$

After defining this solution, we may define the solution $\varphi_3(x,\lambda) := \varphi_{3\lambda}(x)$ of equation (1.8) on $[h_2,h_3]$ by means of the solution $\varphi_2(x,\lambda)$ by the initial conditions

$$u(h_2)=\delta_2^{-1}\varphi_2(h_2,\lambda),\ u'(h_2)=\delta_2^{-1}\varphi_2'(h_2,\lambda). \quad \textbf{(3.8)}$$

Continuing in this manner, we may define the solution $\varphi_{(m+1)}(x,\lambda) := \varphi_{(m+1)\lambda}(x)$ of equation (1.8) on $[h_m,1]$ by means of the solution $\varphi_m(x,\lambda)$ by the initial conditions

$$u(h_m)=\delta_m^{-1}\varphi_m(h_m,\lambda),\ u'(h_m)=\delta_m^{-1}\varphi_m'(h_m,\lambda). \quad \textbf{(3.9)}$$

Therefore, $\varphi(x,\lambda)$ satisfies the equation (1.8) on $[-1,h_1)\cup(h_1,h_2)\cup...\cup(h_m,1]$, the boundary condition (1.9), and the transmission conditions (1.11)-(1.12).

Analogically, first we define the solution $\chi_{(m+1)\lambda}(x) := \chi_{(m+1)}(x,\lambda)$ on $[h_m,1]$ by the initial conditions

$$u(1)=\beta_2'\lambda+\beta_2,\ u'(1)=\beta_1'\lambda+\beta_1. \quad \textbf{(3.10)}$$

Again, after defining this solution, we may define the solution $\chi_{m\lambda}(x) := \chi_m(x,\lambda)$ of the equation (1.8) on $[h_{m-1},h_m]$ by the initial conditions

$$u(h_m)=\delta_m\chi_{m+1}(h_m,\lambda),\ u'(h_m)=\delta_m\chi_{m+1}'(h_m,\lambda). \quad \textbf{(3.11)}$$

Continuing in this manner, we may define the solution $\chi_{1\lambda}(x) := \chi_1(x,\lambda)$ of the equation (1.8) on $[-1,h_1]$ by the initial conditions

$$u(h_1)=\delta_1\chi_2(h_1,\lambda),\ u'(h_1)=\delta_1\chi_2'(h_1,\lambda). \quad \textbf{(3.12)}$$

Therefore, $\chi(x,\lambda)$ satisfies the equation (1.8) on $[-1,h_1)\cup(h_1,h_2)\cup...\cup(h_m,1]$, the boundary condition (1.10), and the transmission conditions (1.11)-(1.12). It is obvious that the Wronskians

$$\omega_i(\lambda) := W_\lambda(\varphi_i,\chi_i;x) := \varphi_i(x,\lambda)\chi_i'(x,\lambda) - \varphi_i'(x,\lambda)\chi_i(x,\lambda),\ \ x\in\Omega_i\ \ \left(i=\overline{1,m+1}\right).$$

are independent of $x\in\Omega_i$ and are entire functions of λ.

Lemma 3.4. For each $\lambda\in C$,

$$\omega_1(\lambda) = \delta_1^2\omega_2(\lambda) = \delta_1^2\delta_2^2\omega_3(\lambda) = \ldots = \left(\prod_{i=1}^m \delta_i^2\right)\omega_{m+1}(\lambda).$$

Proof. By using (3.7), (3.8), (3.9), (3.11) and (3.12), it is easy to show that

$$\begin{aligned} W_\lambda(\varphi_1,\chi_1;h_1) &= \delta_1^2 W_\lambda(\varphi_2,\chi_2;h_1) = \delta_1^2 W_\lambda(\varphi_3,\chi_3;h_2) = \delta_1^2\delta_2^2 W_\lambda(\varphi_3,\chi_3;h_2) \\ &= \delta_1^2\delta_2^2 W_\lambda(\varphi_4,\chi_4;h_3) = \ldots = \left(\prod_{i=1}^m \delta_i^2\right) W_\lambda(\varphi_{m+1},\chi_{m+1};h_m), \end{aligned}$$

so $\omega_1(\lambda) = \delta_1^2\omega_2(\lambda) = \delta_1^2\delta_2^2\omega_3(\lambda) = \ldots = \left(\prod_{i=1}^m \delta_i^2\right)\omega_{m+1}(\lambda).$

Now we may introduce the characteristic function of the considered problem as

$$\omega(\lambda) := \omega_1(\lambda) = \delta_1^2\omega_2(\lambda) = \delta_1^2\delta_2^2\omega_3(\lambda) = \ldots = \left(\prod_{i=1}^m \delta_i^2\right)\omega_{m+1}(\lambda).$$

Theorem 3.3. The eigenvalues of the problem (1.8)-(1.12) are the zeros of the function $\omega(\lambda)$.

Proof. Let $\omega(\lambda_0)=0$. Then $W_{\lambda_0}(\varphi_1,\chi_1;x)=0$ and therefore the functions $\varphi_{1\lambda_0}(x)$ and $\chi_{1\lambda_0}(x)$ are linearly dependent, i.e.

$$\chi_{1\lambda_0}(x) = k_1\varphi_{1\lambda_0}(x),\ \ x\in[-1,h_1]$$

for some $k_1 \neq 0$. From this, it follows that $\chi(x,\lambda_0)$ satisfies also the first boundary condition (1.9), so $\chi(x,\lambda_0)$ is an eigenfunction of the problem (1.8)-(1.12) corresponding to this eigenvalue λ_0.

Now we let $u_0(x)$ be any eigenfunction corresponding to eigenvalue λ_0, but $\omega(\lambda_0) \neq 0$. Then the functions $\varphi_1, \chi_1, \varphi_2, \chi_2, ..., \varphi_{m+1}, \chi_{m+1}$ would be linearly independent on $[-1,h_1]$, $[h_1,h_2]$ and $[h_m,1]$ respectively. Therefore $u_0(x)$ may be represented in the following form

$$u_0(x) = \begin{cases} c_1\varphi_1(x,\lambda_0) + c_2\chi_1(x,\lambda_0), & x \in [-1,h_1), \\ c_3\varphi_2(x,\lambda_0) + c_4\chi_2(x,\lambda_0), & x \in (h_1,h_2), \\ \vdots & \\ c_{2m+1}\varphi_{m+1}(x,\lambda_0) + c_{2m+2}\chi_{m+1}(x,\lambda_0), & x \in (h_m,1]. \end{cases}$$

where at least one of the constants $c_1, c_2, ..., c_{2m+2}$ is not zero. Considering the equations

$$\tau_\upsilon(u_0(x)) = 0, \quad \upsilon = \overline{1,2m+2} \qquad \textbf{(3.13)}$$

as the homogenous system of linear equations of the variables c_1, c_2, c_{2n+2} and taking (3.7), (3.8), (3.9), (3.11) and (3.12) into account, it follows that the determinant of this system is equal to

$$-\left(\prod_{i=1}^{m}\delta_i^2\omega_i(\lambda_0)\right)\omega_{m+1}^m(\lambda_0) \neq 0.$$

Therefore, the system (3.13) has only the trivial solution $c_i = 0\ \left(i = \overline{1,2m+2}\right)$. Thus we get a contradiction, which completes the proof.

Lemma 3.5. If $\lambda=\lambda_0$ is an eigenvalue, then $\varphi(x,\lambda_0)$ and $\chi(x,\lambda_0)$ are linearly dependent.

Proof. Let $\lambda=\lambda_0$ be an eigenvalue. Then by virtue of Theorem 3.3

$$W\left(\varphi_{i\lambda_0},\chi_{i\lambda_0};x\right)=\omega_i\left(\lambda_0\right)=0$$

and hence

$$\chi_{i\lambda_0}\left(x\right)=k_i\varphi_{i\lambda_0}\left(x\right)\quad\left(i=\overline{1,m+1}\right) \tag{3.14}$$

for some $k_1\neq 0,\ k_2\neq 0,...,k_{m+1}\neq 0.$ We must show that $k_1=k_2=...=k_{m+1}$. Suppose, if possible, that $k_m\neq k_{m+1}$.Taking into account the definitions of the solutions $\varphi_i\left(x,\lambda\right)$ and $\chi_i\left(x,\lambda\right)$ from the equalities (3.14), we have

$$\begin{aligned}\tau_{2m+1}\left(\chi_{\lambda_0}\right)&=\chi_{\lambda_0}\left(h_m-0\right)-\delta_m\chi_{\lambda_0}\left(h_m+0\right)=\chi_{m\lambda_0}\left(h_m\right)-\delta_m\chi_{(m+1)\lambda_0}\left(h_m\right)\\&=k_m\varphi_m\left(h_m\right)-\delta_m k_{m+1}\varphi_{m+1}\left(h_m\right)=k_m\delta_m\varphi_{m+1}\left(h_m\right)-\delta_m k_{m+1}\varphi_{m+1}\left(h_m\right)\\&=\delta_m\left(k_m-k_{m+1}\right)\varphi_{m+1}\left(h_m\right)=0.\end{aligned}$$

since $\tau_{2m+1}\left(\chi_{\lambda_0}\right)=0$ and $\delta_m\left(k_m-k_{m+1}\right)\neq 0,$ it follows that

$$\varphi_{(m+1)\lambda_0}\left(h_m\right)=0. \tag{3.15}$$

By the same procedure from $\tau_{2n+2}\left(\chi_{\lambda_0}\right)=0$ we can derive that

$$\varphi'_{(m+1)\lambda_0}\left(h_m\right)=0. \tag{3.16}$$

From the fact that $\varphi_{(m+1)\lambda_0}\left(x\right)$ is a solution of the differential equation (1.8) on $\left[h_m,1\right]$ and satisfies the initial conditions (3.15) and (3.16), it follows that $\varphi_{(m+1)\lambda_0}\left(x\right)=0$ identically on $\left[h_m,1\right]$ because of the well-known existence and uniqueness theorem for the initial value problems of the ordinary linear differential equations. Making use of (3.9), (3.14) and (3.15), we may also derive that

$$\varphi_{m\lambda_0}\left(h_m\right)=\varphi'_{m\lambda_0}\left(h_m\right)=0. \qquad \textbf{(3.17)}$$

Continuing in this matter, we may also find that

$$\begin{cases} \varphi_{(m-1)\lambda_0}\left(h_{m-1}\right)=\varphi'_{(m-1)\lambda_0}\left(h_{m-1}\right)=0. \\ \qquad\vdots \\ \varphi_{1\lambda_0}\left(h_1\right)=\varphi'_{1\lambda_0}\left(h_1\right)=0. \end{cases} \qquad \textbf{(3.18)}$$

identically on $\left[h_{m-1},h_m\right],...,\left[-1,h_1\right]$ respectively. Hence $\varphi\left(x,\lambda_0\right)=0$ identically on $\left[-1,h_1\right)\cup\left(h_1,h_2\right)\cup...\cup\left(h_m,1\right]$. But this contradicts with (3.6). Hence $k_m=k_{m+1}$. Analogically we can prove that $k_{m-1}=k_m,...,$ $k_2=k_3$ and $k_1=k_2$.

Corollary 3.3. If $\lambda=\lambda_0$ is an eigenvalue, then both $\varphi\left(x,\lambda_0\right)$ and $\chi\left(x,\lambda_0\right)$ are eigenfunctions corresponding to this eigenvalue.

Lemma 3.6. All eigenvalues λ_n are simple zeros of $\omega(\lambda)$.

Proof. Using the Lagrange's formula (cf. [44]), it can be shown that

$$\left(\lambda-\lambda_n\right)\left[\sum_{j=0}^{m}\left(\prod_{i=0}^{j}\delta_i^2\right)\int_{h_j}^{h_{j+1}}\varphi_\lambda(x)\varphi_{\lambda_n}\left(x\right)dx\right]=\left(\prod_{i=1}^{m}\delta_i^2\right)W\left(\varphi_\lambda,\varphi_{\lambda_n};1\right) \qquad \textbf{(3.19)}$$

for any $\lambda.$ Recall that

$$\chi_{\lambda_n}\left(x\right)=k_n\varphi_{\lambda_n}\left(x\right),\quad x\in\left[-1,h_1\right)\cup\left(h_1,h_2\right)\cup...\cup\left(h_m,1\right].$$

for some $k_n\neq 0,$ $n=1,2,...$. Using this equality for the right side of (3.19), we have

$$W\left(\varphi_\lambda,\varphi_{\lambda_n};1\right)=\frac{1}{k_n}W\left(\varphi_\lambda,\chi_{\lambda_n};1\right)=\frac{1}{k_n}\left(\lambda_n R_1^{'}\left(\varphi_\lambda\right)+R_1\left(\varphi_\lambda\right)\right)$$
$$=\frac{1}{k_n}\left[\omega(\lambda)-(\lambda-\lambda_n)R_1^{'}\left(\varphi_\lambda\right)\right]$$
$$=(\lambda-\lambda_n)\frac{1}{k_n}\left[\frac{\omega(\lambda)}{\lambda-\lambda_n}-R_1^{'}\left(\varphi_\lambda\right)\right].$$

Substituting this formula in (3.19) and letting $\lambda\to\lambda_n$, we get

$$\sum_{j=0}^{m}\left(\prod_{i=0}^{j}\delta_i^2\right)\int_{h_j}^{h_{j+1}}\varphi_{\lambda_n}^2\left(x\right)dx=\frac{\prod_{i=1}^{m}\delta_i^2}{k_n}\left(\omega^{'}\left(\lambda_n\right)-R_1^{'}\left(\varphi_{\lambda_n}\right)\right). \qquad \textbf{(3.20)}$$

Now putting

$$R_1^{'}\left(\varphi_{\lambda_n}\right)=\frac{1}{k_n}R_1^{'}\left(\chi_{\lambda_n}\right)=\frac{\rho}{k_n}$$

in (3.20) we get $\omega'\left(\lambda_n\right)\neq 0$.

