İSTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL OF SCIENCE
ENGINEERING AND TECHNOLOGY

SYMMETRY IN SOME OVERDETERMINED PROBLEMS

M.Sc. THESIS

Nurlan ABDIKAIMOV

Department of Mathematics Engineering

Mathematics Engineering Programme

Thesis Advisor: Assoc. Prof. Dr. Ceni BABAOĞLU

MAY 2015

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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

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Matematik Mühendisliği Anabilim Dalı

Matematik Mühendisliği Programı

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MAYIS 2015

Nurlan ABDIKAIMOV, a **M.Sc.** student of ITU **Graduate School of Science Engineering and Technology** 509111081 successfully defended the **thesis** entitled **"SYMMETRY IN SOME OVERDETERMINED PROBLEMS"**, which he prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

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To my mother in heaven,

FOREWORD

I would like to thank my supervisor Assoc. Prof. Dr. Ceni Babaoğlu for giving me valuable advice and support during my whole master degree study and thesis work period. I am also grateful to my father Canıbek Abdikaimov and my girlfriend Nazima Elalova for their moral support.

May 2015

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SYMMETRY IN SOME OVERDETERMINED PROBLEMS

SUMMARY

In the present study, overdetermined symmetry problems for elliptic and parabolic cases are studied for one-phase, two-phase and multi-phase versions. The study contains five main sections.

In the first section, a general introduction is given including some information about symmetry in elliptic and parabolic multi-phase overdetermined problems with nonlinear governing equations. Existing results in literature are also argued briefly.

The second section is devolved to the background. Related terminology with basic definitions and theorems about Laplace and heat equations are reviewed.

In the third section, symmetry for an elliptic overdetermined problem, related to balls is analysed. A general overview of existing results is given. First, Laplace equation with nonlinear boundary conditions is considered for both one-phase and two-phase cases. For one-phase case the nature of the problem is indicated. The difficulties of the problem is observed for the two-phase case. Then, p-Laplacian operator is considered and in order to generalize the results, exact definitions and forms of the Green's functions for the ball are written. Multi-phase version of the problem for the p-Laplacian is examined in detail. These results also cover the symmetry for the Laplace equation for multi-phases.

In the fourth section, symmetry for a parabolic overdetermined problem, related to heat balls is analysed. The one-phase, two-phase and multi-phase versions with nonlinear overdetermined boundary conditions are proved. Along with these ideas spherical space symmetry problems for the solutions of similar overdetermined problems are also stated and proved.

BAZI AŞIRI BELİRGİN PROBLEMLERDE SİMETRİ

ÖZET

Bu yüksek lisans tez çalışmasında iki temel konu ele alınmıştır. Bunlardan ilki eliptik aşırı belirgin problemleri, ikincisi ise parabolik aşırı belirgin problemleri kapsamaktadır.

Eliptik yapıda temel bir problem ele alınmak istendiğinde

$$\begin{aligned} \Delta u = -c_0\delta_0, &\qquad \Omega \quad \text{bölgesinde,} \\ u = 0, &\qquad \partial\Omega \quad \text{sınırında.} \end{aligned}$$

denklemleri uygun bir seçimdir. Burada Δ Laplasyeni, δ_0 dirac delta fonksiyonunu, $\partial\Omega$ ise Ω bölgesinin sınırını göstermektedir.

Yukarıdaki verilen probleme

$$\partial_\nu u = F(|x|), \qquad \partial\Omega \quad \text{sınırında,}$$

şeklinde bir ek sınır koşulu getirdiğimizde problem aşırı belirgin bir hal alacaktır. Burada, F, sonradan belirlenecek bir fonksiyon olmak üzere, ν iç normal doğrultusunu göstermektedir. Ayrıca, $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ bölgesi sınırlı C^1 sınıfından seçilmiştir.

Yukarıda verilen sınır değer probleminde Ω bölgesinin hangi koşullar altında küre olabileceği ilginç bir soru olarak karşımıza çıkmaktadır.

Aşırı belirgin bir eliptik problemde görülen ek sınır koşulu altında bölgenin G Green fonksiyonu ele alındığında $|\nabla G|$ sınırda sabit bir değer olarak elde edilir. Bu ise bize Ω bölgesinin bir küre olduğu sonucunu verir.

Diğer yandan, iki fazlı bir yapıda u^+ ve u^- olarak iki Green fonksiyonu ile çalışılmaktadır. Ω bölgesi içinde örneğin orijinde kutup noktasına sahip olan u^+ Green fonksiyonu ve Ω bölgesinin dışında sonsuzda kutuba sahip olan u^- Green fonksiyonu arasında bir ilişki elde edilebilir. Bu ilişki Ω bölgesinin sınırında Green fonksiyonlarının normalleri cinsinden aşağıdaki gibi yazılabilir:

$$\partial_\nu u^+ = F(\partial_\nu u^-), \qquad \partial\Omega \quad \text{sınırında.}$$

Burada F verilen bir fonksiyon olup, hangi özellikleri sağlaması durumunda Ω bölgesinin bir küre olabileceği sorusuna cevap aranmaktadır.

İki fazlı durumdan çok fazlı duruma geçildiğinde, aşırı belirgin problem için simetrinin gösterilmesi biraz daha zahmetli ve detaylı hesaplar içermektedir. Bu çalışmada, bu hesapların yanısıra çok fazlı durumda aşırı belirgin $\Delta_p u$, p-Laplace denklemi için sonuçlar genelleştirilmiştir.

Bir parabolik problemin incelenmesinde en temel denklem olan ısı denklemi

$$\begin{cases} Hu = -\delta_0, & D_\lambda \quad \text{bölgesinde}, \\ u = 0, & \partial D_\lambda \quad \text{sınırında}, \end{cases}$$

şeklinde verilen sınır değer problemi olarak incelenebilir. Burada $H = \Delta - \partial_t$ ısı operatörü olup, D_λ ısı yuvarını, ∂D_λ ise ısı yuvarının sınırını göstermektedir.

Parabolik sınır değer problemine ek olarak

$$|\partial_{n_x} u| = \lambda \frac{|x|}{2t},$$

sınır koşulunun tanımlanması problemi aşırı belirgin bir sisteme dönüştürür. Isı denklemindeki u çözümünün ek sınır koşulunu da sağladığı gösterilebilir. Burada $\partial_{n_x} u$, n_x doğrultusundaki doğrultu türevidir.

Tezde çalışılan ikinci temel konu, yukarıda verilen parabolik sınır değer probleminde Ω bölgesinin hangi koşullar altında bir ısı yuvarı olabileceğidir. Parabolik problem yapı itibari ile ısı yuvarları ile ilgili olduğundan teknik olarak eliptik problemden farklı yöntemler kullanmamızı gerektirir. Örneğin, elliptik bir sistemde maksimum prensibi sonuç verirken, parabolik bir sistemde Hopf lemmasından faydalanmamız gerekmektedir. Parabolik problemde de eliptik problemde olduğu gibi çok fazlı ortamda genel sonuçlar elde edilmiş, özel durumlarda bu sonuçların tek fazlı ve iki fazlı ortamlardaki sonuçları kapsadığı gösterilmiştir.

Bu yüksek lisans tezinde, eliptik ve parabolik durumlar için aşırı belirgin simetri problemleri tek fazlı, iki fazlı ve çok fazlı durumlar için ele alınmıştır. Çalışma beş ana bölümden oluşmaktadır.

Birinci bölümde, konuya genel bir giriş yapılmış, eliptik ve parabolik çok fazlı aşırı belirgin problemlerde simetri problemi ile ilgili sınır değerlerin doğrusal olmayan fonksiyonlardan oluştuğu durumda bazı bilgiler verilmiştir. Konuyla ilgili literatürde mevcut olan bazı sonuçlar da kısaca özetlenmiştir.

İkinci bölümde, Laplace ve ısı denklemleri tanıtılmış, ilgili terminoloji gözden geçirilmiş; Diverjans Teoremi, Maksimum Prensibi, Hopf Lemması gibi temel tanım ve teoremlere yer verilmiştir.

Üçüncü bölümde, bir aşırı belirgin eliptik problem için küreler ele alınarak simetri analiz edilmiştir. Literatürde mevcut olan sonuçlar göz önünde bulundurularak, öncelikle, lineer olmayan sınır koşulları altında Laplace denklemi tek fazlı ortamda incelenmiştir. Ardından, problem iki fazda ele alınmıştır. Tek fazlı durum için problemin yapısı ile ilgili fikirler verilirken, iki fazlı durumda ise problemin zorlukları irdelenmiştir. Daha sonra, sonuçları genelleştirmek adına, p-Laplace operatörü ele alınmış, küre için Green fonksiyonlarının tanımları ve formları açık şekilde yazılmıştır. P-Laplace denklemi için çok fazlı versiyon detaylı olarak incelenmiştir. Elde edilen sonuçların aynı zamanda Laplace denkleminin çok fazlı durumundaki simetri özelliğini de kapsadığı belirtilmiştir.

Dördüncü bölümde, parabolik aşırı belirgin bir problem için ısı yuvarları ele alınarak simetri analiz edilmiştir. Doğrusal olmayan sınır koşulları altında aşırı belirgin tek fazlı, iki fazlı ve çok fazlı durumlarda simetri kanıtlanmıştır. Diğer bir değişle, hangi ek sınır koşulları altında bölgenin bir ısı yuvarı olabileceği detaylı bir şekilde incelenmiştir. Bu fikirler ile birlikte benzer problemlerde küresel simetri de ifade ve ispat edilmiştir.

1. INTRODUCTION

In this thesis, we study symmetry for both elliptic and parabolic versions of multi-phase overdetermined problems.

First we consider an elliptic overdetermined symmetry problem where it is usual to consider the problem in a given domain, involving only one function, and an extra boundary condition. If the Green's function G of a domain, has the property that $|\nabla G|$ is constant on the boundary, then one expects the domain to be a ball. The proof of this theorem, for C^1 domains follows a simple argument, given in [1]. On the other hand, recent years has seen a lot of mathematical problems where there are more than one phase entering into the game, or the physical model. One such example is the so-called multi-phase flows, where several liquids are present. In such problems, there is a different governing equation on the "free boundary", that in general, is a nonlinear equation. More exactly, suppose for a bounded C^1 domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) the Green's function u^+ with pole at some interior point (origin, say), and the Green's function u^- of the exterior domain with pole at infinity we have the boundary gradient condition

$$\partial_\nu u^+ = F(\partial_\nu u^-) \qquad \text{on } \partial\Omega,$$

where F is a given function, with certain properties, and ν is the inward normal direction. Can we conclude that Ω is a ball? We prove a multi-phase version of this problem, with general governing conditions, and with the p-Laplacian, see [2].

Second main result is for the parabolic version, which is related to heat balls. Let u be a solution to problem (1.1).

$$\begin{cases} Hu = -\delta_0, & \text{in} \quad D_\lambda, \\ u = 0, & \text{on} \quad \partial D_\lambda. \end{cases} \tag{1.1}$$

Here δ_0 is the Dirac delta function and D_λ denotes the heat ball. Then one can show that u satisfies the extra boundary gradient condition (1.2).

$$|\partial_{n_x} u| = \lambda \frac{|x|}{2t}. \tag{1.2}$$

It is also well known [3] that under mild conditions the heat balls are the only domains admitting such a solution with extra boundary gradient condition (1.2).

In a series of papers [4], [5] and [6], the authors have considered various related symmetry problems. E.g. in [4], the authors consider this problem under some regularity conditions. In a recent paper [6] the authors considered the same problem under very weak regularity conditions.

However, it is less known that the extra boundary gradient condition can be given as a nonlinear equation, e.g.

$$|\partial_{n_x} u| = F(|x|, t).$$

The question is what kind of nonlinearity F one has to choose in order to achieve a similar symmetry result.

The parabolic setting introduces a substantial difficulty on what kind of functions F we should choose in order to conclude that the heat balls are the only solutions to the problem. It should be noted that F would depend on the (x,t) variable. A natural choice is that F depends on the level sets of the fundamental solution $w(x,t)$ of the heat equation

$$F = F(w(x,t)).$$

We generalize the above symmetry problem by considering nonlinearities F with certain assumptions. We also introduce two- and multi-phase versions of this problem. Along with these ideas we also state and prove spherical space symmetry problems for the solutions of similar overdetermined problems.

2. BACKGROUND

2.1 Divergence theorem

Theorem 2.1 Let Ω be a bounded domain with C^1 boundary $\partial\Omega$ and let ν denote the unit outward normal to $\partial\Omega$. For any vector field w in $C^1(\bar{\Omega})$. Then we have equation (2.1), where ds indicates the $(n-1)$-dimensional area element in $\partial\Omega$.

$$\int_{\Omega} \text{div } w \, dx = \int_{\partial\Omega} w \cdot \nu \, ds \tag{2.1}$$

In particular if u is a $C^2(\bar{\Omega})$ function we have, by taking $w = Du$ in equation (2.1), we can have (2.2).

$$\int_{\Omega} \triangle u \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, ds. \tag{2.2}$$

2.2 Laplace's equation

Among the most important of all partial differential equations are undoubtedly Laplace's equation (2.3) and Poisson's equation (2.4).

$$\triangle u = 0 \quad \text{Laplace's equation,} \tag{2.3}$$

$$-\triangle u = f \quad \text{Poisson's equation.} \tag{2.4}$$

In both (2.3) and (2.4) equations, $x \in U$ and the unknown is $u : \bar{U} \to \mathbb{R}, u = u(x)$, where $U \subset \mathbb{R}^n$ is a given open set. In (2.4) the function $f : U \to \mathbb{R}$ is also given. The *Laplacian* of u is $\triangle u = \sum_{i=1}^{n} u_{x_i x_i}$. A C^2 function u satisfying equation (2.3) is called a harmonic function. The function (2.5) defined for $x \in \mathbb{R}^n$, $x \neq 0$, is the fundamental solution of Laplace's equation [7].

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi} \ln|x| & (n = 2) \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & (n \geq 3) \end{cases} \tag{2.5}$$

Consider now an open set $U \subset \mathbb{R}^n$ and suppose u is a harmonic function within U. We next derive the important mean-value formulas, which declare that $u(x)$ equals both the average of u over the sphere $\partial B(x,r)$ and the average of u over the entire ball $B(x,r)$, provided $B(x,r) \subset U$.

Theorem 2.2 (Mean-value formulas for Laplace's equation). If $u \in C^2(U)$ is harmonic, then we have (2.6) for each ball $B(x,r) \subset U$.

$$u(x) = \fint_{\partial B(x,r)} u\, ds = \fint_{B(x,r)} u\, dy \tag{2.6}$$

Proof. 1. Set

$$\phi(r) := \fint_{\partial B(x,r)} u(y) dS(y) = \fint_{\partial B(0,1)} u(x+rz) dS(z).$$

Then

$$\phi'(r) = \fint_{\partial B(0,1)} Du(x+rz) \cdot z dS(z),$$

and consequently, using Green's formulas, we compute

$$\begin{aligned}
\phi'(r) &= \fint_{\partial B(x,r)} Du(y) \cdot \frac{y-x}{r} dS(y) \\
&= \fint_{\partial B(x,r)} \frac{\partial u}{\partial \nu} dS(y) \\
&= \frac{r}{n} \fint_{B(x,r)} \triangle u(y) dy = 0.
\end{aligned}$$

Hence ϕ is constant, and so

$$\phi(r) = \lim_{t \to 0} \phi(t) = \lim_{t \to 0} \fint_{\partial B(x,t)} u(y) dS(y) = u(x).$$

2. Observe next that our employing polar coordinates gives

$$\begin{aligned}
\int_{B(x,r)} u dy &= \int_0^r \left(\int_{\partial B(x,s)} u dS \right) ds \\
&= u(x) \int_0^r n\alpha(n) s^{n-1} ds = \alpha(n) r^n u(x).
\end{aligned}$$

Theorem 2.3 (Converse to mean-value property). If $u \in C^2(U)$ satisfies

$$u(x) = \fint_{\partial B(x,r)} u\, dS$$

for each ball $B(x,r) \subset U$, then u is harmonic.

Proof. If $\triangle \not\equiv 0$, there exists some ball $B(x,r) \subset U$ such that, say, $\triangle u > 0$ within $B(x,r)$. But then for ϕ as above,

$$0 = \phi'(r) = \frac{r}{n} \fint_{B(x,r)} \triangle u(y)\, dy > 0,$$

a condradiction.

Theorem 2.4 (Strong maximum principle). Suppose $u \in C^2(U) \cap C(\bar{U})$ is harmonic within U.

(i) Then

$$\max_{\bar{U}} u = \max_{\partial U} u.$$

(ii) Furthermore, if U is connected and there exists a point $x_0 \in U$ such that

$$u(x_0) = \max_{\bar{U}} u,$$

then u is constant within U.

Assertion (i) is the maximum principle for Laplace's equation and (ii) is the strong maximum principle. Replacing u by $-u$, we recover also similar assertions with "min" replacing "max".

Proof. Suppose there exists a point $x_0 \in U$ with $u(x_0) = M \ := max_{\bar{U}} u$. Then for $0 < r < \text{dist}(x_0, \partial U)$, the mean-value property asserts

$$M = u(x_0) = \fint_{B(x_0,r)} u\, dy \leq M.$$

As equality holds only if $u \equiv M$ within $B(x_0,r)$, we see $u(y) = M$ for all $y \in B(x,r)$. Hence the set $x \in U \mid u(x) = M$ is both open and relatively closed in U, and thus equals U if U is connected. This proves assertion (ii), from which (i) follows.