Definition 3.1. The geometric multiplicity of an eigenvalue λ of the problem (1.8)-(1.12) is the dimension of its eigenspace, i.e. the number of its linearly independent eigenfunctions.

Theorem 3.4. All eigenvalues of the problem (1.8)-(1.12) are geometrically simple.

Proof. If f and g are two eigenfunctions for an eigenvalue λ_0 of (1.8)-(1.12) then (1.9) implies that $f(-1)=cg(-1)$ and $f'(-1)=cg'(-1)$ for some constant $c\in \mathbb{C}$. By the uniqueness theorem for solutions of ordinary differential equation and the transmission conditions (1.11)-(1.12), we have that $f=cg$ on $\left[-1,h_1\right]$, $\left[h_1,h_2\right]$ and $\left[h_m,1\right]$. Thus the geometric multiplicity of λ_0 is one.

3.2 Asymptotic Approximate Formulas of $\omega(\lambda)$ for Four Distinct Cases

We start by proving some lemmas.

Lemma 3.7. Let $\varphi(x,\lambda)$ be the solutions of equation (1.8) defined in Section 3.1, and let $\lambda = s^2$. Then the following integral equations hold for $k=0,1$:

$$\varphi_{1\lambda}^{(k)}(x) = \alpha_2 \left(\cos s(x+1)\right)^{(k)} - \alpha_1 \tfrac{1}{s}\left(\sin s(x+1)\right)^{(k)}$$
$$+\tfrac{1}{s}\int_{-1}^{x}\left(\sin s(x-y)\right)^{(k)} q(y)\varphi_{1\lambda}(y)\,dy,$$

$$\varphi_{(i+1)\lambda}^{(k)}(x) = \tfrac{1}{\delta_i}\varphi_{i\lambda}(h_i)\left(\cos s(x-h_i)\right)^{(k)} + \tfrac{1}{s}\tfrac{1}{\delta_i}\varphi_{i\lambda}'(h_i)\left(\sin s(x-h_i)\right)^{(k)}$$
$$+\tfrac{1}{s}\int_{h_i}^{x}\left(\sin s(x-y)\right)^{(k)} q(y)\varphi_{(i+1)\lambda}(y)\,dy,\ i=\overline{1,m}, \tag{3.21}$$

where $(\cdot)^{(k)} = \frac{d^k}{dx^k}(\cdot)$.

Proof. It is enough to substitute $s^2\varphi_{1\lambda}(y)+\varphi_{1\lambda}''(y)$, $s^2\varphi_{2\lambda}(y)+\varphi_{2\lambda}''(y),\ldots,$ $s^2\varphi_{(m+1)\lambda}(y)+\varphi_{(m+1)\lambda}''(y)$ instead of $q(y)\varphi_{1\lambda}(y)$, $q(y)\varphi_{2\lambda}(y)$, $q(y)\varphi_{(m+1)\lambda}(y)$ in the integral terms of the (3.21), respectively, and integrate by parts twice.

Lemma 3.8. Let $\lambda = s^2$, $\operatorname{Im} s = t$. Then the functions $\varphi_{i\lambda}(x)$ have the following asymptotic formulas for $|\lambda|\to\infty$, which hold uniformly for $x\in\Omega_i$ $\left(\textit{for } i=\overline{1,m+1} \text{ and } k=0,1.\right)$:

$$\varphi_{j\lambda}^{(k)}(x) = \frac{\alpha_2}{\prod_{i=0}^{j-1}\delta_i}\left(\cos s(x+1)\right)^{(k)} + O\left(|s|^{k-1}e^{|t|(x+1)}\right) \tag{3.22}$$

if $\alpha_2 \neq 0$,

$$\varphi_{j\lambda}^{(k)}(x) = -\frac{\alpha_1}{s\prod_{i=0}^{j-1}\delta_i}\left(\sin s(x+1)\right)^{(k)} + O\left(|s|^{k-2} e^{|t|(x+1)}\right) \qquad \textbf{(3.23)}$$

if $\alpha_2 = 0$.

Proof. Since the proof of the formulas for $\varphi_{1\lambda}(x)$ is identical to Titchmarsh's proof to similar results for $\varphi_{\lambda}(x)$ (see [55], Lemma 1.7 p. 9-10), we may formulate them without proving them here.

Since the proof of the formulas for $\varphi_{2\lambda}(x)$ and $\varphi_{3\lambda}(x)$ are identical to Kadakal's and Mukhtarov's proof to similar results for $\varphi_{\lambda}(x)$ (see [22], Lemma 3.2 p. 1373-1375), we may formulate them without proving them here. But the similar formulas for $\varphi_{4\lambda}(x),...,\varphi_{(m+1)\lambda}(x)$ need individual consideration, since the last solutions are defined by the initial conditions of these special nonstandart forms. We shall only prove the formula (3.22) for $k=0$ and $m=3$.

Let $\alpha_2 \neq 0$. Then according to (3.22) for $m=2$

$$\varphi_{3\lambda}(h_3) = \frac{\alpha_2 \cos s(h_3+1)}{\delta_1\delta_2} + O\left(|s|^{-1} e^{|t|(h_3+1)}\right)$$

and

$$\varphi_{3\lambda}'(h_3) = -\frac{\alpha_2 s \sin(h_3+1)}{\delta_1\delta_2} + O\left(e^{|t|(h_3+1)}\right).$$

Substituting these asymptotic expressions into (3.21), we get

$$\varphi_{4\lambda}(x) = \frac{\alpha_2 \cos s(x+1)}{\delta_1\delta_2\delta_3} + \frac{1}{s}\int_{h_3}^{x} \sin s(x-y)q(y)\varphi_{4\lambda}(y)dy + O\left(|s|^{-1} e^{|t|(x+1)}\right). \qquad \textbf{(3.24)}$$

Multiplying through by $e^{-|t|(x+1)}$, and denoting

$$F_{4\lambda}(x) := e^{-|t|(x+1)}\varphi_{4\lambda}(x)$$

we have

$$F_{4\lambda}(x) := \frac{\alpha_2 \cos s(x+1)}{\delta_1\delta_2\delta_3} e^{-|t|(x+1)} + \frac{1}{s}\int_{h_3}^{x} \sin s(x-y) q(y) e^{-|t|(x+1)} F_{4\lambda}(y)\,dy + O\left(|s|^{-1}\right).$$

Denoting $M := \max_{x\in[h_3,1]} |F_{4\lambda}(x)|$ from the last formula, it follows that

$$M(\lambda) \le \frac{8|\alpha_2|}{|\delta_1\delta_2\delta_3|} + \frac{M(\lambda)}{|s|}\int_{h_3}^{1} q(y)\,dy + \frac{M_0}{|s|}$$

for some $M_0 > 0$. From this, it follows that $M(\lambda) = O(1)$ as $\lambda \to \infty$, so

$$\varphi_{4\lambda}(x) = O\left(e^{|t|\left[(x-h_3)+(h_3-h_2)+(h_2-h_1)+(h_1+1)\right]}\right).$$

Substituting this back into the integral on the right side of (3.24) yields (3.22) for $k=0$ and $m=3$. The other cases may be considered analogically.

Theorem 3.5. Let $\lambda = s^2$, $t = \operatorname{Im} s$. Then the characteristic function $\omega(\lambda)$ has the following asymptotic formulas :

Case 1 : If $\beta_2' \ne 0$, $\alpha_2 \ne 0$, then

$$\omega(\lambda) = \beta_2'\alpha_2 s^3\left(\prod_{i=1}^{m}\delta_i\right)\sin 2s + O\left(|s|^2 e^{2|t|}\right). \qquad \textbf{(3.25)}$$

Case 2 : If $\beta_2' \ne 0$, $\alpha_2 = 0$, then

$$\omega(\lambda) = \beta_2'\alpha_1 s^2\left(\prod_{i=1}^{m}\delta_i\right)\cos 2s + O\left(|s|^2 e^{2|t|}\right). \qquad \textbf{(3.26)}$$

Case 3： If $\beta_2' = 0$, $\alpha_2 \neq 0$, then

$$\omega(\lambda) = \beta_1' \alpha_2 s^2 \left(\prod_{i=1}^{m} \delta_i \right) \cos 2s + O\left(|s|^2 e^{2|t|} \right). \tag{3.27}$$

Case 4： If $\beta_2' = 0$, $\alpha_2 = 0$, then

$$\omega(\lambda) = -\beta_1' \alpha_1 s \left(\prod_{i=1}^{m} \delta_i \right) \sin 2s + O\left(|s|^2 e^{2|t|} \right). \tag{3.28}$$

Proof. The proof is completed by substituting (3.22) and (3.23) into the representation

$$\begin{aligned}\omega(\lambda) &= \left(\prod_{i=1}^{m} \delta_i^2 \right) \omega_{m+1}(\lambda) = \left(\prod_{i=1}^{m} \delta_i^2 \right) \left[\varphi_{(m+1)\lambda}(1) \chi_{(m+1)\lambda}'(1) - \varphi_{(m+1)\lambda}'(1) \chi_{(m+1)\lambda}(1) \right] = \\ &= \left(\prod_{i=1}^{m} \delta_i^2 \right) \left[\left(\lambda\beta_1' + \beta_1 \right) \varphi_{(m+1)\lambda}(1) - \left(\lambda\beta_2' + \beta_2 \right) \varphi_{(m+1)\lambda}'(1) \right].\end{aligned} \tag{3.29}$$

Corollary 3.4. The eigenvalues of the problem (1.8)-(1.12) are bounded below.

Proof. Putting $s = it \quad (t > 0)$ in the above formulas, it follows that $\omega(-t^2) \to \infty$ as $t \to \infty$. Therefore, $\omega(\lambda) \neq 0$ for λ negative and sufficiently large.

3.3. Asymptotic Formulas for Eigenvalues and Eigenfunctions

Now we can obtain the asymptotic approximation formulas for the eigenvalues of the considered problem (1.8)-(1.12).

Since the eigenvalues coincide with the zeros of the entire function $\omega_{m+1}(\lambda)$, it follows that they have no finite limit. Moreover, we know from Corollaries 3.1 and

3.4 that all eigenvalues are real and bounded below. Hence, we may renumber them as $\lambda_0 \le \lambda_1 \le \lambda_2 \le \dots$, listed according to their multiplicity.

Theorem 3.6. The eigenvalues $\lambda_n = s_n^2$, $n = 0,1,2,\dots$ of the problem (1.8)-(1.12) have the following asymptotic formulas for $n \to \infty$:

Case 1： If $\beta_2' \ne 0, \alpha_2 \ne 0,$ then

$$s_n = \frac{\pi(n-1)}{2} + O\left(\frac{1}{n}\right). \tag{3.30}$$

Case 2： If $\beta_2' \ne 0, \alpha_2 = 0,$ then

$$s_n = \frac{\pi\left(n-\frac{1}{2}\right)}{2} + O\left(\frac{1}{n}\right). \tag{3.31}$$

Case 3： If $\beta_2' = 0, \alpha_2 \ne 0,$ then

$$s_n = \frac{\pi\left(n-\frac{1}{2}\right)}{2} + O\left(\frac{1}{n}\right). \tag{3.32}$$

Case 4： If $\beta_2' = 0, \alpha_2 = 0,$ then

$$s_n = \frac{\pi n}{2} + O\left(\frac{1}{n}\right). \tag{3.33}$$

Proof. We shall only consider the first case. The other cases may be considered similarly. Denoting $\omega_1(s)$ and $\omega_2(s)$ the first and O-term of the right of (3.25) repectively, we shall apply the well-known Rouché's theorem, which asserts that if $f(s)$ and $g(s)$ are analytic inside and on a closed contour C, and $|g(s)| < |f(s)|$ on C, then $f(s)$ and $f(s)+g(s)$ have the same number zeros inside C, provided that each zero is counted according to their multiplicity. It is readily shown that $\left|\overline{\omega}_1(s)\right| > \left|\overline{\omega}_2(s)\right|$ on the contours

$$C_n := \left\{ s \in \mathrm{C} \left\| s \right| = \frac{\left(n+\frac{1}{2}\right)\pi}{2} \right\}$$

for sufficiently large n.