2.3 Heat equation

The heat equation (2.7) and the nonhomogeneous heat equation (2.8) are given, where $t > 0$ and $x \in U$, where $U \subset \mathbb{R}^n$ is open.

$$u_t - \triangle u = 0 \tag{2.7}$$

$$u_t - \triangle u = f \tag{2.8}$$

The unknown is $u : \bar{U} \times [0,\infty) \to \mathbb{R}, u = u(x,t)$, and the Laplacian $\triangle$ is taken with respect to the spatial variables $x = (x_1,...,x_n) : \triangle u = \triangle_x u = \sum_{i=1}^{n} u_{x_i x_i}$. In equation (2.8) the function $f : U \times [0,\infty) \to \mathbb{R}$ is given. The function (2.9) is called the fundamental solution of the heat equation [7].

$$\Phi(x,t) := \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & (x \in \mathbb{R}^n, t > 0) \\ 0 & (x \in \mathbb{R}^n, t < 0) \end{cases} \tag{2.9}$$

For the heat operator $H = \Delta - \partial_t$, we have the following fact.

Fact 1. If $Hu = -\delta_0$, and $u_r(x,t) = r^n u(rx, r^2 t)$, then $Hu_r = -\delta_0$. To see this, let $\varphi \in C_0^\infty$ and $\tilde{H} = \Delta + \partial_t$. Then we have

$$\begin{aligned} < Hu_r, \varphi > &= < u_r, \tilde{H}\varphi > = r^n \iint u(rx, r^2 t) \tilde{H}\varphi(x,t) dx dt \\ &= r^{-2} \iint u(\xi,\tau) \tilde{H}\varphi(\frac{\xi}{r}, \frac{\tau}{r^2}) d\xi d\tau = \iint u(\xi,\tau) \tilde{H}\varphi(\xi,\tau) d\xi d\tau \\ &= < u, \tilde{H}\varphi > = < Hu, \varphi > = < -\delta_0, \varphi > . \end{aligned}$$

Fact 2 (Dini continuity). A modulus of continuity α is called a Dini modulus of continuity if

$$\int_{0^+} \frac{\alpha(r)}{r} dr < \infty$$

and a function h is called Dini continuous if h has a Dini modulus of continuity [8].

Fact 3 (Hopf's Lemma). Let $C_\delta(x,t)$ be $B_\delta(x) \times (t - \delta^2, t + \delta^2)$ and let $u \geq 0$ be a weak solution to the heat equation and $u > 0$ in the interior of $\Omega \cap C_\delta(x_0,t_0)$ and that $u(x_0,t_0) = 0$. Assume that Ω is $C^{1,Dini}$ at (x_0,t_0). $C^{1,Dini}$ means the boundary is C^1 with a Dini continuous normal. Then

$$\partial_{n_x} u(x_0,t_0) \geq C > 0,$$

where n_x denotes inward space normal direction and C is a constant [8].

3. AN ELLIPTIC OVERDETERMINED PROBLEM

3.1 One-phase case (Exploratory)

Suppose we are given the overdetermined problem (3.1) for some bounded C^1 domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$).

$$\begin{aligned} &\Delta u = -c_0 \delta_0 \quad \text{in} \quad \Omega, \\ &u = 0 \quad \text{on} \quad \partial\Omega, \\ &\partial_\nu u = F(|x|) \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{3.1}$$

A bounded domain is of class C^1 if its boundary may be locally represented as the graph of finitely many C^1 functions, i.e. continuous functions that have continuous first order derivatives.

A question that has challenged several mathematicians is whether Ω is a ball. A natural follow-up question would then be the uniqueness of the solution. If we replace the boundary normal derivative with $|\nabla u|(x) = F(|x|)$, then the smoothness assumption on the boundary can be relaxed considerably. For example one can allow this boundary gradient condition to hold a.e. on the boundary, and still ask the same question [3]. However, in the discussion to follow below, we will only consider C^1 domains.

The departing point of any analysis of this problem would be to set the right conditions on F, so that an appropriate ball can be a solution. Indeed, a simple integration gives

$$\int_{\partial\Omega} F ds = \int_{\partial\Omega} \partial_\nu u ds = \int_\Omega -\Delta u dx = c_0.$$

In particular if Ω is the ball $B_R(0)$ then

$$F(R)\ \omega_n\ R^{n-1} = c_0,$$

where ω_n is the surface area of the unit sphere. This suggests that if for any R we have the above condition fulfilled then the ball $B_R(0)$ and its Green's function would be a solution to our problem. In particular there would be no solution if the condition above is not satisfied for any ball $B_R(0)$. The uniqueness also fails when the above condition holds for more than one R. For example if

$$F(t) := c_0 t^{1-n}/\omega_n$$

then all balls with centers at the origin are solutions to our problem.

In proving that a solution must be a ball, one can use standard argument of scaling and comparison between the solution and the scaled version of it. So let us start with a solution u and the corresponding Ω to our problem (3.1). Let us first set $c_0 = \omega_n$ for simplicity. Then the requirement that the appropriate ball is a solution is $F(R) = R^{1-n}$ for at least one R. Suppose that this condition fails. Then one can easily see that there cannot be any solution (u,Ω) to our problem. Indeed, suppose $F(R) > R^{1-n}$, for all R. Let us take the smallest ball $B_r(0) \supset \Omega$ and its Green's function G_r. Then by comparison principle $u \leq G_r$ in Ω and hence

$$r^{1-n} = \partial_\nu G_r(z) \geq \partial_\nu u(z) = F(r)$$

where z is a touching point between the boundary of Ω and the sphere $|x| = r$. This contradicts that $F(R) > R^{1-n}$ for all R. A similar argument (taking largest ball from inside) also shows the failure of existence when the reverse condition $F(R) < R^{1-n}$ holds.

Let us now look for further conditions that forces solutions to be spherical. Assume we have a solution (u,Ω) and also that (G_r, B_r) is the corresponding ball solution. If $B_r \setminus \Omega \neq \emptyset$ then we may scale so that $B_{tr} \subset \Omega$ and it touches the boundary of Ω at z, and that $t > 1$. Then one can easily show that $v(x) := t^{n-2}G_r(tx)$ satisfies $\Delta v = -\omega_n$ and hence by comparison principle $v \leq u$ in B_{tr}. In particular

$$\partial_\nu (v-u)(z) \leq 0$$

where ν is the inward normal direction, resulting in

$$t^{n-1}F(t|z|) \leq F(|z|).$$

If we assume that $r^{n-1}F(r)$ is strictly increasing (or just increasing for $C^{1,\mathrm{dini}}$ domains, by the use of Hopf's lemma) then the above inequality results in

$$t^{n-1}F(t|z|) \leq F(|z|) = \frac{|z|^{n-1}}{|z|^{n-1}}F(|z|) < t^{n-1}F(t|z|).$$

It is noteworthy that the above condition on F can be relaxed considerably. Indeed, it would be enough to assume that $T(r) := r^{n-1}F(r) - 1$ vanishes at only one point and that $T(r) < 0$ for small values of r and $T(R) > 0$ for large values of R. To see this we need a different argument than scaling. So let us again consider the largest ball B_r inside our domain and the smallest one B_R containing it. A similar comparison argument as above gives that the inward normal derivative of $u - G_r$ and $G_R - u$ are non-negative ($r < R$). In particular we will have

$$\partial_\nu(u - G_r) \geq 0, \quad \partial_\nu(G_R - u) \leq 0$$

$$F(r) \geq r^{1-n}, \qquad F(R) \leq R^{1-n}$$

or that $T(r) := r^{n-1}F(r) - 1 \geq 0$ and $T(R) := R^{n-1}F(R) - 1 \leq 0$. But this along with the conditions on T implies that T vanishes at two points, at least. A contradiction.

3.2 Two-phase case

In this section we will consider the two-phase counterpart of the symmetry problem. We will need to set the conditions on the function F to ensure that our arguments will go through.

Definition 3.1 (Nonlinear joining condition.) We will denote by $F = F(t)$ a continuous increasing function with the property (3.2).

$$F(t) = \begin{cases} < b\, t^{n-1} & \text{for } t \text{ large enough} \\ > b\, t^{n-1} & \text{for } t \text{ small enough} \\ = b\, t^{n-1} & \text{for just one } t \end{cases} \tag{3.2}$$

Here

$$b = \begin{cases} c_0/\omega_n\, (n-2)^{2-n}\, c^{1-n}, & n \geq 3 \\ c_0/(2\pi\, \bar{c}), & n = 2 \end{cases}$$

and $\bar{c}$ denotes a constant.

Let us further set $u^+ := \max(u,0)$ and $u^- := \min(u,0)$, δ_0 denotes the Dirac delta function.

Theorem 3.1 Let Ω be a bounded smooth domain in $\mathbb{R}^n$ with ν the normal vector on $\partial\Omega$ pointing towards the interior of Ω. Suppose we have a solution to the problem (3.3).

$$
\begin{aligned}
&\Delta u = -c_0\delta_0 \qquad \text{in} \quad \mathbb{R}^n \backslash \partial\Omega,\\
&u = 0 \qquad \text{on} \quad \partial\Omega,\\
&u \to -c \quad \text{as} \quad |x| \to \infty, \quad n \geq 3\\
&u \to -\infty \quad \text{as} \quad |x| \to \infty, \quad n = 2\\
&\partial_\nu u^+ = F(\partial_\nu u^-)
\end{aligned}
\tag{3.3}
$$

Then, Ω is a sphere centered at the origin.

Proof. Let us take the largest sphere $B_{R_1}(0) \subset \Omega$ centered at the origin, touching $\partial\Omega$ at some point x^1. Denoting the Green's function corresponding to the sphere with radius R_1 by G_{R_1}, we have

$$
\begin{aligned}
&\Delta(u^+ - G^+_{R_1}) = 0 \quad \text{in} \quad B_{R_1}(0),\\
&u^+ - G^+_{R_1} \geq 0 \quad \text{on} \quad \partial B_{R_1}(0),
\end{aligned}
$$

and by maximum principle of harmonic functions, we get

$$
G^+_{R_1} \leq u^+ \quad \text{in} \quad B_{R_1}(0).
$$

If we consider the normals at the point x^1,

$$
\partial_\nu G^+_{R_1}(x^1) \leq \partial_\nu u^+(x^1),
$$

and by the last equation (3.3) we have (3.4).

$$
\partial_\nu G^+_{R_1}(x^1) \leq \partial_\nu u^+(x^1) = F\left(\partial_\nu u^-(x^1)\right) \tag{3.4}
$$

On the other hand, we may take

$$
\begin{aligned}
&G^-_{R_1} = c\left(\frac{|x|^{2-n}}{R_1^{2-n}} - 1\right),\\
&G^-_{R_1} - u^- \to 0 \qquad \text{as} \quad |x| \to \infty,
\end{aligned}
$$

and have

$$G_{R_1}^- \le u^- \le 0 \quad \text{on} \quad \partial\Omega,$$
$$G_{R_1}^- \le u^- \quad \text{in} \quad \Omega^c.$$

By considering the normals at the point x^1,

$$\partial_\nu u^-(x^1) \le \partial_\nu G_{R_1}^-(x^1).$$

Since $F(t)$ is increasing the inequality (3.5) is right.

$$F\left(\partial_\nu u^-(x^1)\right) \le F\left(\partial_\nu G_{R_1}^-(x^1)\right). \tag{3.5}$$

Thus, (3.4) and (3.5) give (3.6).

$$\partial_\nu G_{R_1}^+(x^1) \le F\left(\partial_\nu G_{R_1}^-(x^1)\right) \tag{3.6}$$

By symmetry property of the Green's function, the inequality (3.6) holds for every $x \in \partial B_{R_1}$.

Now, let us take the smallest sphere $B_{R_2}(0) \supset \Omega$ centered at the origin, touching $\partial\Omega$ at some point x^2. Denoting the Green's function corresponding to the sphere with radius R_2 by G_{R_2}, a similar reasoning gives inequality (3.7).

$$F\left(\partial_\nu G_{R_2}^-(x^2)\right) \le \partial_\nu G_{R_2}^+(x^2) \quad \text{on} \quad \partial B_{R_2} \tag{3.7}$$

For $n \ge 3$,

$$G_R^+(x) = \frac{c_0}{\omega_n}\left(|x|^{2-n} - R_1^{2-n}\right)$$
$$G_R^-(x) = c\left(R_1^{n-2}|x|^{2-n} - 1\right).$$

Hence we have equations (3.8) and (3.9).

$$\partial_\nu G_{R_1}^+(x^1) = \frac{c_0}{\omega_n}(n-2)R_1^{1-n}, \tag{3.8}$$
$$\partial_\nu G_{R_1}^-(x^1) = c\,(n-2)R_1^{-1}. \tag{3.9}$$

Equations (3.8) and (3.9) together with (3.6) gives (3.10).

$$(n-2)\frac{c_0}{\omega_n} R_1^{1-n} \le F\left(c\,(n-2)R_1^{-1}\right)$$

If we let $r_1 = c(n-2)R_1^{-1}$, then the inequality (3.10) becomes a result.

$$F(r_1) \geq b\, r_1^{n-1}. \tag{3.10}$$

Similarly, for $r_2 = c(n-1)R_2^{-1}$, we obtain an inequality (3.11).

$$F(r_2) \leq b\, r_2^{n-1}. \tag{3.11}$$

Equations (3.10) and (3.11) contradicts (3.2).

For $n = 2$, we have

$$G_R^+(x) = \frac{c_0}{2\pi}\left(\ln|x|^{-1} - \ln R_1^{-1}\right),$$
$$G_R^-(x) = \bar{c}\left(\ln|x|^{-1} - \ln R_1^{-1}\right),$$

and equations (3.12), (3.13).

$$\partial_\nu G^+ R_1(x^1) = \frac{c_0}{2\pi} R_1^{-1}, \tag{3.12}$$
$$\partial_\nu G^- R_1(x^1) = \bar{c}\, R_1^{-1}. \tag{3.13}$$

Equations (3.12) and (3.13) together with (3.6) gives

$$\frac{c_0}{2\pi} R_1^{-1} \leq F(\bar{c}\, R_1^{-1}).$$

If we let $r_1 = \bar{c}\, R_1^{-1}$, then we have an inequality (3.14).

$$F(r_1) \geq b\, r_1. \tag{3.14}$$

Similarly, for $r_2 = \bar{c}\, R_2^{-1}$, we obtain (3.15).

$$F(r_2) \leq b\, r_2. \tag{3.15}$$

Equations (3.14) and (3.15) contradicts (3.2).

It follows easily that the function $F(t)$ can be assumed to depend on $|x|$ as well. Then, $F(|x|,t)$ is an increasing function in t having the same property given by (3.2) for $n \geq 2$. Hence, the inequality (3.10) becomes (3.16).

$$F\left(\frac{a}{r_1}, r_1\right) \geq b\, r_1^{n-1} \tag{3.16}$$

The inequality (3.11) becomes (3.17).

$$F\left(\frac{a}{r_2}, r_2\right) \leq b\, r_2^{n-1} \tag{3.17}$$

Equations (3.16) and (3.17) contradicts (3.2).

3.3 The Radial Green's and Capacitor function

In this section we compute explicitly the Green's functions and the capacitor potential of a sequence of spherical rings, that we will use as barriers from below and from above in the sequel.

Define the p-Laplacian

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u) \qquad (1 < p < \infty).$$

For a given positive integer m, and positive real numbers

$$0 < r_1 < r_2 < \cdots < r_m < \infty, \qquad 0 = \alpha_1 < \alpha_2 < \cdots < \alpha_m < \alpha_{m+1} < \infty$$

consider $B_{r_i}(0)$ and the corresponding Green's/capacitor functions G_i:

$$\Delta_p G_i = 0, \quad \text{in } B_{r_i} \setminus B_{r_{i-1}}, \qquad i = 2, \cdots, m$$

$$G_i = -\alpha_i, \quad \text{on } \partial B_{r_i}, \qquad G_i = -\alpha_{i-1}, \quad \text{on } \partial B_{r_{i-1}}, \quad i = 2, \cdots, m$$

and for $i = 1$ and $i = m+1$

$$\Delta_p G_1 = -c_0\delta_0, \quad \text{in } B_{r_1}, \qquad G_1 = 0, \quad \text{on } \partial B_{r_1}$$

and

$$\Delta_p G_{m+1} = 0, \quad \text{in } \mathbb{R}^n \setminus B_{r_m}, \quad G_{m+1} = -\alpha_m, \quad \text{on } \partial B_{r_m},$$

$$\lim_{x\to\infty} G_{m+1} = -\alpha_{m+1}, \quad p < n, \quad \lim_{x\to\infty} |x|^{\frac{n-p}{p-1}} G_{m+1} = -\alpha_{m+1} \quad p > n.$$

We can compute explicitly all these functions, and we obtain results (3.18 - 3.24).
Case: $n \neq p$

$$G_1 = \frac{c_0}{\omega_n}\left| |x|^{\frac{p-n}{p-1}} - r_1^{\frac{p-n}{p-1}} \right|, \tag{3.18}$$

$$G_i = (-\alpha_i + \alpha_{i-1})\,\frac{|x|^{\frac{p-n}{p-1}} - r_i^{\frac{p-n}{p-1}}}{r_i^{\frac{p-n}{p-1}} - r_{i-1}^{\frac{p-n}{p-1}}} - \alpha_i, \quad i = 2, \cdots, m, \tag{3.19}$$

$$G_{m+1} = (\alpha_{m+1} - \alpha_m)\, r_m^{\frac{n-p}{p-1}} |x|^{\frac{p-n}{p-1}} - \alpha_{m+1}, \quad p < n, \tag{3.20}$$

$$G_{m+1} = -\alpha_{m+1}\, r_m^{\frac{n-p}{p-1}} |x|^{\frac{p-n}{p-1}}, \quad p > n. \tag{3.21}$$

Case: $n = p$

$$G_1 = \frac{c_0}{\omega_n}\,(\ln|x|^{-1} - \ln r_1^{-1}), \tag{3.22}$$

$$G_i = (-\alpha_i + \alpha_{i-1})\,\frac{\ln|x|^{-1} - \ln r_i^{-1}}{\ln r_i^{-1} - \ln r_{i-1}^{-1}} - \alpha_i, \quad i = 2, \cdots, m, \tag{3.23}$$

$$G_{m+1} = -\alpha_{m+1}\,(\ln|x|^{-1} - \ln r_m^{-1}) - \alpha_m. \tag{3.24}$$

3.4 Multi-phase case

In this section we consider the multi-phase version of overdetermined problem discussed in sections 3.1 and 3.2. To start we need some definitions.