Let $\lambda_0 \le \lambda_1 \le \lambda_2 \le \ldots$ be zeros of $\omega(\lambda)$ and $\lambda_n = s_n^2$. Since inside the contour C_n, $\overline{\omega}_1(s)$ has zeros at points $s = 0$ and $s = \frac{k\pi}{2}$, $k = \pm 1, \pm 2, \ldots, \pm n$.

$$s_n = \frac{(n-1)\pi}{2} + \delta_n \qquad \textbf{(3.34)}$$

where $\delta_n = O(1)$ for sufficiently large n. By substituting this in (3.25), we derive that $\delta_n = O\left(\frac{1}{n}\right)$, which completes the proof.

The next approximation for the eigenvalues may be obtained by the following procedure. For this, we shall suppose that $q(y)$ is of bounded variation in $[-1,1]$. Firstly we consider the case $\beta_2' \ne 0$ and $\alpha_2 \ne 0$. Putting $x = h_1, x = h_2, \ldots, x = h_m$ in (3.21) and then substituting in the expression of $\varphi'_{(m+1)\lambda}$, we get that

$$\varphi'_{(m+1)\lambda}(1) = s\frac{\alpha_2}{\prod_{i=1}^{m}\delta_i}\sin 2s - \frac{\alpha_1}{\prod_{i=1}^{m}\delta_i}\cos 2s + \sum_{j=0}^{m}\frac{1}{\prod_{i=j+1}^{m+1}\delta_i}\int_{h_j}^{h_{j+1}}\cos\left(s(1-y)\right)q(y)\varphi_{(j+1)\lambda}(y)\,dy$$

where $\delta_{m+1} = 1$.

Substituting (3.22) into the right side of the last integral equality then gives

$$\varphi'_{(m+1)\lambda}(1)=\frac{s\alpha_2}{\prod_{i=1}^{m}\delta_i}\sin 2s-\frac{\alpha_1}{\prod_{i=1}^{m}\delta_i}\cos 2s$$
$$+\frac{\alpha_2}{\prod_{i=1}^{m}\delta_i}\sum_{j=0}^{m}\int_{h_j}^{h_{j+1}}\cos\left(s(1-y)\right)\cos\left(s(1+y)\right)q(y)\,dy+O\left(|s|^{-1}e^{2|t|}\right).$$

On the other hand, from (3.22), it follows that

$$\varphi_{(m+1)\lambda}(1)=\frac{\alpha_2}{\prod_{i=1}^{m}\delta_i}\cos 2s+O\left(|s|^{-1}e^{2|t|}\right).$$

Putting these formulas into (3.29), we have

$$\omega(\lambda)=\frac{s^3\beta_2'\alpha_2}{\prod_{i=1}^{m}\delta_i}\sin 2s+s^2\left[\left(\frac{\beta_1'\alpha_2+\beta_2'\alpha_1}{\prod_{i=1}^{m}\delta_i}\right)\cos 2s\right.$$
$$\left.-\sum_{j=0}^{m}\frac{\beta_2'}{\prod_{i=j+1}^{m+1}\delta_i}\int_{h_j}^{h_{j+1}}\cos\left(s(1-y)\right)q(y)\,\varphi_{(j+1)\lambda}(y)\,dy\right]+O\left(|s|^{-1}e^{2|t|}\right).$$

Putting (3.34) in the last equality we find that

$$\sin(2\delta_n)=-\frac{\cos(2\delta_n)}{s_n}\left[\frac{\beta_1'}{\beta_2'}+\frac{\alpha_1}{\alpha_2}-\frac{1}{2\prod_{i=1}^{m}\delta_i}\int_{-1}^{1}q(y)\,dy-\frac{1}{2\prod_{i=1}^{m}\delta_i}\int_{-1}^{1}\cos(2s_ny)\,q(y)\,dy\right]$$
$$+O\left(|s_n|^{-2}\right)$$

(3.35)

Recalling that $q(y)$ is of bounded variation in $[-1,1]$, and applying the well-known Riemann-Lebesque Lemma (see [70], p. 48, Theorem 4.12) to the second integral on the right in (3.35), this term is $O\left(\frac{1}{n}\right)$. As a result, from (3.34) it follows that

$$\delta_n = -\frac{1}{\pi(n-1)}\left[\frac{\beta_1'}{\beta_2'}+\frac{\alpha_1}{\alpha_2}-\frac{1}{2\prod_{i=1}^{m}\delta_i}\int_{-1}^{1}q(y)\,dy\right]+O\left(\frac{1}{n^2}\right).$$

Substituting in (3.30), we have

$$s_n = \frac{\pi(n-1)}{2}-\frac{1}{\pi(n-1)}\left[\frac{\beta_1'}{\beta_2'}+\frac{\alpha_1}{\alpha_2}-\frac{1}{2\prod_{i=1}^{m}\delta_i}\int_{-1}^{1}q(y)\,dy\right]+O\left(\frac{1}{n^2}\right).$$

Similar formulas in the other cases are as follows:

In case 2:

$$s_n = \frac{\pi\left(n-\frac{1}{2}\right)}{2}-\frac{1}{\pi\left(n-\frac{1}{2}\right)}\left[\frac{\beta_1'}{\beta_2'}+\frac{1}{2\prod_{i=1}^{m}\delta_i}\int_{-1}^{1}q(y)\,dy\right]+O\left(\frac{1}{n^2}\right).$$

In case 3:

$$s_n = \frac{\pi\left(n-\frac{1}{2}\right)}{2}+\frac{1}{\pi\left(n-\frac{1}{2}\right)}\left[\frac{\beta_2}{\beta_1'}-\frac{\alpha_1}{\alpha_2}+\frac{1}{2\prod_{i=1}^{m}\delta_i}\int_{-1}^{1}q(y)\,dy\right]+O\left(\frac{1}{n^2}\right).$$

In case 4:

$$s_n = \frac{\pi n}{2} + \frac{1}{\pi n}\left[\frac{\beta_2}{\beta_1'} + \frac{1}{2\prod_{i=1}^{m}\delta_i}\int_{-1}^{1} q(y)\,dy\right] + O\left(\frac{1}{n^2}\right).$$

Recalling that $\varphi(x,\lambda_n)$ is an eigenfunction according to the eigenvalue λ_n and by putting (3.30) into the (3.22) we obtain that

$$\varphi_{j\lambda_n}(x) = \frac{\alpha_2}{\prod_{i=0}^{j-1}\delta_i}\cos\left(\frac{\pi(n-1)(x+1)}{2}\right) + O\left(\frac{1}{n}\right),\ j = \overline{1, m+1}$$

in the first case which holds uniformly for $x \in [-1, h_1) \cup (h_1, h_2) \cup \ldots \cup (h_m, 1]$.

Similar formulas in the other cases are as follows:

In case 2

$$\varphi_{j\lambda_n}(x) = -\frac{2\alpha_1}{\pi\left(n-\frac{1}{2}\right)\prod_{i=0}^{j-1}\delta_i}\sin\left(\frac{\pi\left(n-\frac{1}{2}\right)(x+1)}{2}\right) + O\left(\frac{1}{n^2}\right),\ j = \overline{1, m+1}.$$

In case 3

$$\varphi_{j\lambda_n}(x) = \frac{\alpha_2}{\prod_{i=0}^{j-1}\delta_i}\cos\left(\frac{\pi\left(n-\frac{1}{2}\right)(x+1)}{2}\right) + O\left(\frac{1}{n}\right),\ j = \overline{1, m+1}$$

In case 4

$$\varphi_{j\lambda_n}(x) = -\frac{2\alpha_1}{\pi n \prod_{i=0}^{j-1} \delta_i} \sin\left(\frac{\pi n(x+1)}{2}\right) + O\left(\frac{1}{n^2}\right), \ j = \overline{1, m+1}.$$

All these asymptotic formulas hold uniformly for $x \in [-1, h_1) \cup (h_1, h_2) \cup ... \cup (h_m, 1]$.

3.4. Completeness of Eigenfunctions

Let A be the operator as defined in Section 3.1.

Theorem 3.7. The spectrum of A consist only of eigenvalues, i.e., $\sigma(A) = \sigma_\rho(A)$.

Proof. Let η is not an eigenvalue. Consider the operator equation $(A - \eta I)U = F$ for arbitrary $F = \begin{pmatrix} f(x) \\ f_1 \end{pmatrix} \in H$. This equation is equivalent to the inhomogeneous differential equation

$$-u'' + q(x)u = f(x), \ x \in [-1, h_1) \cup (h_1, h_2) \cup ... \cup (h_m, 1] \qquad \textbf{(3.36)}$$

subject to inhomogeneous boundary conditions

$$\tau_1 u = 0, \ \tau_2 u = f_1 \qquad \textbf{(3.37)}$$

and transmission conditions

$$\tau_{2i+1} u = \tau_{2i+2} u = 0, \ i = \overline{1, m} \qquad \textbf{(3.38)}$$

where

$$U = \begin{pmatrix} u(x) \\ -R_1'(u) \end{pmatrix} \in D(A).$$

Making use of the definitions of the functions $\varphi_{i\lambda}(x)$ and $\chi_{i\lambda}(x)$ $\left(i=\overline{1,m}\right)$ we find that the general solution of the equation (3.36) has the following representation:

$$u(x)=\begin{cases} \frac{\chi_{i\lambda}(x)}{\omega_i(\eta)}\int\limits_{h_{i-1}}^{x}\varphi_{i\eta}(y)\,f(y)\,dy+\frac{\varphi_{i\lambda}(x)}{\omega_i(\eta)}\int\limits_{x}^{h_i}\chi_{i\eta}(y)\,f(y)\,dy \\ \quad +C_{1i}\varphi_{i\eta}(x)+C_{2i}\chi_{i\eta}(x),\ \text{for } x\in(h_{i-1},h_i), \\ \qquad i=1,2,...,m \end{cases} \tag{3.39}$$

where C_{1i}, C_{2i} are arbitrary constants. Substituting (3.39) in (3.37)-(3.38) we see that the unknown constants C_{ji} $(j=1,2;\ i=1,2,...,m)$ are uniquely solvable, i.e. $U=\begin{pmatrix} u(x) \\ -R_1'(u) \end{pmatrix}$ is uniquely solvable. Therefore the resolvent operator $R(\eta,A)=(A-\eta I)^{-1}$ is defined on whole H. Moreover, by virtue of Theorem 3.2 and well-known Closed Graph Theorem we get that $R(\eta,A)$ is bounded, i.e. η is a regular value of A. The proof is complete.

Theorem 3.8. The resolvent operator $R(\eta,A)$ is compact in the Hilbert space H.

Proof. Let $\lambda_1\le\lambda_2\le...$ are eigenvalues of A and $P_1,P_2,...$ are orthogonal projections onto corresponding eigen-spaces, respectively. Since A is self-adjoint operator with discrete spectrum we can write the spectral resolution of the resolvent operator $R(\eta,A)$ by

$$R(\eta,A)=\sum_{n=1}^{\infty}\frac{1}{\lambda_n-\delta}P_n. \tag{3.40}$$

By virtue of Theorem 3.6 we have $\frac{1}{\lambda_n-\delta}=O\left(\frac{1}{n^2}\right)$ for $n\to\infty$. Therefore the series (3.40) is strongly convergent. It is obvious that the orthogonal projections P_n, $n=1,2,...$ are compact operators, since each of which are of finite rank.

Consequently the sum of the series (3.40) is also compact in H. The proof is complete.

Now we are ready to formulate the following properties by using the above results, the well-known spectral theorems for self-adjoint operators with discrete spectrum and the same techniques as used in [16].

Let $U_n = \begin{pmatrix} u_n(x) \\ -R_1'(u) \end{pmatrix}$ be a maximal set of orthogonal eigenelements of A.

Theorem 3.9 (Parseval's equality). For $U \in H$

$$\|U\|^2 = \sum_{n=1}^{\infty} \left|\langle U, U_n \rangle\right|^2 .$$

Theorem 3.10 (Expansion in terms of eigenelements). For $D(A) \subset H$

$$U = \sum_{n=1}^{\infty} \langle U, U_n \rangle U_n$$

with the series being absolutely and uniformly convergent in the first component and absolutely convergent in the second component.

Denote by $\overset{m}{\underset{j=0}{\oplus}} L_2\left(h_j, h_{j+1}\right)$ the direct sum of Hilbert spaces $L_2\left(-1, h_1\right)$, $L_2\left(h_1, h_2\right), \dots, L_2\left(h_m, 1\right)$.

Corollary 3.5 (Expansion in terms of eigenfunctions). The eigenfunctions $u_n(x)$, $n = 1, 2, \dots$ of the problem (1.8)-(1.12) are complete in $\overset{m}{\underset{j=0}{\oplus}} L_2\left(h_j, h_{j+1}\right)$, i.e. for every $f \in \overset{m}{\underset{j=0}{\oplus}} L_2\left(h_j, h_{j+1}\right)$,

$$f(x)=\sum_{n=1}^{\infty}\left(\sum_{j=0}^{m}\left(\prod_{i=0}^{j}\delta_i^2\right)\int_{h_j}^{h_{j+1}} f(y)u_n(y)dy\right)u_n(x)$$

in the sense of strong convergence in $\overset{m}{\underset{j=0}{\oplus}} L_2\left(h_j, h_{j+1}\right)$.

4. THE REGULARIZED TRACE FORMULA FOR A DIFFERENTIAL OPERATOR WITH UNBOUNDED OPERATOR COEFFICIENT

The results of this chapter are gathered in an article written by "E. Şen, A. Bayramov and K. Oruçoğlu" and accepted for publication in the journal "Advanced Studies in Contemporary Mathematics".

4.1. The regularized trace of L

Let R_λ^0 and R_λ be the resolvents of the operators L_0 and $L,$ respectively. From (1.14) we get that the series $\sum_{k=1}^{\infty}\frac{1}{|\lambda_k-\lambda|}$ and $\sum_{k=1}^{\infty}\frac{1}{|\mu_k-\lambda|}$ are convergent for $\lambda \neq \lambda_k,\ \mu_k$ $(k=1,2,...).$ In this case R_λ^0 and R_λ are nuclear operators and

$$tr\left(R_\lambda - R_\lambda^0\right) = \sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k-\lambda} - \frac{1}{\mu_k-\lambda}\right). \tag{4.1}$$

Let $|\lambda| = b_p = 2^{-1}\left(\mu_{n_p+1} + \mu_{n_p}\right).$ It is easy to see that for large value of p the inequalities $\mu_{n_p} < b_p < \mu_{n_p+1}$ and $\lambda_{n_p} < b_p < \lambda_{n_p+1}$ are satisfied. The series $\sum_{k=1}^{\infty}\frac{1}{\lambda_k-\lambda}$ and $\sum_{k=1}^{\infty}\frac{1}{\mu_k-\lambda}$ are uniform convergent on the circle $|\lambda| = b_p.$ Therefore from (1.15) and (4.1), we get

$$\sum_{k=1}^{n_p}\left(\lambda_k - \mu_k\right) = -\frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda tr\left(R_\lambda - R_\lambda^0\right)d\lambda. \tag{4.2}$$

On the other hand, from the formula $R_\lambda = R_\lambda^0 - R_\lambda Q R_\lambda^0,$ the equality

$$R_\lambda - R_\lambda^0 = \sum_{j=1}^{N} (-1)^j R_\lambda^0 \left(QR_\lambda^0\right)^j + (-1)^{N+1} R_\lambda \left(QR_\lambda^0\right)^{N+1} \qquad \textbf{(4.3)}$$

is obtained for every natural number N. From (4.2) and (4.3) we can get

$$\sum_{k=1}^{n_p} (\lambda_k - \mu_k) = \sum_{j=1}^{p} M_p^j + M_{pN}, \qquad \textbf{(4.4)}$$

Here

$$M_p^j = \frac{(-1)^j}{2\pi ij} \int_{|\lambda|=b_p} tr\left[\left(QR_\lambda^0\right)^j\right] d\lambda, \;\; j = 1, 2, ... \qquad \textbf{(4.5)}$$

and

$$M_{pN} = \frac{(-1)^N}{2\pi i} \int_{|\lambda|=b_p} \lambda tr\left[R_\lambda \left(QR_\lambda^0\right)^{N+1}\right] d\lambda. \qquad \textbf{(4.6)}$$

Let $\left\{\psi_q(x)\right\}_1^\infty$ be the orthonormal eigenfunctions corresponding to the eigenvalues $\left\{\mu_q\right\}_1^\infty$ respectively. Since the orthonormal eigenfunctions according to the eigenvalues $\nu_k + \gamma_j$ $\left(k = 1, 2, ...; j = 1, 2, ...\right)$ of the operator L_0 are $\alpha_k \sin\sqrt{\nu_k}\phi_j$ $\left(k = 1, 2, ...; j = 1, 2, ...\right)$ respectively then

$$\psi_q(x) = \alpha_{k_q} \sin\sqrt{\nu_{k_q}}\phi_{j_q}, \;\; q = 1, 2, ... \qquad \textbf{(4.7)}$$

here $\nu_1 < \nu_2 < ... < \nu_k < ...$ are positive roots of the equation $\sqrt{\nu}\cos\sqrt{\nu} + b\sin\sqrt{\nu} = 0$ and

$$\alpha_k = \frac{\sqrt{2}}{\sqrt{1 + b^{-1}\cos^2\sqrt{\nu_k}}}.$$

Since QR_λ^0 is a nuclear operator for every $\lambda \in \rho(L_0)$ and $\{\psi_q(x)\}_1^\infty$ is an orthonormal basis of the space H_1 then from (4.5) and (4.7) we get

$$\begin{aligned} M_p^1 &= -\frac{1}{2\pi i}\int_{|\lambda|=b_p} tr\left(QR_\lambda^0\right) d\lambda = \sum_{q=1}^{n_p}\int_0^1 \left(Q(x)\psi_q(x), \psi_q(x)\right)_H dx \\ &= \sum_{q=1}^{n_p} \alpha_{k_q}^2 \int_0^1 \sin^2 \sqrt{\nu_{k_q}}\, x. \left(Q(x)\phi_{j_q}, \phi_{j_q}\right)_H dx. \end{aligned} \qquad \textbf{(4.8)}$$

If the operator function $Q(x)$ satisfies the conditions (1) and (2), the multiple series

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\int_0^1 \left(Q(x)\phi_{j_q}, \phi_{j_q}\right)_H \alpha_{k_q}^2 \sin^2 \sqrt{\nu_{k_q}}\, xdx$$

is absolutely convergent. Therefore from (4.8) we get

$$\lim_{p\to\infty} M_p^1 = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\int_0^1 \left(Q(x)\phi_j, \phi_j\right)_H \alpha_k^2 \sin^2 \sqrt{\nu_k}\, xdx. \qquad \textbf{(4.9)}$$

Let

$$T_p(x) = \sum_{k=1}^{p} \alpha_k^2 \sin^2 \sqrt{\nu_k}\, xdx. \qquad \textbf{(4.10)}$$

The following equality is proved in [25]:

$$T_p(x) = 2\sum_{k=1}^{p} \sin^2\left(k - \frac{1}{2}\right)\pi xdx + T_p^1(x), \quad x \in [0,1] \qquad \textbf{(4.11)}$$

where for large values of $p,$ the function $T_p^1(x)$ satisfies the equalities

$$\left|T_p^1(x)\right| < const.p^{\varepsilon-1}, \quad x \in \left[0, 1 - p^{-\varepsilon}\right), \qquad \textbf{(4.12)}$$

$$\left|T_p^1(x)\right| < const.p^{\varepsilon-1}, \quad x \in \left[1-p^{-\varepsilon}, 1\right], \tag{4.13}$$

and ε is a constant number belonging to the interval $\left(\frac{1}{2}, 1\right)$.

From (4.12) and (4.13) we have

$$\begin{aligned}
&\lim_{p\to\infty}\left|\int_0^1 trQ(x)T_p^1(x)\,dx\right| \\
&= \lim_{p\to\infty}\left|\int_0^{1-p^{-\varepsilon}} trQ(x)T_p^1(x)\,dx + \int_{1-p^{-\varepsilon}}^1 trQ(x)T_p^1(x)\,dx\right| \\
&\leq \lim_{p\to\infty}\left|\int_0^{1-p^{-\varepsilon}} p^{\varepsilon-1}dx + \int_{1-p^{-\varepsilon}}^1 p^{1-\varepsilon}dx\right| = 0.
\end{aligned} \tag{4.14}$$

Theorem 4.1. If the operator function $Q(x)$ satisfies the conditions (1), (2) and $\gamma_j \sim aj^{\alpha}$ $\left(a > 0, \alpha > \frac{2+2\sqrt{2}}{\sqrt{2}-1}\right)$ as $j \to \infty$ then

$$\lim_{p\to\infty} M_p^j = 0, \quad j \geq 2, \tag{4.15}$$

$$\lim_{p\to\infty} M_{pN} = 0, \quad N > 3\delta^{-1} = 3\frac{\alpha-2}{\alpha+2}. \tag{4.16}$$

Proof. For $j = 2$ from (4.5), we write

$$\begin{aligned}
M_p^2 &= \frac{1}{4\pi i}\int_{|\lambda|=b_p} tr\left(QR_\lambda^0\right)^2 d\lambda \\
&= \frac{1}{4\pi i}\int_{|\lambda|=b_p} \sum_{k=1}^{\infty}\left(\left(QR_\lambda^0\right)^2 \psi_k(x), \psi_k(x)\right)_{H_1} d\lambda.
\end{aligned} \tag{4.17}$$

Moreover, we know that

$$QR_\lambda^0 \psi_{k_1} = \left(\mu_{k_1} - \lambda\right)^{-1} Q\psi_{k_1}$$

and

$$\left(QR_\lambda^0\right)^2\psi_{k_1}=\left(\mu_{k_1}-\lambda\right)^{-1}QR_\lambda^0\left(Q\psi_{k_1}\right)$$
$$=\left(\mu_{k_1}-\lambda\right)^{-1}\sum_{k_2=1}^{\infty}\left(\mu_{k_2}-\lambda\right)^{-1}\left(Q\psi_{k_1},\psi_{k_2}\right)_{H_1}Q\psi_{k_2}. \qquad \textbf{(4.18)}$$

From (4.17) and (4.18), we have

$$M_p^2=\frac{1}{4\pi i}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\left[\int_{|\lambda|=b_p}\frac{d\lambda}{\left(\lambda-\mu_{k_1}\right)\left(\lambda-\mu_{k_2}\right)}\right]\left(Q\psi_{k_1},\psi_{k_2}\right)_{H_1}\left(Q\psi_{k_2},\psi_{k_1}\right)_{H_1}. \qquad \textbf{(4.19)}$$

It is easy to see that for $k_1\le n_p,\ k_2\le n_p$ and $k_1>n_p,\ k_2>n_p$

$$\int_{|\lambda|=b_p}\frac{d\lambda}{\left(\lambda-\mu_{k_1}\right)\left(\lambda-\mu_{k_2}\right)}=0. \qquad \textbf{(4.20)}$$

Then, from (4.19) and (4.20), we have

$$\left|M_p^2\right|=\left|\frac{1}{2\pi i}\sum_{k=1}^{n_p}\sum_{j=n_p+1}^{\infty}\left[\int_{|\lambda|=b_p}\frac{d\lambda}{\left(\lambda-\mu_k\right)\left(\lambda-\mu_j\right)}\right]\left(Q\psi_k,\psi_j\right)_{H_1}\left(Q\psi_j,\psi_k\right)_{H_1}\right|$$
$$=\sum_{k=1}^{n_p}\sum_{j=n_p+1}^{\infty}\left(\mu_j-\mu_k\right)^{-1}\left|\left(\psi_k,Q\psi_j\right)_{H_1}\right|^2$$
$$\le\sum_{j=n_p+1}^{\infty}\left(\mu_j-\mu_{n_p}\right)^{-1}\sum_{k=1}^{\infty}\left|\left(\psi_j,Q\psi_k\right)_{H_1}\right|^2 \qquad \textbf{(4.21)}$$
$$=\sum_{j=n_p+1}^{\infty}\left(\mu_j-\mu_{n_p}\right)^{-1}\left\|Q\psi_j\right\|_{H_1}^2\le\left\|Q\right\|_{H_1}^2\sum_{j=n_p+1}^{\infty}\left(\mu_j-\mu_{n_p}\right)^{-1}.$$