Definition 3.2 For the multi phase case we will define $F_i(A,B),\quad i = 1, \cdots, m$ with the properties (3.26 - 3.30).

$$F_i(A,B) < F_i(A_1,B_1), \quad \text{for} \quad (A,B) < (A_1,B_1), \quad i = 1, \cdots, m \tag{3.25}$$

$$F_1(A,B) \geq CB^{\frac{1+\alpha}{p-1}} - A \quad (0 < \alpha < n-2) \quad \text{for } A,B \text{ large enough} \tag{3.26}$$

$$F_i(A,B) \geq CB - A \quad \text{for } A,B \text{ large enough}, \quad i = 2, \cdots, m \tag{3.27}$$

$$F_i(A,B) \leq CB^{\frac{n-1}{p-1}} - A \quad \text{for } A,B \text{ small enough}, \quad i = 1, \cdots, m-1 \tag{3.28}$$

$$F_m(A,B) \leq CB^{\alpha} - A \quad (\alpha < 1) \quad \text{for } A,B \text{ small enough}, \tag{3.29}$$

Here $(A,B) < (A_1,B_1)$ means either $A < A_1$ or $B < B_1$ or both and C is a constant.

Theorem 3.2 Let Ω_i $(i = 1, \cdots, m)$ be bounded smooth domains in $\mathbb{R}^n$ with $\Omega_{i-1} \subset \Omega_i$, $0 = \alpha_1 < \alpha_2 < \cdots < \alpha_{m+1}$, and suppose there exist u_i $(i = 1, \cdots, m)$ solving the following problem (3.30) along with the boundary gradient condition (3.31).

$$\Delta_p u_1 = -c_0 \delta_0 \quad \text{in} \quad \Omega_1,$$
$$\Delta_p u_i = 0, \quad \text{in} \quad \Omega_i \backslash \Omega_{i-1}, \quad (i = 2, \cdots, m)$$

$$\begin{aligned}
&\Delta_p u_{m+1} = 0, \quad \text{in} \quad \mathbb{R}^n \backslash \Omega_m, \\
&u_i = u_{i-1} = -\alpha_{i-1}, \quad \text{on} \quad \partial\Omega_{i-1}, \quad (i = 2, \cdots, m+1) \\
&u_{m+1} \to -\alpha_{m+1} \quad \text{as} \quad |x| \to \infty, \quad 1 < p < n, \\
&|x|^{\frac{n-p}{p-1}} u_{m+1} \to \; -\alpha_{m+1} \quad \text{as} \quad |x| \to \infty, \quad n < p, \\
&u_{m+1}/\ln|x|^{-1} \to \; -\alpha_{m+1} \quad \text{as} \quad |x| \to \infty, \quad n = p.
\end{aligned} \tag{3.30}$$

$$F_i(\partial_\nu u_i, \partial_\nu u_{i+1}) = 0, \quad i = 1, \cdots, m \qquad \textbf{(3.31)}$$

Then, Ω_i $(i = 1, \cdots, m)$ are balls centered at the origin.

Remark 3.1 In the above theorem the Dirac source δ_0 can be replaced by a Dirichlet data on $B_s(0) \subset \Omega$ (for some s) and/or Ω_m with ball $B_{r_m}(0)$. Then one may need to modify the assumptions on the functions F_i slightly.

Proof. We split the proof into two cases.

Case A: $n \neq p$.

Step 1: (Largest ball from inside.)

Let us first consider the largest ball $B_{r_i} \subset \Omega_i$ $(i = 1, 2, \cdots, m)$ and denote G_i, the capacitor potential for each ring-shaped region $B_{r_i} \setminus B_{r_{i-1}}$ $(i = 2, 3, \cdots, m)$. For B_{r_1} we let G_1 denote the Green's function with source $-c_0\delta_0$ and for $\mathbb{R}^n \setminus B_{r_m}$ we let G_m be the harmonic function in $\mathbb{R}^n \setminus B_{r_m}$ with $G_{m+1} = -\alpha_m$ on ∂B_{r_m} and $G_{m+1} = u_{m+1}$ at infinity. For G_1 and G_i, we have

$$G_1 \leq u_1 \quad \text{in} \quad B_{r_1}(0),$$
$$G_i \leq u_i \quad \text{in} \quad B_{r_i}(0) \backslash \Omega_{i-1}.$$

Let $x^i \in \partial B_{r_i} \cap \partial \Omega_i$. Then we have inequalities (3.33 - 3.36).

$$\partial_\nu G_1 \leq \partial_\nu u_1 \quad \text{at} \quad x^1, \qquad \textbf{(3.32)}$$

$$\partial_\nu G_i \geq \partial_\nu u_i \quad \text{at} \quad x^{i-1}, \qquad \textbf{(3.33)}$$

$$\partial_\nu G_i \leq \partial_\nu u_i \quad \text{at} \quad x^i, \quad i = 2, 3, \cdots, m, \qquad \textbf{(3.34)}$$

$$\partial_\nu G_{m+1} \geq \partial_\nu u_{m+1} \quad \text{at} \quad x^m. \qquad \textbf{(3.35)}$$

By using these inequalities and considering the monotonicity of F_i, see (3.25), we get inequality (3.36).

$$0 = F_i(\partial_\nu u_i, \partial_\nu u_{i+1}) \geq F_i(\partial_\nu G_i, \partial_\nu G_{i+1}), \quad i = 1, \cdots, m \qquad \textbf{(3.36)}$$

Let us define equations (3.37), (3.38) and (3.39) , for $i = 2, \cdots, m-1$, and $y = (y_1, \cdots, y_m)$.

$$T_i(y) := F_i\left(\frac{A_i\, y_i^{-1}}{(y_i/y_{i-1})^{\frac{n-p}{p-1}} - 1}, \frac{B_i\, y_i^{-1}}{1 - (y_i/y_{i+1})^{\frac{n-p}{p-1}}}\right) \qquad \textbf{(3.37)}$$

and

$$T_1(y) := F_1\left(A_1\, y_1^{\frac{1-n}{p-1}},\ \frac{B_1\, y_1^{-1}}{1-(y_1/y_2)^{\frac{n-p}{p-1}}}\right), \tag{3.38}$$

$$T_m(y) := F_m\left(\frac{A_m\, y_m^{-1}}{(y_m/y_{m-1})^{\frac{n-p}{p-1}}-1},\ B_m\, y_m^{-1}\right). \tag{3.39}$$

Here

$$\begin{aligned}
&A_1 = \frac{c_0}{\omega_n}\,\frac{|n-p|}{p-1}, \quad B_1 = \alpha_2\,\frac{|n-p|}{p-1},\\
&A_i = (\alpha_i - \alpha_{i-1})\,\frac{n-p}{p-1}, \quad B_i = (\alpha_{i+1}-\alpha_i)\,\frac{n-p}{p-1}, \quad i = 2,\cdots,m-1,\\
&A_m = (\alpha_m - \alpha_{m-1})\,\frac{n-p}{p-1}, \quad B_m = (\alpha_{m+1}-\alpha_m)\,\frac{n-p}{p-1}, \quad n>p,\\
&A_m = (\alpha_m - \alpha_{m-1})\,\frac{p-n}{p-1}, \quad B_m = \alpha_{m+1}\,\frac{p-n}{p-1}, \quad n<p.
\end{aligned}$$

Finally define, $T(y) = (T_1(y),\cdots,T_m(y))$. We need to remark that

$$T_{i-1}(y) \geq T_{i-1}(y'), \quad T_{i+1}(y) \geq T_{i+1}(y') \qquad \text{if} \quad y_i < y_i' \qquad \text{and} \quad y_j = y_j', \quad j \neq i.$$

Let us also use the notation $T(y) \leq 0$ if the inequality holds for all components T_i. From (3.36) we have that $T(\bar{r}) \leq 0$, where $\bar{r} = (r_1,\cdots,r_m)$ are the radii of the balls. Next consider the domain

$$D := \{y : T(y) \leq 0\}.$$

The idea is to prove that for $y = (y_1,\cdots,y_m) \in D$, we have $y_i > s_0 > 0$ for some s_0, and for all $i = 1,\cdots,m$.

From now on we will also let C_i be constants, that might change value, depending only on the ingredients such as $n, m, c_0, \cdots$.

It should be noted that while working with the largest balls from inside, we will take y_i small, so that y_i^{-1} is large and will use the assumptions on $F_i(A,B)$, $i = 1,\cdots,m$ where A and B are large enough.