Let $\delta=\frac{\alpha-2}{\alpha+2}$. Then by (1.15) we get

$$\sum_{k=n_p+1}^{\infty}\frac{1}{\mu_k-\mu_{n_p}}<\sum_{k=n_p+1}^{\infty}\frac{1}{d_1\left(k^{1+\delta}-n_p^{1+\delta}\right)}$$

$$
\begin{aligned}
&= \frac{1}{d_1\left(\left(n_p+1\right)^{1+\delta}-n_p^{1+\delta}\right)} + d_1^{-1}\sum_{i=1}^{\infty}\int_{n_p+i}^{n_p+i+1}\frac{dx}{\left(n_p+i+1\right)^{1+\delta}-n_p^{1+\delta}} \\
&\le d_1^{-1}n_p^{-\delta} + d_1^{-1}\sum_{i=1}^{\infty}\int_{n_p+i}^{n_p+i+1}\frac{dx}{x^{1+\delta}-n_p^{1+\delta}} \qquad \textbf{(4.22)} \\
&= d_1^{-1}\left[n_p^{-\delta} + \int_{n_p+i}^{\infty}\frac{dx}{x^{1+\delta}-n_p^{1+\delta}}\right] < d_1^{-1}\left[n_p^{-\delta}+\delta^{-1}n_p^{-\frac{\delta^2}{1+\delta}}\right].
\end{aligned}
$$

From (4.21) and (4.22) we get

$$
\lim_{p\to\infty} M_p^2 = 0. \qquad \textbf{(4.23)}
$$

In a similar form it can be proved that the inequality

$$
\left|M_p^3\right| \le 4\left\|Q\right\|_{H_1}^3 d_1^{-2}\delta^{-2}\left(n_p^{-\frac{2\delta^2}{1+\delta}} + 2n_p^{\frac{1-2\delta^2}{1+\delta}}\right)
$$

is true. From here, we get

$$
\lim_{p\to\infty} M_p^3 = 0, \quad \delta > \frac{1}{\sqrt{2}}. \qquad \textbf{(4.24)}
$$

From (4.5) we get

$$
\begin{aligned}
\left|M_{pj}\right| &\le \frac{1}{2\pi j}\int_{|\lambda|=b_p}\left|tr\left(QR_\lambda^0\right)^j\right|\left|d\lambda\right| \le \frac{1}{2\pi j}\int_{|\lambda|=b_p}\left\|QR_\lambda^0\right\|_{\sigma_1(H_1)}^j\left|d\lambda\right| \\
&\le \int_{|\lambda|=b_p}\left\|QR_\lambda^0\right\|_{\sigma_1(H_1)}\left\|QR_\lambda^0\right\|_{H_1}^{j-1}\left|d\lambda\right| \qquad \textbf{(4.25)} \\
&\le \int_{|\lambda|=b_p}\left\|Q\right\|_{H_1}^j\left\|R_\lambda^0\right\|_{\sigma_1(H_1)}\left\|R_\lambda^0\right\|_{H_1}^{j-1}\left|d\lambda\right|.
\end{aligned}
$$

We shall now estimate $\left\|R_\lambda^0\right\|_{\sigma_1(H_1)}$ and $\left\|R_\lambda^0\right\|_{H_1}$ on the circle $|\lambda| = b_p$. For $\lambda \notin \{\mu_k\}_1^\infty$, since R_λ^0 is normal operator then

$$\left\|R_\lambda^0\right\|_{\sigma_1(H_1)} = \sum_{k=1}^{\infty} \frac{1}{\left|\mu_k - \lambda\right|} \qquad [13, p.121].$$

Since $\left|\lambda\right| = b_p = 2^{-1}\left(\mu_{n_p} + \mu_{n_p+1}\right)$ then

$$\begin{aligned} \left\|R_\lambda^0\right\|_{\sigma_1(H_1)} &\le \sum_{k=1}^{\infty} \frac{1}{\left|\left|\lambda\right| - \mu_k\right|} \\ &\le \sum_{k=1}^{n_p} \frac{2}{\mu_{n_p} - \mu_k} + \sum_{k=n_p+1}^{\infty} \frac{2}{\mu_k - \mu_{n_p}}. \end{aligned} \qquad \textbf{(4.26)}$$

By using (1.15), we obtain

$$\sum_{k=1}^{n_p} \frac{1}{\mu_{n_p+1} - \mu_k} < \frac{n_p}{\mu_{n_p+1} - \mu_{n_p}} < \frac{n_p}{d_1\left[\left(n_p+1\right)^{1+\delta} - n_p^{1+\delta}\right]} < d_1 n_p^{1-\delta}. \qquad \textbf{(4.27)}$$

From (4.22), (4.26) and (4.27), we get

$$\left\|R_\lambda^0\right\|_{\sigma_1(H_1)} \le \frac{6}{d_1\delta} n_p^{1-\delta}. \qquad \textbf{(4.28)}$$

Since the eigenvalues of the nuclear operator R_λ are $\left\{\left(\lambda_k - \lambda\right)^{-1}\right\}_1^{\infty}$ then

$$\left\|R_\lambda\right\|_{H_1} = \max_k \left\{\left|\lambda_k - \lambda^{-1}\right|\right\}_1^{\infty} \qquad [13, p.46].$$

From here and (1.15) we can get

$$\left\|R_\lambda\right\|_{H_1} \le const.n_p^{-\delta}\ , \quad \delta = \frac{\alpha - 2}{\alpha + 2}. \qquad \textbf{(4.29)}$$

Since $R_\lambda = R_\lambda^0$ for $Q \equiv 0$ according to (4.29)

$$\left\|R_\lambda^0\right\|_{H_1} \leq const.n_p^{-\delta} \ . \qquad \textbf{(4.30)}$$

By (1.14) we have

$$b_p < const.n_p^{1+\delta}. \qquad \textbf{(4.31)}$$

From (4.5), (4.28), (4.30) and (4.31) we get

$$\left|M_p^j\right| \leq const.\int_{|\lambda|=b_p} n_p^{1-\delta} n_p^{-\delta(j-1)} \left|d\lambda\right| \leq const.n_p^{2-\delta(j-1)}.$$

As seen, if $j > 1+2\delta^{-1}$ then

$$\lim_{p\to\infty} M_p^j = 0. \qquad \textbf{(4.32)}$$

If $\delta > \frac{1}{\sqrt{2}}$ then from (4.23), (4.24) and (4.32) we get formula (4.15) for $j \geq 2.$ We now prove formula (4.16). From (4.6), (4.28), (4.29), (4.30) and (4.31) we have

$$\begin{aligned}\left|M_{pN}\right| &\leq \frac{1}{2\pi}\int_{|\lambda|=b_p} |\lambda| \left|tr\left[R_\lambda\left(QR_\lambda^0\right)^{N+1}\right]\right| |d\lambda| \\ &\leq b_p \int_{|\lambda|=b_p} \left\|R_\lambda\left(QR_\lambda^0\right)^{N+1}\right\|_{\sigma_1(H_1)} |d\lambda| \\ &\leq b_p \int_{|\lambda|=b_p} \left\|R_\lambda\right\|_{H_1} \left\|Q\right\|_{H_1}^{N+1} \left\|R_\lambda^0\right\|_{H_1}^{N} \left\|R_\lambda^0\right\|_{\sigma_1(H_1)} |d\lambda| \\ &\leq const.n_p^{3-N\delta}.\end{aligned}$$

From here, we get

$$\lim_{p\to\infty} M_{pN} = 0, \ N > 3\delta^{-1}.$$

Theorem is proved.

The main result of this article is given by the following theorem.

Theorem 4.2. If the operator function $Q(x)$ satisfies the conditions (1)-(3) and $\gamma_j \sim aj^{\alpha}$ $\left(a>0, \alpha>\frac{2+2\sqrt{2}}{\sqrt{2}-1}\right)$ as $j \to \infty$ then

$$\lim_{p\to\infty}\sum_{k=1}^{n_p}(\lambda_k-\mu_k)=\frac{1}{4}\left[trQ(1)-trQ(0)\right].$$

The limit on the left side of this equality is called the regularized trace of the operator L.

Proof. From (4.4), (4.9), (4.10), (4.11), (4.14), (4.15) and (4.16) we obtain

$$\begin{aligned}
\lim_{p\to\infty}\sum_{k=1}^{n_p}(\lambda_k-\mu_k) &= -\lim_{p\to\infty}\sum_{k=1}^{p} trQ(x)\cos\left((2k-1)\pi x\right)dx \\
&= -\frac{1}{2}\sum_{k=1}^{\infty}\left[\int_0^1 trQ(x)\cos(k\pi x)\,dx-(-1)^k\int_0^1 trQ(x)\cos(k\pi x)\,dx\right] \\
&= -\frac{1}{4}\sum_{k=1}^{\infty}\left\{\left[\int_0^1 trQ(x)\sqrt{2}\cos(k\pi x)\,dx\right]\sqrt{2}\cos(k\pi .0)\right. \\
&\quad \left.-\left[\int_0^1 trQ(x)\sqrt{2}\cos(k\pi x)\,dx\right]\sqrt{2}\cos(k\pi .1)\right\} \\
&= -\frac{1}{4}\left[trQ(0)-trQ(1)\right].
\end{aligned}$$

This proves theorem.

5. COMPLETENESS OF EIGENFUNCTIONS OF DISCONTINUOUS STURM-LIOUVILLE PROBLEMS

The results of this chapter were objected to the article written by E. Şen, O. Mukhtarov and K. Oruçoğlu that was accepted for publication in the journal "Iranian Journal of Science and Technology Transaction A: Science". Also, the results obtained in this chapter were presented at the conference "XVIII. Ulusal Mekanik Kongresi, 26-30 August 2013, Celal Bayar University, Manisa" and were published in the conference proceedings under the title "Sturm-Liouville probleminin rezolvent operatörü ve özfonksiyonları" (pp. 560-569, with O. Mukhtarov and K. Oruçoğlu).

5.1. Statement of the Boundary-Value Problem as an Eigenvalue Problem in a Suitable Hilbert Space

If we use the following representations

$$\begin{cases} (u)_1 := \beta_1 u(1) - \beta_2 u'(1), \\ (u)'_1 := \alpha_1 u(1) - \alpha_2 u'(1) \end{cases} \tag{5.1}$$

it is easy to see that for $u,\ v \in C^1[-1,1]$, we have

$$\rho[u(1)v'(1) - u'(1)v(1)] = (u)_1(v)'_1 - (u)'_1(v)_1 . \tag{5.2}$$

Now we shall define the inner product of two component elements

$$T := \begin{pmatrix} T_1(x) \\ T_2 \end{pmatrix},\ T_1(x) \in L_2[-1,1], T_2 \in \mathbb{C};$$

$$G := \begin{pmatrix} G_1(x) \\ G_2 \end{pmatrix},\ G_1(x) \in L_2[-1,1], G_2 \in \mathbb{C};$$

in the linear space $L_2[-1,1] \oplus \mathbb{C}$ by the formula

$$\langle T, G \rangle_{p,r,\rho} := \int_{-1}^{h_1} T_1(x)\overline{G_1(x)}r(x)dx + \int_{h_1}^{h_2} T_1(x)\overline{G_1(x)}r(x)dx + \int_{h_2}^{1} T_1(x)\overline{G_1(x)}r(x)dx + \frac{p(1)}{\rho}T_2\overline{G_2}.$$

Then it can be easily shown that the inner product space

$$H_{p,r,\rho} := (L_2[-1,1] \oplus \mathbb{C}, \langle \bullet, \bullet \rangle_{p,r,\rho})$$

is a Hilbert space. In this space, let us define the operator $K:\ H_{p,r,p} \to H_{p,r,p}$ by the equality

$$K \begin{pmatrix} T_1(x) \\ (T_1)_1' \end{pmatrix} := \begin{pmatrix} \ell T_1 \\ -(T_1)_1 \end{pmatrix} \qquad \textbf{(5.3)}$$

on the domain of definition $D(K)$ consisting of all $T \in H_{p,r,\rho}$ which satisfies the following conditions:

(i) T_1 and T_1' are absolutely continuous functions in the intervals [-1, h_1), (h_1, h_2) and (h_2 ,1].

(ii) There exists finite limit values $T_1(h_1 \pm 0)$, $T_1'(h_1 \pm 0)$, $T_1(h_2 \pm 0)$ and $T_2'(h_2 \pm 0)$.

(iii) $T_1(-1) = 0$.

(iv) $\delta_1 T_1(h_1+0) = \gamma_1 T_1(h_1-0)$, $\delta_2 T_1'(h_1+0) = \gamma_2 T_1'(h_1-0)$,

$\delta_3 T_1(h_2+0) = \gamma_3 T_1(h_2-0)$,

$\delta_4 T_1'(h_2+0) = \gamma_4 T_1'(h_2-0)$ and $T_2 = (T_1)_1'$. **(5.4)**

Then we can write the boundary value problem (1.16)-(1.22) as an operator-equation:

$$KU = \lambda U \quad \left(U := \begin{pmatrix} u(x) \\ (u)'_1 \end{pmatrix} \in D(K) \right). \qquad \textbf{(5.5)}$$

Thus we stated the boundary-value problem (1.16)-(1.22) as an eigenvalue problem for a linear operator which is defined in a Hilbert space.

Lemma 5.1. If $\delta_1\delta_2 p(h_1-0) = \gamma_1\gamma_2 p(h_1+0)$ and $\delta_3\delta_4 p(h_2-0) = \gamma_3\gamma_4 p(h_2+0)$ then the operator K is symmetric.

Proof. Let $T, G \in D(K)$. If we use the well-known Lagrange formula [44] we find the following equality

$$\langle KT, G\rangle_{p,r,\rho} := \int_{-1}^{1} (\ell T_1)(x)\overline{G_1(x)} r(x)dx + \frac{p(1)}{\rho}((-T_1)_1)(G_1)'_1 = \int_{-1}^{h_1} T_1(x)\overline{(\ell G_1)(x)} r(x)dx$$

$$+\int_{h_1}^{h_2} T_1(x)\overline{(\ell G_1)(x)} r(x)dx + \int_{h_2}^{1} T_1(x)\overline{(\ell G_1)(x)} r(x)dx$$

$$+p(h_1-0)W(T_1,G_1;h_1-0) - p(-1)W(T_1,G_1;-1)$$

$$+p(h_2-0)W(T_1,G_1;h_2-0) - p(h_1+0)W(T_1,G_1;h_1+0) + p(1)W(T_1,G_1;1)$$

$$-p(h_2+0)W(T_1,G_1;h_2+0) - \frac{p(1)}{\rho}(T_1)_1\overline{(G_1)}'_1$$

$$= \left[\langle T, G\rangle_{p,r,\rho} - \frac{p(1)}{\rho}\overline{(G_1)}_1\overline{(T_1)}'_1 \right] +$$

$$\left[p(h_1-0)W(T_1,G_1;h_1-0) - p(h_1+0)W(T_1,G_1;h_1+0) \right]$$

$$+\left[p(h_2-0)W(T_1,G_1;h_2-0) - p(h_2+0)W(T_1,G_1;h_2+0) \right] - \frac{p(1)}{\rho}[(T_1)_1(G_1)'_1 - (T_1)'_1(G_1)_1]$$

$$=$$

$$\left[\int_{-1}^{h_1} T_1(x)\overline{(\ell G_1)(x)} r(x)dx + \int_{h_1}^{h_2} T_1(x)\overline{(\ell G_1)(x)} r(x)dx + \int_{h_2}^{1} T_1(x)\overline{(\ell G_1)(x)} r(x)dx) + \right.$$

$$\left. \frac{p(1)}{\rho}(T_1)'_1((-G_1)_1) \right]$$

$$-p(-1)W(T_1,\overline{G_1};-1) + p(h_1-0)W(T_1,\overline{G_1};h_1-0) - p(h_1+0)W(T_1,\overline{G_1};h_1+0)$$

$$+p(h_2-0)W(T_1,\overline{G_1};h_2-0)-p(h_2+0)W(T_1,\overline{G_1};h_2+0)$$
$$+p(1)\left[W(T_1,\overline{G_1};1)-\frac{1}{\rho}\left(\left(T_1\right)_1\left(\overline{G_1}\right)'_1-\left(T_1\right)'_1\left(\overline{G_1}\right)_1\right)\right]. \quad \textbf{(5.6)}$$

Here by

$$W(T_1,G_1;x):=T_1(x)G_1'(x)-T_1(x)'G_1(x) \quad \textbf{(5.7)}$$

we denote the Wronskians of the functions T_1 and G_1. T_1(x) and $\overline{G_1(x)}$ satisfy the boundary condition (1.17). Thus we have the following equality

$$W(T_1,\overline{G_1};-1)=0. \quad \textbf{(5.8)}$$

From the fact that the functions T_1 and $\overline{G_1}$ satisfy the transmission conditions (1.19)-(1.22), we obtain the equality

$$p(h_2-0)W(T_1,\overline{G_1};h_2-0)=p(h_2-0)\left[T_1(h_2-0)\overline{G_1}'(h_2-0)-T_1'(h_2-0)\overline{G_1}(h_2-0)\right]$$
$$=\frac{\gamma_3\gamma_4}{\delta_3\delta_4}p(h_2+0)\left\{\left(\frac{\delta_3}{\gamma_3}T_1(h_2+0)\right)\left(\frac{\delta_4}{\gamma_4}\overline{G_1}'(h_2+0)\right)\right.$$
$$\left.-\left(\frac{\delta_3}{\gamma_3}T_1'(h_2+0)\right)\left(\frac{\delta_4}{\gamma_4}\overline{G_1}(h_2+0)\right)\right\}=p(h_2+0)W(T_1,\overline{G_1};h_2+0) \quad \textbf{(5.9)}$$

and similarly we have

$$p(h_1-0)W(T_1,\overline{G_1};h_1-0)=p(h_1+0)W(T_1,\overline{G_1};h_1+0). \quad \textbf{(5.10)}$$

Consequently, we get

$$\left\langle KT,G\right\rangle_{p,r,\rho}=\left\langle T,KG\right\rangle_{p,r,\rho}.$$

Thus we obtain that the operator K is symmetric in the Hilbert space $H_{p,r,p}$.

5.2. Resolvent Operator

In this section we show that each number $\lambda \in \mathbb{C}$ which is not an eigenvalue of the operator K is a regular value of the operator K. Also we will investigate the resolvent operator

$$R(\lambda, K) := (K - \lambda I)^{-1}.$$

Here I is the unit operator. For an arbitrary element $T \in H_{p,r,\rho}$ let us write the operator equation

$$(K - \lambda I)U = T \tag{5.11}$$

as a non-homogenous boundary-value problem

$$\frac{1}{r(x)}\{-(p(x)U_1')' + q(x)U_1\} - \lambda U_1 = T_1(x),\ x \in [-1, h_1) \cup (h_1, h_2) \cup (h_2, 1], \tag{5.12}$$

$$U_1(-1) = 0, \tag{5.13}$$

$$(\beta_1 U_1(1) - \beta_2 U_1'(1)) + \lambda(\alpha_1 U_1(1) - \alpha_2 U_1'(1)) = T_2, \tag{5.14}$$

$$\gamma_1 U_1(h_1 - 0) - \delta_1 U_1(h_1 + 0) = 0, \tag{5.15}$$

$$\gamma_2 U_1'(h_1 - 0) - \delta_2 U_1'(h_1 + 0) = 0, \tag{5.16}$$

$$\gamma_3 U_1(h_2 - 0) - \delta_3 U_1(h_2 + 0) = 0, \tag{5.17}$$

$$\gamma_4 U_1'(h_2 - 0) - \delta_4 U_1'(h_2 + 0) = 0, \tag{5.18}$$

which is equivalent to (5.11). Firstly, we will state the following lemma:

Lemma 5.2. Let $f(\lambda)$ and $g(\lambda)$ be entire functions. Then the equation

$$\frac{1}{r(x)}\left\{-(p(x)u')'+q(x)u\right\}=\lambda u,\ x\in\left[d_1,d_2\right] \qquad \textbf{(5.19)}$$

has a unique solution $u(x,\lambda)$ which satisfy the boundary conditions

$$u(d_i)=f(\lambda), u'(d_i)=g(\lambda)\ (i=1\ or\ 2). \qquad \textbf{(5.20)}$$

This solution is an entire function of λ for each $x\in\left[d_1,d_2\right]$.

The proof of this lemma is similar to the Theorem 1.5 in the book of [55]. Now, using this lemma let us define two solutions $\varphi(x,\lambda)$ and $\chi(x,\lambda)$ of the differential equation (1.16).

Let us denote the solution of the differential equation (1.16) by $\varphi_1(x,\lambda)$ satisfying the initial conditions

$$u(-1)=0, \quad u'(-1)=1$$

in the interval $[-1,h_1)$. After defining the function $\varphi_1(x,\lambda)$ we can define the solution of differential equation (1.16) in the interval $[h_1,h_2)$ satisfying the initial conditions

$$u(h_1)=\frac{\gamma_1}{\delta_1}\varphi_1(h_1-0,\lambda),\ u'(h_1)=\frac{\gamma_2}{\delta_2}\varphi_1'(h_1-0,\lambda). \qquad \textbf{(5.21)}$$

Let us denote this solution by $\varphi_2(x,\lambda)$. Similarly let us denote the solution of differential equation (1.16) by $\varphi_3(x,\lambda)$ in the interval $[h_2,1]$ satisfying the initial conditions

$$u(h_2)=\frac{\gamma_3}{\delta_3}\varphi_2(h_2-0,\lambda),\ u'(h_2)=\frac{\gamma_4}{\delta_4}\varphi_2'(h_2-0,\lambda). \quad \textbf{(5.22)}$$

Similarly, let us denote the solution of the differential equation (1.16) by $\chi_3(x,\lambda)$ in the interval $(h_2,1]$ satisfying the initial conditions

$$u(1)=\alpha_2\lambda+\beta_2, u'(1)=\alpha_1\lambda+\beta_1 \quad \textbf{(5.23)}$$

After defining this solution let us denote the solution of the differential equation (1.16) by $\chi_2(x,\lambda)$ in the interval $(h_1,h_2]$ satisfying the initial conditions

$$u(h_2)=\frac{\delta_3}{\gamma_3}\chi_3(h_2+0,\lambda),\ u'(h_2)=\frac{\delta_4}{\gamma_4}\chi_3'(h_2+0,\lambda). \quad \textbf{(5.24)}$$

Similarly, let us denote the solution of the differential equation (1.16) by $\chi_1(x,\lambda)$ in the interval $[-1,h_1]$ satisfying the initial conditions

$$u(h_1)=\frac{\delta_1}{\gamma_1}\chi_2(h_1+0,\lambda),\ u'(h_1)=\frac{\delta_2}{\gamma_2}\chi_2'(h_1+0,\lambda). \quad \textbf{(5.25)}$$

Lemma 5.2 implies that the functions $\varphi_i(x,\lambda)$, $\chi_i(x,\lambda)$ (i =1,2,3) are entire functions of λ. Now, we can define the functions φ and χ as follows:

$$\varphi(x,\lambda)=\begin{cases}\varphi_1(x,\lambda), x\in[-1,h_1),\\ \varphi_2(x,\lambda), x\in(h_1,h_2),\\ \varphi_3(x,\lambda), x\in(h_2,1].\end{cases}$$

$$\chi(x,\lambda)=\begin{cases}\chi_1(x,\lambda), x\in[-1,h_1),\\ \chi_2(x,\lambda), x\in(h_1,h_2),\\ \chi_3(x,\lambda), x\in(h_2,1].\end{cases}$$

It is obvious that these functions satisfy the equation (1.16) and transmission conditions. Moreover the solution $\varphi(x,\lambda)$ satisfies the boundary condition (1.17) and $\chi(x,\lambda)$ satisfies the boundary condition (1.18). We shall use the following notations in the rest of the paper:

$$\omega_i := W_\lambda(\varphi_i,\chi_i;x) \text{ (i=1,2,3)},$$

$$\omega(x,\lambda) := W_\lambda(\varphi,\chi;x) = \begin{cases} \omega_1(x,\lambda), x\in[-1,h_1), \\ \omega_2(x,\lambda), x\in(h_1,h_2), \\ \omega_3(x,\lambda), x\in(h_2,1]. \end{cases}$$

Lemma 5.3. For all $\lambda\in\mathbb{C}$ which is not an eigenvalue of the problem (1.16)-(1.22) and for all $x\in[-1,h_1)\cup(h_1,h_2)\cup(h_2,1]$ we have $\omega(x,\lambda)\neq 0$.

Proof. This lemma can be proven similarly using the same method as in [36].

Corollary 5.1. Let us assume that $\lambda\in\mathbb{C}$ is not an eigenvalue of the problem (1.16)-(1.22). Then the functions $\varphi_1(x,\lambda)$, $\chi_1(x,\lambda)$ are linearly independent in the interval $[-1,h_1)$, the functions $\varphi_2(x,\lambda)$, $\chi_2(x,\lambda)$ are linearly independent in the interval (h_1,h_2) and the functions $\varphi_3(x,\lambda)$, $\chi_3(x,\lambda)$ are linearly independent in the interval $(h_2,1]$.