Let $y \in D$, then we extract from (3.27) for F_m and from (3.36) the inequality (3.40).

$$\frac{A_m\, y_m^{-1}}{(y_m/y_{m-1})^{\frac{n-p}{p-1}}-1} \leq C\, B_m\, y_m^{-1}. \tag{3.40}$$

and

$$\frac{1}{(y_m/y_{m-1})^{\frac{n-p}{p-1}}-1} \leq C_1,$$

so that

$$\frac{1}{1-(y_{m-1}/y_m)^{\frac{n-p}{p-1}}} \leq 1+C_1.$$

Now we will use this for the next step, F_{m-1}, and see that

$$\frac{1}{(y_{m-1}/y_{m-2})^{\frac{n-p}{p-1}}-1} \leq C_2,$$

and as before

$$\frac{1}{1-(y_{m-2}/y_{m-1})^{\frac{n-p}{p-1}}} \leq 1+C_2.$$

Iterating this all the way down to $i=1$ we obtain inequality (3.41).

$$\frac{1}{1-(y_1/y_2)^{\frac{n-p}{p-1}}} \leq 1+C_{m-1} \tag{3.41}$$

On the other hand, the equation (3.26) gives inequality (3.42).

$$A_1\, y_1^{\frac{1-n}{p-1}} \leq C \left(\frac{B_1\, y_1^{-1}}{1-(y_1/y_2)^{\frac{n-p}{p-1}}}\right)^{\frac{1+\alpha}{p-1}} \tag{3.42}$$

(3.41) and (3.42) gives us

$$y_1^{\frac{n-2-\alpha}{p-1}} \geq C_m,$$

and we conclude with the inequality (3.43) uniformly for all $r_i,\ i=1,\cdots,m$.

$$y_1 \geq C_{m+1}, \tag{3.43}$$

This proves that all y_i are confined within the convex cone

$$D \subset \{y_1 > s_0,\ y_i > s_0 y_{i-1},\ i=2,\cdots,m\},$$

for some constant $s_0>0$.

Let us now take the smallest element ρ in D, i.e. if for any y with $T(y) \leq 0$ we have $\rho \leq y$. In particular this means that there is an element $\rho \in D$ with $T(\rho)=0$. Indeed, if this fails, then for some i we have $T_i(\rho) \leq 0$. If we decrease ρ_i to $\rho_i - \varepsilon$, for small enough $\varepsilon > 0$, and set $\rho^\varepsilon = (\rho_1,\cdots,\rho_i-\varepsilon,\cdots,\rho_m)$, then by continuity $T_i(\rho^\varepsilon)<0$. It is also apparent that changing ρ_i will only give rise to changes of the value T_i, T_{i-1}, T_{i+1}, for $i=2,\cdots,m-1$. For $i=1$, the changes occur only for two elements T_1, T_2, and for $i=m$ the changes occur only for two elements T_{m-1}, T_m.

Using monotonicity of T_i, it is seen that we should have $T_{i-1}(\rho^{\varepsilon}) < 0$ and $T_{i+1}(\rho^{\varepsilon}) < 0$. Hence the minimality of ρ is violated. Thus, for an element $\rho \in D$ we must have $T(\rho) = 0$.

Step 2: (Smallest ball from outside.)

Let us now take a reverse situation. Let B_{R_i} be the smallest ball containing Ω_i, with the corresponding Green's functions G_i. Then a similar argument as in the previous case shows that $G_i \geq u_i$ and considering the monotonicity of F_i, see (3.25), we get inequality (3.44).

$$0 = F_i(\partial_\nu u_i, \partial_\nu u_{i+1}) \leq F_i(\partial_\nu G_i, \partial_\nu G_{i+1}), \quad i = 1, \cdots, m. \tag{3.44}$$

Now we use a similar iteration as we did in the earlier case. As in the previous case, we define $T(y) = (T_1(y), \cdots, T_m(y))$ and $T(\overline{R}) \geq 0$, where $\overline{R} = (R_1, \cdots, R_m)$ are the radii of the balls. We need to show the estimate (3.40) and the further ones. Next consider the domain

$$D' := \{y : T(y) \geq 0\}.$$

We will take y_i large, so that y_i^{-1} is small. We start with F_1, using (3.28) and (3.44), we obtain

$$C\left(\frac{B_1\, y_1^{-1}}{1-(y_1/y_2)^{\frac{n-p}{p-1}}}\right)^{\frac{n-1}{p-1}} \leq A_1\, y_1^{\frac{1-n}{p-1}}.$$

Hence we have

$$\frac{1}{1-(y_1/y_2)^{\frac{n-p}{p-1}}} \leq C_1\ ,$$

and consequently

$$\frac{1}{(y_2/y_1)^{\frac{n-p}{p-1}}-1} \leq C_1 - 1.$$

In analogy with Step 1, we can use this estimate, along with (3.44) to derive a similar estimate

$$\frac{1}{(y_3/y_2)^{\frac{n-p}{p-1}}-1} \leq C_2 - 1.$$

Iterating this up to $i = m$, we obtain

$$\frac{1}{(y_m/y_{m-1})^{\frac{n-p}{p-1}}-1} \leq C_{m-1} - 1.$$

For y_m we use the estimate for F_m, and have

$$C\left(B_m\, y_m^{-1}\right)^{\alpha} \leq \frac{A_m\, y_m^{-1}}{(y_m/y_{m-1})^{\frac{n-p}{p-1}}-1}$$

$$y_m^{1-\alpha} \leq C_m$$

that simplifies to inequality (3.45).

$$y_m \leq C_{m+1} \tag{3.45}$$

Then according to our analysis above we have that

$$D' \subset \{y_m < s_0,\ y_i > s_0 y_{i-1},\ i = m, \cdots, 2\},$$

where the latter cone is bounded. Now, a similar argument used in the previous case, gives us that the largest element $\rho' \in D'$ must be so that $T(\rho') = 0$.

Step 3: (Putting things together.)

From the above two cases we see that we will have two values ρ, ρ' for which T becomes zero. Since F_i are strictly increasing this gives us a contradiction that we were looking for.

Case B: $n = p$.

The same argument can be used for $n = p$. We need to show (3.43) and (3.45), the rest of the prove will follow as in the previous steps. For $n = p$, while working with the largest balls from inside, we start with F_m, use (3.27) and (3.36), we obtain

$$\frac{(\alpha_m - \alpha_{m-1})y_m^{-1}}{\ln(y_m/y_{m-1})} \leq C\,\bar{c}\, y_m^{-1},$$

which gives

$$\frac{1}{\ln(y_m/y_{m-1})} \leq C_1.$$

Using this for the next step and iterating up to i=1 gives

$$\frac{1}{\ln(y_2/y_1)} \leq C_{m-1},$$

and with the assumption on F_1 we get

$$\frac{c_0}{\omega_n}\, y_1^{-1} \leq C\left(\frac{\alpha_2 y_1^{-1}}{\ln(y_2/y_1)}\right)^{\frac{1+\alpha}{p-1}},$$

$$y_1^{1-\frac{1+\alpha}{p-1}} \geq C_m,$$

which gives $y_1 \geq C_{m+1}$. While working with the smallest balls from outside, we start with F_1, we get

$$\frac{C\,\alpha_2\,y_1^{-1}}{\ln(y_2/y_1)} \leq \frac{c_0}{\omega_n}\,y_1^{-1},$$

which gives

$$\frac{1}{\ln(y_2/y_1)} \leq C_1,$$

by iteration

$$\frac{1}{\ln(y_m/y_{m-1})} \leq C_{m-1}.$$

Finally by using the assumption on F_m, we obtain

$$C\left(\bar{c}\,y_m^{-1}\right)^{\alpha} \leq \frac{(\alpha_m - \alpha_{m-1})y_m^{-1}}{\ln(y_m/y_{m-1})},$$

$$y_m^{1-\alpha} \leq C_m,$$

which gives $y_m \leq C_{m+1}$. Hence the rest of the proof becomes straightforward.

4. A PARABOLIC OVERDETERMINED PROBLEM

We will give the general notation used in the parabolic version of our problem together with some definitions. Ω will be a domain in $\mathbb{R}^{n+1}$. All functions considered will be assumed to be C_x^1 up to the boundary of their domain of definition. $u^+ = \max(u,0)$ and $u^- = \min(u,0)$. $w(x,t)$ will denote the heat kernel (4.1) for $(x,t) = (x_1, x_2, \cdots, x_n, t) \in \mathbb{R}^n \times \mathbb{R}^+$.

$$w(x,t) = (4\pi t)^{-n/2}\exp(-\frac{|x|^2}{4t}) \qquad \textbf{(4.1)}$$

For $t \leq 0$, we let $w(x,t) \equiv 0$. For $\lambda > 0$, we define heat balls

$$D_\lambda = \{(x,t) : w(x,t) > \lambda\}.$$

$Q(x_0,t_0,r)$ is the cylinder $\{(x,t) : |x - x_0| < r, \quad 0 < t_0 - t < r^2\}$. For a bounded domain $\Omega \subset \mathbb{R}^{n+1}$, we define the parabolic boundary $\partial_{par}\Omega$ to be the set of all points $(x_0,t_0) \in \partial\Omega$ such that for any $\varepsilon > 0$, the cylinder $Q(x_0,t_0,\varepsilon)$ contains points not in Ω.

Condition 4.1 Throughout this section the boundaries of the domains will be assumed to be $C_{x,t}^{1Dini,1/2}$ type, i.e., the boundary is C^1 in the variable x with a Dini continuous normal; $C^{1/2}$ in the t direction.

Condition 4.2 All the domains in this section are assumed to contain some interval of the type $\{(0,t) : t \in (0,T)\}$, so that we can take heat balls in these domains. The domains considered here are bounded and have the property that there exist heat balls containing them. It is also assumed that the largest heat ball inside the domain and the smallest heat ball outside the domain touch the boundary $\partial_{par}\Omega$ at some points (x_1,t_1) and (x_2,t_2) with $t_1, t_2 > 0$.

4.1 One-phase case

Suppose we are given the overdetermined problem for the heat equation (4.2) under the Conditions 4.1 and 4.2.

$$\begin{cases} Hu = -\delta_0 \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial_{par}\Omega, \\ \partial_{n_x} u = \frac{|x|}{t} F(w(x,t)) \quad \text{on} \quad (\partial_{par}\Omega) \backslash \{t=0\}, \end{cases} \tag{4.2}$$

Here F is a continuous function and w is the heat kernel (4.1). If $(x,t) \in \partial D_\lambda$, then

$$|\partial_{n_x} w(x,t)| = \lambda \frac{|x|}{2t}.$$

The Green's function of D_λ can be written as

$$G_\lambda = w(x,t) - \lambda.$$

Let us now set the right conditions on F for our problem (4.2), so that an appropriate heat ball can be a solution. Since

$$\partial_{n_x} G_\lambda = \lambda \frac{|x|}{2t}, \quad \text{on} \quad \partial D_\lambda,$$

we need $F(\lambda) = \lambda/2$ to hold for at least one λ. Then D_λ is a solution to our one-phase problem (4.2). In particular, if $F(\lambda) \neq \lambda/2$ for any λ, then we cannot have a solution of the type D_λ.

Remark 4.1 The same problem in the elliptic case was considered before. It can be noted that in this case the function F in (4.2) was taken as a function of $|x|$, the level sets of the fundamental solution.

Before stating the first theorem for the one-phase case, we need the following condition.

Condition 4.3 $F(s)$ is a continuous function having properties in (4.3).

$$F(s) = \begin{cases} < s/2 \quad \text{for } s > s_0, \\ > s/2 \quad \text{for } s < s_0, \\ = s_0/2 \quad \text{for just one } s_0. \end{cases} \tag{4.3}$$

Theorem 4.1 Suppose that there exists a non-constant u solving the overdetermined problem (4.2) under the Conditions 4.1 and 4.2. Then, Ω is a heat ball.

Proof. Let us take the largest heat ball $D_{\bar{\lambda}}$ in Ω which touches the boundary at some point $(\bar{x},\bar{t})$ with $\bar{t}>0$ (by Condition 4.2) and consider its corresponding Green's function $G_{\bar{\lambda}}$. For the touching point $(\bar{x},\bar{t})$ on the boundary we write $w(\bar{x},\bar{t})=\bar{\lambda}$. By strong comparison principle, we have $G_{\bar{\lambda}}<u$ and by Hopf's boundary lemma, we get $\partial_{n_x}G_{\bar{\lambda}}<\partial_{n_x}u$ which gives

$$\frac{1}{2}\frac{|\bar{x}|}{\bar{t}}\bar{\lambda}=\partial_{n_x}G_{\bar{\lambda}}<\partial_{n_x}u=\frac{|\bar{x}|}{\bar{t}}F(\bar{\lambda})$$

i.e., we have (4.4).

$$\frac{\bar{\lambda}}{2}<F(\bar{\lambda}) \tag{4.4}$$

If we take the smallest heat ball $D_{\tilde{\lambda}}$ from outside, touching the boundary at some point $(\tilde{x},\tilde{t})$ and consider the corresponding Green's function $G_{\tilde{\lambda}}$, we obtain

$$\frac{1}{2}\frac{|\tilde{x}|}{\tilde{t}}\tilde{\lambda}=\partial_{n_x}G_{\tilde{\lambda}}>\partial_{n_x}u=\frac{|\tilde{x}|}{\tilde{t}}F(\tilde{\lambda}),$$

i.e., we obtain (4.5).

$$\frac{\tilde{\lambda}}{2}>F(\tilde{\lambda}) \tag{4.5}$$

On the other hand, since $D_{\bar{\lambda}}\subset D_{\tilde{\lambda}}$, we have (4.6).

$$\bar{\lambda}>\tilde{\lambda} \tag{4.6}$$

Equations (4.4), (4.5) and (4.6) contradicts with (4.3) and hence we cannot have strict inequalities. Then, we can conclude that $D_{\bar{\lambda}}=D_{\tilde{\lambda}}$, i.e., Ω coincides with $D_{\bar{\lambda}}$ and $D_{\tilde{\lambda}}$. We get the conclusion that Ω is a heat ball.

Theorem 4.2 Suppose that there is a non-constant solution u to the following overdetermined problem (4.7) and the boundary gradient condition (4.8) under the Conditions 4.1 and 4.2, together with (4.9).

$$\begin{cases} Hu=-\delta_0 & \text{in} \quad \Omega, \\ u=0 & \text{on} \quad \partial_{par}\Omega, \\ u=0 & \text{on} \quad (\{t=0\}\cap\Omega)\setminus\{x=0\}, \end{cases} \tag{4.7}$$

$$\partial_{n_x}u=F(|x|,t) \quad \text{on} \quad \partial_{par}\Omega \tag{4.8}$$

$$F(|x|,t) < r^{n+1}F(r|x|,r^2t), \quad \text{where} \quad r > 1. \tag{4.9}$$

Then, $\Omega \cap (t = \tau)$ is a ball for all τ.

Proof. Let Ω be a domain fulfilling the Conditions 4.1 and 4.2, also let u be a solution to (4.7). Then, let us suppose that $\Omega \cap (t = \tau)$ is not a ball. We consider a rotation in space $\tilde{\Omega} \neq \Omega$ of Ω, and let $\tilde{u}$ be the same rotation of u. Define the scaled function

$$\tilde{u}_r(x,t) := r^n \tilde{u}(rx, r^2t), \quad \text{where} \quad r > 1.$$

which satisfies (4.7) in $\tilde{\Omega}^r := \{x/r : x \in \tilde{\Omega}\}$. We choose $r > 1$ to be the smallest possible so that $\tilde{\Omega}^r \subset \Omega$ and $\partial\tilde{\Omega}^r \cap \partial\Omega \neq \emptyset$. By using the strong comparison principle, $\tilde{u}_r < u$ in $\tilde{\Omega}^r$ and we have $\partial_{n_x}\tilde{u}_r < \partial_{n_x}u = \partial_{n_x}\tilde{u}$. For a touching point $(x_1,t_1) \in \partial\tilde{\Omega}^r \cap \partial\Omega$, we get

$$r^{n+1}F(r|x_1|,r^2t_1) = r^{n+1}\partial_{n_x}\tilde{u}(rx_1,r^2t_1) = \partial_{n_x}\tilde{u}_r < \partial_{n_x}u = F(|x_1|,t_1)$$

and the last inequality contradicts with (4.9).

4.2 Two-phase case

In the two-phase case, Green functions are

$$G^+ = w(x,t) - \lambda, \quad G^- = c\left(\frac{w(x,t)}{\lambda} - 1\right).$$

The normal derivatives can be written as given in equations (4.10). [9]

$$\partial_{n_x}G^+ = \frac{|x|}{2t}w(x,t), \quad \partial_{n_x}G^- = \frac{c}{\lambda}\frac{|x|}{2t}w(x,t) \tag{4.10}$$

Theorem 4.3 Suppose that there is a non-constant solution u to the problem (4.11) under the Conditions 4.1 and 4.2, along with the extra boundary condition (4.12), where c is a positive constant.

$$\begin{cases} Hu = -\delta_0 & \text{in} \quad \mathbb{R}^n \times \mathbb{R}^+ \setminus \partial_{par}\Omega, \\ u = 0 & \text{on} \quad \partial_{par}\Omega, \\ u^- \to -c & \text{as} \quad x \to \infty \end{cases} \tag{4.11}$$

$$\partial_{n_x}u^+ = w(x,t)\ \partial_{n_x}u^- \quad \text{on} \quad (\partial_{par}\Omega)\backslash\{t = 0\}, \tag{4.12}$$

Then, Ω is a heat ball.

Proof. We will give the proof in three steps.