Corollary 5.1 implies that for all $\lambda\in\mathbb{C}$ which is not an eigenvalue of the problem (1.16)-(1.22) we can write the general solution of the differential equation (1.16) as

$$u(x,\lambda) = \begin{cases} C_1\varphi_1(x,\lambda)+D_1\chi_1(x,\lambda), x\in[-1,h_1), \\ C_2\varphi_2(x,\lambda)+D_2\chi_2(x,\lambda), x\in(h_1,h_2), \\ C_3\varphi_3(x,\lambda)+D_3\chi_3(x,\lambda), x\in(h_2,1], \end{cases}$$

where C_i, D_i (i=1,2,3) are arbitrary constants. Then applying the method of variation of constants [44] we can write the general solution of the non-homogenous equation (5.12) for $x\in[-1,h_1)$ as

$$U_1(x,\lambda)=\chi_1(x,\lambda)\int_{-1}^{x}\frac{\varphi_1(y,\lambda)}{\omega_1(y,\lambda)}T_1(y)dy$$
$$+\varphi_1(x,\lambda)\int_{x}^{h_1}\frac{\chi_1(y,\lambda)}{\omega_1(y,\lambda)}T_1(y)dy+C_1\varphi_1(x,\lambda)+D_1\chi_1(x,\lambda), \quad \textbf{(5.26)}$$

for $x\in(h_1,h_2)$ as

$$U_1(x,\lambda)=\chi_2(x,\lambda)\int_{h_1}^{x}\frac{\varphi_2(y,\lambda)}{\omega_2(y,\lambda)}T_1(y)dy+\varphi_2(x,\lambda)\int_{x}^{h_1}\frac{\chi_2(y,\lambda)}{\omega_2(y,\lambda)}T_1(y)dy$$
$$+C_2\varphi_2(x,\lambda)+D_2\chi_2(x,\lambda), \quad \textbf{(5.27)}$$

and for $x\in(h_2,1]$ as

$$U_1(x,\lambda)=\chi_3(x,\lambda)\int_{h_2}^{x}\frac{\varphi_3(y,\lambda)}{\omega_3(y,\lambda)}T_1(y)dy+\varphi_3(x,\lambda)\int_{x}^{1}\frac{\chi_3(y,\lambda)}{\omega_3(y,\lambda)}T_1(y)dy$$
$$+C_3\varphi_3(x,\lambda)+D_3\chi_3(x,\lambda). \quad \textbf{(5.28)}$$

Using the equalities (5.26)-(5.28) and writing the general solution of (5.12) in the conditions (5.13)-(5.18) then we can find the constants C_i, D_i. If we write the expression (5.26) in the boundary condition (5.13) then we obtain the equality $D_1\chi(-1,\lambda)=0$. Since λ is not an eigenvalue we have $\chi(-1,\lambda)\neq 0$. Therefore we get $D_1=0$. If we write the expression (5.28) in the boundary condition (5.14) we find $C_3=\dfrac{T_2}{\omega_3(1,\lambda)}$. Let us consider the values of constants D_1 and C_3. If we write the expressions (5.26)-(5.28) in the transmission conditions (5.15)-(5.18) we can find the constants C_1,C_2,D_2,D_3 by using the following system of linear equations:

$$\gamma_1\varphi_1(h_1,\lambda)C_1-\delta_1\chi_2(h_1,\lambda)D_2=$$
$$\gamma_1\chi_1(h_1,\lambda)\int_{-1}^{h_1}\frac{\varphi_1(y,\lambda)}{\omega_1(y,\lambda)}T_1(y)dy+\delta_1\varphi_2(h_1,\lambda)\int_{h_1}^{h_2}\frac{\chi_2(y,\lambda)}{\omega_2(y,\lambda)}T_1(y)dy+\delta_1C_2\varphi_2(h_1,\lambda),$$

$$\gamma_2\varphi_1{}'(h_1,\lambda)C_1-\delta_2\chi_2{}'(h_1,\lambda)D_2$$
$$=-\gamma_2\chi_1{}'(h_1,\lambda)\int_{-1}^{h_1}\frac{\varphi_1(y,\lambda)}{\omega_1(y,\lambda)}T_1(y)dy+\delta_2\varphi_2{}'(h_1,\lambda)\int_{h_1}^{h_2}\frac{\chi_2(y,\lambda)}{\omega_2(y,\lambda)}T_1(y)dy$$
$$+\delta_2C_2\varphi_2{}'(h_1,\lambda),$$

$$\gamma_3\varphi_2(h_2,\lambda)C_2-\delta_3\chi_3(h_2,\lambda)D_3=-\gamma_3\chi_2(h_2,\lambda)\int_{h_1}^{h_2}\frac{\varphi_2(y,\lambda)}{\omega_2(y,\lambda)}T_1(y)dy+$$
$$\delta_3\varphi_3(h_2,\lambda)\int_{h_2}^{1}\frac{\chi_3(y,\lambda)}{\omega_3(y,\lambda)}T_1(y)dy+\frac{\delta_3T_2\varphi_3(h_2,\lambda)}{\omega_3(1,\lambda)}+\gamma_3D_2\chi_2(h_2,\lambda),$$

$$\gamma_4\varphi_2{}'(h_2,\lambda)C_2-\delta_4\chi_3{}'(h_2,\lambda)D_3$$
$$=-\gamma_4\chi_2{}'(h_2,\lambda)\int_{h_1}^{h_2}\frac{\varphi_2(y,\lambda)}{\omega_2(y,\lambda)}T_1(y)dy+\delta_4\varphi_3{}'(h_2,\lambda)\int_{h_2}^{1}\frac{\chi_3(y,\lambda)}{\omega_3(y,\lambda)}T_1(y)dy$$
$$+\frac{\delta_4T_2\varphi_3{}'(h_2,\lambda)}{\omega_3(1,\lambda)}+\gamma_4D_2\chi_2{}'(h_2,\lambda).$$

The determinant of this system equals to $-\delta_1\delta_2\delta_3\delta_4\omega_2(h_1,\lambda)\omega_3(h_2,\lambda)\neq 0$. Hence we have a unique solution for the above system of linear equations. Using the definitions of the functions $\varphi_i(x,\lambda),\chi_i(x,\lambda)$ and from the above system of linear equations we obtain

$$C_1=\int_{h_1}^{h_2}\frac{\chi_2(y,\lambda)}{\omega_2(y,\lambda)}T_1(y)dy+\int_{h_2}^{1}\frac{\chi_3(y,\lambda)}{\omega_3(y,\lambda)}T_1(y)dy+\frac{T_2}{\omega_3(y,\lambda)},$$
$$C_2=\int_{h_2}^{1}\frac{\chi_3(y,\lambda)}{\omega_3(y,\lambda)}T_1(y)dy+\frac{T_2}{\omega_3(y,\lambda)},$$
$$D_2=\int_{-1}^{h_1}\frac{\varphi_1(y,\lambda)}{\omega_1(y,\lambda)}T_1(y)dy,\ D_3=\int_{h_1}^{h_2}\frac{\varphi_2(y,\lambda)}{\omega_2(y,\lambda)}T_1(y)dy.$$

By putting these values of constants C_i, D_i in the expressions (5.26)-(5.28) we find the following formula for the solution of the problem (5.12)-(5.18) in the whole $[-1,h_1)\cup(h_1,h_2)\cup(h_2,1]$:

$$U_1=\chi(x,\lambda)\int_{-1}^{x}\frac{\varphi(y,\lambda)}{\omega(y,\lambda)}T_1(y)dy+\varphi(x,\lambda)\int_{-1}^{x}\frac{\chi(y,\lambda)}{\omega(y,\lambda)}T_1(y)dy+\frac{T_2\varphi(x,\lambda)}{\omega_3(1,\lambda)}.$$

Theorem 5.1. Each $\lambda \in \mathbb{C}$ which is not an eigenvalue of the problem (1.16)-(1.22) is a regular value of the operator K that defined by equalities (5.3), (5.4), and the resolvent operator $R(\lambda, K): H_{p,r,\rho} \to H_{p,r,\rho}$ is a compact operator.

Proof. Using the following representation

$$G_1(x, y; \lambda) = \begin{cases} \dfrac{\chi(x,\lambda)\varphi(y,\lambda)}{\omega(y,\lambda)}, & -1 \le y \le x \le 1; \\ & x, y \ne h_i \, (i = 1, 2), \\ \dfrac{\varphi(x,\lambda)\chi(y,\lambda)}{\omega(y,\lambda)} & -1 \le x \le y \le 1; \\ & x, y \ne h_i \, (i = 1, 2), \end{cases}$$

we can rewrite the last formula as

$$U_1(x,\lambda) = \int_{-1}^{1} G_1(x, y; \lambda) T_1(y) dy + \frac{T_2}{\omega(1,\lambda)} \varphi(x,\lambda).$$

Therefore we obtain the following formula for the resolvent operator $R(\lambda, K)$:

$$R(\lambda, K)T = \begin{pmatrix} \displaystyle\int_{-1}^{1} G_1(x, y; \lambda) T_1(y) dy + \frac{T_2}{\omega(1,\lambda)} \varphi(x,\lambda) \\ \displaystyle\int_{-1}^{1} \left(G_1(\bullet, y; \lambda)\right)_1' T_1(y) dy + \frac{T_2}{\omega(1,\lambda)} \left(\varphi(\bullet,\lambda)\right)_1' \end{pmatrix}.$$

If we define the operators $B_\lambda : L_2[-1,1] \to L_2[-1,1]$, $\widetilde{B_\lambda} : H_{p,r,\rho} \to H_{p,r,\rho}$ and $S_\lambda : H_{p,r,\rho} \to H_{p,r,\rho}$ by the equalities

$$B_\lambda T_1 := \int_{-1}^{1} G_1(x, y; \lambda) T_1(y) dy \,,$$

$$\widetilde{B_\lambda} T := \begin{pmatrix} B_\lambda T_1 \\ (B_\lambda T_1)_1' \end{pmatrix},$$

$$S_\lambda T := \begin{pmatrix} \frac{T_2}{\omega(1,\lambda)}\varphi(x,\lambda) \\ \frac{T_2}{\omega(1,\lambda)}\left(\varphi(\bullet,\lambda)\right)'_1 \end{pmatrix},$$

we can write the resolvent operator $R(\lambda, K)$ as $R(\lambda,K) = \widetilde{B_\lambda} + S_\lambda$. The operator B_λ is compact in the Hilbert space $L_2[-1,1]$ [28]. Hence the operator $\widetilde{B_\lambda}$ is compact in the Hilbert space $H_{p,r,\rho}$. It is clear that the operator S_λ is compact in the Hilbert space $H_{p,r,\rho}$. Therefore for each $\lambda \in \mathbb{C}$ which is not an eigenvalue of the problem (1.16)-(1.22) the operator $R(\lambda, K)$ is also a compact operator in the Hilbert space $H_{p,r,\rho}$.

5.3. Expansion in Series of System of Eigenfunctions

Theorem 5.2. The operator K which is defined by the equalities (5.3)-(5.4) is a self-adjoint operator in the Hilbert space $H_{p,r,\rho}$.

Proof. It is clear that the operator K is densely defined in the Hilbert space $H_{p,r,\rho}$. Also, for all $\lambda \in \mathbb{C}$ which satisfy $\operatorname{Im}\lambda \neq 0$ Theorem 5.1 implies that the ranges of the operators $K - \lambda I$ and $K - \overline{\lambda} I$ coincide with whole Hilbert space $H_{p,r,\rho}$. Namely, the equalities $(K - \lambda I)D(K) = H_{p,r,\rho}$ and $(K - \tilde{\lambda} I)D(K) = H_{p,r,\rho}$ hold true. Also Lemma 5.1 implies that the operator K is symmetric. Therefore, the well-known theorem about extension of symmetric operators implies that the operator K is self-adjoint [28].

Corollary 5.2. All eigenvalues of the boundary-value problem (1.16)-(1.22) are real.

Note: Since $p(x)$, $q(x)$ and $r(x)$ are real valued functions, the coefficients of the conditions (1.17)-(1.22) are real numbers and all eigenvalues are real we may assume that the all eigenfunctions of the problem (1.16)-(1.22) are real valued functions.

Corollary 5.3. If λ_1 and λ_2 are two different eigenvalues of the problem (1.16)-(1.22) and $u_1(x)$, $u_2(x)$ are eigenfunctions corresponding to these eigenvalues respectively then:

$$\int_{-1}^{1} u_1(x)u_2(x)r(x)dx = \frac{-p(1)}{\rho}(u_1)_1'(u_2)_1' . \quad \textbf{(5.29)}$$

Proof. Since the operator K is self-adjoint the appropriate eigenelements (corresponding to different eigenvalues λ_1 and λ_2) $U_1 = \begin{pmatrix} u_1(x) \\ (u_1)_1' \end{pmatrix}$ and $U_2 = \begin{pmatrix} u_2(x) \\ (u_2)_1' \end{pmatrix}$ are orthogonal in the space $H_{p,r,\rho}$. Namely the equality (5.29) holds.

In the Hilbert space $H_{p,r,\rho}$, the operator K which is defined by the equalities (5.3), (5.4) has countable number of real eigenvalues, the algebraic multiplicity of each eigenvalue is finite, the sequence of eigenvalues has a lower bound and doesn't have a finite accumulation point. Regarding to each eigenvalue is counted according to its algebraic multiplicity, we can write the sequence of eigenvalues as $\lambda_1 \le \lambda_2 \le ...$. Let us denote the appropriate-normed eigenelements as

$$\varphi_n := \begin{pmatrix} \phi_n(x) \\ (\phi_n)_1' \end{pmatrix}, \left(\| \varphi_n \|_{H_{p,r,\rho}} = 1,\ n = 1,2,... \right).$$

Then Theorem 5.1, Theorem 5.2 and the well-known Hilbert-Schmidt Theorem implies the following theorem [53].

Theorem 5.3. For each element $T \in H_{p,r,\rho}$ the Fourier series $\sum_{n=1}^{\infty} c_n \varphi_n, c_n = \langle T, \varphi_n \rangle_{H_{p,r,\rho}}$ will be converge to

$$T = \sum_{n=1}^{\infty} \langle T, \varphi_n \rangle_{H_{p,r,\rho}} \varphi_n \quad \textbf{(5.30)}$$

in the Hilbert space $H_{p,r,\rho}$.

Corollary 5.4. Each function $f \in L_2[-1,1]$ can be written as a series expansion of the eigenfunction system $\{\phi_n\}$, $n = 1, 2, \ldots$ of the boundary-value problem (1.16)-(1.22) as

$$f(x) = \sum_{n=1}^{\infty} \left(\int_{-1}^{1} f(y)\phi_n(y)r(y)dy \right) \phi_n(x)$$

in the Hilbert space $L_2([-1,1], r)$.

Proof. It is enough to get the element $T \in H_{p,r,\rho}$ as $T = \begin{pmatrix} f(x) \\ 0 \end{pmatrix}$ in the formula (5.30).

Corollary 5.5. For each $f \in L_2[-1,1]$ the following equalities hold:

$$\sum_{n=1}^{\infty} [(\phi_n)_1']^2 = \frac{\rho}{p(1)}, \tag{5.31}$$

$$\sum_{n=1}^{\infty} (\phi_n)_1' \phi_n(x) = 0. \tag{5.32}$$

Proof. Let us rewrite the formula (5.30) as

$$\begin{pmatrix} T_1(x) \\ T_2 \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty} \langle T, \varphi_n \rangle_{H_{p,r,\rho}} \phi_n(x) \\ \sum_{n=0}^{\infty} \langle T, \varphi_n \rangle_{H_{p,r,\rho}} (\phi_n)_1' \end{pmatrix}. \tag{5.33}$$

Now, putting $T = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in the formula (5.33) we get

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{p(1)}{\rho} (\phi_n)_1' \phi_n(x) \\ \sum_{n=0}^{\infty} \frac{p(1)}{\rho} [(\phi_n)_1']^2 \end{pmatrix}.$$

Namely, we obtain the equalities (5.31) and (5.32).

Corollary 5.6. For each $f \in L_2[-1,1]$ the equality

$$\sum_{n=0}^{\infty} \left(\int_{-1}^{1} f(y)\phi_n(y)dy \right) (\phi_n)_1' = 0$$

holds.

Proof. It is enough to rewrite formula (5.33) for the element $T = \begin{pmatrix} f(x) \\ 0 \end{pmatrix}$.

6. CONCLUSIONS AND RECOMMENDATIONS

It is little known about asymptotic behaviour of the eigenvalues and eigenfunctions of the Sturm-Liouvillle problems with eigenvalue dependent boundary conditions if the number of points of discontinuity is more than one. In the second and third chapters we sought an answer to this question. To this aim, we investigated spectral properties of Sturm-Liouville problems with eigenvalue dependent boundary conditions at two or finitely many points of discontinuity. If we take all transmission coefficients equal to each other and weight function equals identically to one we get the continuous case.

In the fourth chapter we investigated spectrum and the resolvent operator of a boundary-value problem which includes an unbounded operator coefficient in differential equation. Lastly, we obtained a regularized trace formula for differential operator equation.

For future works, we plan to obtain trace formulas for n-th order differential operators with unbounded operator coefficient with mixed type and/or periodic boundary conditions.

In the fifth chapter we studied the completeness of eigenfunctions of a Sturm-Liouville problem with eigenvalue-dependent boundary conditions and transmission conditions at two interior points. As a main result we showed that each square integrable function can be written as a series expansion of the eigenfunctions of the related boundary-value problem. In the special case that the transmission coefficients equal to each other and $r(x) = p(x) \equiv 1$ in the results obtained in this work coincide with corresponding results in the classical continuous Sturm-Liouville operator.

APPENDICES

APPENDIX A: Some basic definitions and theorems in functional analysis

APPENDIX B: Big $-O$ notation

APPENDIX A: Some basic definitions and theorems in functional analysis

Definition (Normed space, norm). A normed space X is a vector space with a norm defined on it. Here a norm on a (real or complex) vector space X is a real-valued function on X whose value at an $x \in X$ is denoted by $\|x\|$ and which has the properties

i.) $\|x\| \geq 0$

ii.) $\|x\| = 0 \Leftrightarrow x = 0$

iii.) $\|\alpha x\| = |\alpha| \|x\|$

iv.) $\|x + y\| \leq \|x\| + \|y\|$

here x and y are arbitrary vectors in X and α is any scalar [27].

Definition (Inner product space, Hilbert space). An inner product space (or pre-Hilbert space) is a vector space X with an inner product defined on X. A Hilbert space is a complete inner product space. Here, an inner product on X is a mapping of $X \times X$ into the scalar field K of X; that is, with every pair of vectors x and y there is associated a scalar which is written $\langle x, y \rangle$ and called the inner product of x and y, such that for all vectors x, y, z and scalar α we have

i.) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

ii.) $\langle \alpha x, \mathrm{y} \rangle = \alpha \langle x, y \rangle$

iii.) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

iv.) $\begin{array}{l} \langle x, x \rangle \geq 0 \\ \langle x, x \rangle = 0 \Leftrightarrow x = 0. \end{array}$

An inner product on X defines a norm on X given by $\|x\| = \sqrt{\langle x, x \rangle}$. Hence inner product spaces are normed spaces [27].

Definition (Dense set, separable space). A subset M of a metric space X is said to be dense in X if $\bar{M} = X$.

X is said to be separable if it has a countable subset which is dense in X [27].

Definition (Self-adjoint, normal operators). A bounded linear operator $T : H \to H$ on a Hilbert space H is said to be self-adjoint if $T = T^*$ and normal if $TT^* = T^*T$. The Hilbert-adjoint operator T^* of T is defined by the equality $\langle Tx, y\rangle = \langle x, T^* y\rangle$.

If T is self-adjoint the formula becomes $\langle Tx, y\rangle = \langle x, Ty\rangle$.

If T is self-adjoint then T is normal.

Every self-adjoint linear operator is symmetric [27].

Definition (Symmetric operators). A densely defined linear operator T in a Hilbert space H is symmetric if and only if $T \subset T^*$ [27].

Definition (Resolvent operator). Let $X \neq \{0\}$ be a complex normed space and $T : D(T) \to X$ a linear operator with domain $D(T) \subset X$. With T we associate the operator $T_\lambda = T - \lambda I$, where λ is a complex number and I is the identity operator on $D(T)$. If T_λ has an inverse, we denote it by $R_\lambda(T)$, that is, $R_\lambda(T) = T_\lambda^{-1} = (T - \lambda I)^{-1}$ and call it resolvent operator of T or simply resolvent of T. $R_\lambda(T)$ helps to solve the equation $T_\lambda x = y$. Thus $x = T_\lambda^{-1} y = R_\lambda(T) y$ provided $R_\lambda(T)$ exists [27].

Definition (Regular value, resolvent set, spectrum). Let $X \neq \{0\}$ be a complex normed space and $T : D(T) \to X$ a linear operator with domain $D(T) \subset X$. A regular value λ of T is a complex number such that $R_\lambda(T)$ exists, bounded and defined on a set which is dense in X.

The resolvent set $\rho(T)$ of T is the set of all regular values λ of T. Its complement $\sigma(T) = \mathbb{C} - \rho(T)$ in the complex plane $\mathbb{C}$ is called the spectrum of T. The discrete spectrum or point spectrum $\sigma_p(T)$ is the set such that $R_\lambda(T)$ does not exist. A $\lambda \in \sigma_p(T)$ is called an eigenvalue of T [27].

Theorem (Compactness). In a finite dimensional normed space X, any subset $M \subset X$ is compact if and only if M is closed and bounded.

Definition (Compact linear operator). Let X and Y be normed spaces. An operator $T : X \to Y$ is called a compact linear operator (or completely continuous linear operator) if T is linear and if for every bounded subset M of X, the image $T(M)$ is relatively compact, that is the closure $\overline{T(M)}$ is compact [27].

Theorem (Compactness criterion). Let X and Y be normed spaces and $T: X \to Y$ a linear operator. Then T is compact if and only if it maps every bounded sequence $\{x_n\}$ in X onto a sequence $\{Tx_n\}$ in Y which has a convergent subsequence [27].

Definition (Uniform convergence). A sequence $\{f_n\}$ of functions defined on a set E is said to converge uniformly on E if given $\varepsilon > 0$, there is an N such that for all $x \in E$ and all $n \geq N$, we have $|f(x) - f_n(x)| < \varepsilon$ [47].

Definition (Positive operator). A bounded self-adjoint linear operator $T: H \to H$ is said to be positive, written $T \geq 0$ if and only if $(Tx, x) \geq 0$ for all $x \in H$ [27].

Definition (Identity operator). Given a Hilbert space H. Let $Ix = x$ for all $x \in H$. Then I is called the identity operator [27].

Theorem (Completeness of eigenfunctions). Let $\{\phi_n\}$ be any complete sequence of orthonormal vectors in a Hilbert space H, and let $\{\psi_n\}$ be any sequence of orthonormal vectors in H that satisfies the inequality

$$\sum_{n=1}^{\infty} \|\psi_n - \phi_n\|^2 < \infty$$

then the ψ_n are complete in H [26].

Definition ($L_2[a,b]$ space). The space of square integrable functions on the interval $[a,b]$ (i.e., $\int_a^b |f(x)|^2 \, dx < \infty$).

Definition ($L_1[a,b]$ space). The space of integrable functions on the interval $[a,b]$

(i.e., $\int_a^b |f(x)| \, dx < \infty$).

Definition ($C^1[a,b]$ space). The space of continuously differentiable functions on the interval $[a,b]$.

Definition (Weak derivative). A generalization of the concept of the derivative of a function for functions not assumed differentiable, but only integrable, i.e. to lie in the L_1 space.

Definition (Weakly measurable function). Let H be a Hilbert space with countable base. A function $f : X \to H$ is called weakly measurable if for every functional h on H the composite $h \circ f$ is measurable [28].

Theorem (Closed graph). The graph of the function f is closed if and only if f is continuous.

APPENDIX B: Big$-O$ notation

For $t>0$ and a real number p

$$f(t)=O(t^p) \text{ as } t\to 0 \Leftrightarrow t^{-p}\left|f(t)\right| \text{ is bounded as } t\to 0.$$

In addition we define

$$f(t)=g(t)+O(t^p) \Leftrightarrow f(t)-g(t)=O(t^p).$$

Similarly we may replace $t\to 0$ by $t\to\infty$. For example, $f\left(t\right)=\sqrt{t^2+1}=O(t)$ as $t\to\infty$ or $f(t)-t=\mathrm{O}(t^{-1})$ as $t\to\infty.$

The symbol $O(1)$. The symbol $O(1)$ signifies a function $f\left(x,\lambda\right)$ of x and λ, defined for all sufficiently large λ, which is bounded for $a\le x\le b$ as $\lambda\to\infty.$ The symbol $O(1)/\lambda^p$ is also written as $O(\lambda^{-p}).$

The following important properties of the symbol $O(1)$ can be easily verified:

$$O(1)+O(1)=O(1); \quad O(1)O(1)=O(1); \quad \int_a^b O(1)dx=O(1)$$

for any finite $a,b.$ Again, if α and β are real numbers with $\alpha\le\beta,$ then

$$O\left(\frac{1}{\lambda^\alpha}\right)+O\left(\frac{1}{\lambda^\beta}\right)=O\left(\frac{1}{\lambda^\alpha}\right).$$

Finally, if $q(x)$ is any bounded function of $x,$ then by Taylor's formula we have, as $\lambda\to\infty$

$$[\lambda-q(x)]^\alpha=\lambda^\alpha\left[\frac{1-q\left(x\right)}{\lambda}\right]^\alpha=\lambda^\alpha-\alpha q(x)\lambda^{\alpha-1}+O(\lambda^{\alpha-2}).$$

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