Step 1: (Largest heat ball from inside) Let us take the largest heat ball in Ω, touching $\partial_{par}\Omega$ at some point $(\bar{x},\bar{t})$. We denote the Green's function corresponding to the level surface $\bar{\lambda} = w(\bar{x},\bar{t})$ by G_1, and define G_1^+, G_1^- by (4.10). By strong comparison principle, we get $G_1^+ < u^+$ in the largest heat ball. By considering the normals at the point $(\bar{x},\bar{t})$ and using the Hopf's lemma, we get (4.13).

$$\partial_{n_x} G_1^+(\bar{x},\bar{t}) < \partial_{n_x} u^+(\bar{x},\bar{t}) = w(\bar{x},\bar{t})\ \partial_{n_x} u^-(\bar{x},\bar{t}) \tag{4.13}$$

On the other hand, we have $G_1^- < u^-$ in the largest heat ball from inside and inequality (4.14).

$$\partial_{n_x} u^-(\bar{x},\bar{t}) < \partial_{n_x} G_1^-(\bar{x},\bar{t}) \tag{4.14}$$

Equations (4.13) and (4.14) give us inequality (4.15).

$$\frac{1}{w(\bar{x},\bar{t})}\ \partial_{n_x} G_1^+(\bar{x},\bar{t}) < \partial_{n_x} G_1^-(\bar{x},\bar{t}) \tag{4.15}$$

Using (4.10) in (4.15), we get (4.16).

$$1 < c \tag{4.16}$$

Step 2:(Smallest heat ball from outside.) Let us take the smallest heat ball containing Ω, touching $\partial_{par}\Omega$ at some point $(\tilde{x},\tilde{t})$. We denote the Green's function corresponding to the level surface $\tilde{\lambda} = w(\tilde{x},\tilde{t})$ by G_2 and define G_2^+, G_2^- by (4.10). By strong comparison principle $G_2^+ > u^+$ in the smallest heat ball. By using Hopf's lemma and considering the normals at the point $(\tilde{x},\tilde{t})$, we get (4.17).

$$\partial_{n_x} G_2^+(\tilde{x},\tilde{t}) > \partial_{n_x} u^+(\tilde{x},\tilde{t}) = w(\tilde{x},\tilde{t})\ \partial_{n_x} u^-(\tilde{x},\tilde{t}) \tag{4.17}$$

On the other hand, we have $G_2^- > u^-$ in the smallest heat ball from outside and inequality (4.18).

$$\partial_{n_x} u^-(\tilde{x},\tilde{t}) > \partial_{n_x} G_2^-(\tilde{x},\tilde{t}) \tag{4.18}$$

Equations (4.17) and (4.18) give us inequality (4.19).

$$\frac{1}{w(\tilde{x},\tilde{t})}\ \partial_{n_x} G_2^+(\tilde{x},\tilde{t}) > \partial_{n_x} G_2^-(\tilde{x},\tilde{t}) \tag{4.19}$$

Using (4.10) in (4.19) gives (4.20).

$$1 > c \tag{4.20}$$

Step 3: (Putting things together.)

Equations (4.16) and (4.20) give us a contradiction. We conclude that the largest heat ball in Ω and the smallest heat ball containing Ω coincide and that Ω is a heat ball itself.

Theorem 4.4 Suppose that there is a non-constant solution u satisfying (4.11) under the Conditions 4.1 and 4.2, along with the extra boundary condition (4.21) and c is a constant.

$$\partial_{n_x} u^+ = \partial_{n_x} u^- - (w(x,t) - c)\frac{|x|}{2t} \quad \text{on} \quad (\partial_{par}\Omega)\backslash\{t = 0\} \tag{4.21}$$

Then, Ω is a heat ball.

Proof. For the largest heat ball from inside, say $D_{\bar{\lambda}} \subset \Omega$, we have

$$\partial_{n_x} G_1^+(\bar{x},\bar{t}) + (w(\bar{x},\bar{t}) - c)\frac{|\bar{x}|}{2\bar{t}} < \partial_{n_x} u^-(\bar{x},\bar{t}) < \partial_{n_x} G_1^-(\bar{x},\bar{t})$$

and by denoting $w(\bar{x},\bar{t}) = \bar{\lambda}$, we get

$$\bar{\lambda} < c.$$

By using the similar argument, while working with the smallest heat ball from outside, say $D_{\tilde{\lambda}} \supset \Omega$, with $w(\tilde{x},\tilde{t}) = \tilde{\lambda}$, we get

$$\tilde{\lambda} > c,$$

which gives $\bar{\lambda} < \tilde{\lambda}$ and this contradicts with the fact that $D_{\bar{\lambda}} \subset \Omega \subset D_{\tilde{\lambda}}$.

Theorem 4.5 Suppose that there is a non-constant solution u to the following problem (4.22) under the Conditions 4.1 and 4.2, along with the extra boundary condition (4.23), where c is a constant and F is an increasing function (4.24) with A denoting any constant.

$$\begin{cases} Hu = -\delta_0 \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial_{par}\Omega, \\ u = 0 \quad \text{on} \quad (\{t = 0\} \cap \Omega) \setminus \{x = 0\}, \\ u^- \to -c \quad \text{as} \quad x \to \infty \end{cases} \tag{4.22}$$

$$\partial_{n_x} u^+ = F(\partial_{n_x} u^-) \quad \text{on} \quad \partial_{par}\Omega. \tag{4.23}$$

$$r^{n+1} F(A) > F(r^{n+1} A), \quad \text{where} \quad r > 1. \tag{4.24}$$

Then, $\Omega \cap (t = \tau)$ is a ball for all τ.

Proof. Let Ω be a domain fulfilling the Conditions 4.1 and 4.2, also let u be a solution to (4.22). Then, let us suppose that $\Omega \cap (t = \tau)$ is not a ball. We consider a rotation in space $\tilde{\Omega} \neq \Omega$ of Ω, and let $\tilde{u}$ be the same rotation of u, obviously $\tilde{u} \neq u$. Then, we define the scaled function

$$\tilde{u}_r(x,t) := r^n \tilde{u}(rx, r^2 t), \quad \text{where} \quad r > 1,$$

which satisfies (4.22) in $\tilde{\Omega}^r$. We choose $r > 1$ to be the smallest possible so that $\tilde{\Omega}^r \subset \Omega$ and $\partial\tilde{\Omega}^r \cap \partial\Omega \neq \emptyset$. By using the strong comparison principle, $\tilde{u}_r^+ < u^+$ in $\tilde{\Omega}^r$ and we have $\partial_{n_x} \tilde{u}_r^+ < \partial_{n_x} u^+$. For a touching point $(x_1, t_1) \in \partial\tilde{\Omega}^r \cap \partial\Omega$, we get inequality (4.25).

$$r^{n+1} \partial_{n_x} \tilde{u}^+(rx_1, r^2 t_1) = \partial_{n_x} \tilde{u}_r^+ < \partial_{n_x} u^+ = F(\partial_{n_x} u^-) \tag{4.25}$$

On the other hand, $\tilde{u}_r^- < u^-$ outside $\tilde{\Omega}^r$ and we have $\partial_{n_x} u^- < \partial_{n_x} \tilde{u}_r^-$. By using the monotonicity of F, we get (4.26) for the same touching point (x_1, t_1) on the boundary of $\tilde{\Omega}^r$.

$$F(\partial_{n_x} u^-) < F(\partial_{n_x} \tilde{u}_r^-) = F(r^{n+1} \partial_{n_x} \tilde{u}^-(rx_1, r^2 t_1)) \tag{4.26}$$

Equations (4.25) and (4.26) give us (4.27).

$$r^{n+1} \partial_{n_x} \tilde{u}^+(rx_1, r^2 t_1) < F(r^{n+1} \partial_{n_x} \tilde{u}^-(rx_1, r^2 t_1)) \tag{4.27}$$

By using (4.23), we can write the last equation (4.27) as follows:

$$r^{n+1} F(\partial_{n_x} \tilde{u}^-(rx_1, r^2 t_1)) < F(r^{n+1} \partial_{n_x} \tilde{u}^-(rx_1, r^2 t_1))$$

which contradicts with (4.24).

4.3 Multi-phase case

For a given positive integer m, let

$$0 < \lambda_m < \lambda_{m-1} < \cdots < \lambda_1 < \infty, \qquad 0 = \alpha_1 < \alpha_2 < \cdots < \alpha_m < \alpha_{m+1} < \infty.$$

Consider λ_i and the corresponding Green's/capacitor functions G_i as

$$HG_i = 0, \quad \text{in } D_{\lambda_i} \setminus D_{\lambda_{i-1}}, \qquad i = 2, \cdots, m$$

$$G_i = -\alpha_i, \quad \text{on } \partial D_{\lambda_i}, \qquad G_i = -\alpha_{i-1}, \quad \text{on } \partial D_{\lambda_{i-1}}, \quad i = 2, \cdots, m.$$

For $i = 1$, we have

$$HG_1 = -\delta_0, \quad \text{in } D_{\lambda_1}, \qquad G_1 = 0, \quad \text{on } \partial D_{\lambda_1}$$

and for $i = m+1$, we have

$$HG_{m+1} = 0, \quad \text{in } \mathbb{R}^n \times \mathbb{R}^+ \setminus D_{\lambda_m}, \quad G_{m+1} = -\alpha_m, \quad \text{on } \partial D_{\lambda_m},$$

$$\lim_{x \to \infty} G_{m+1} = -\alpha_{m+1}.$$

We can compute explicitly all these functions, and we obtain results (4.28 - 4.30).

$$G_1 = w(x,t) - \lambda_1, \tag{4.28}$$

$$G_i = (\alpha_i - \alpha_{i-1}) \frac{w(x,t) - \lambda_i}{\lambda_{i-1} - \lambda_i} - \alpha_i, \quad i = 2, \cdots, m, \tag{4.29}$$

$$G_{m+1} = (\alpha_{m+1} - \alpha_m) \frac{w(x,t)}{\lambda_m} - \alpha_{m+1}. \tag{4.30}$$

Computing the normals of these functions gives us results (4.31 - 4.33).

$$\partial_{n_x} G_1 = \frac{|x|}{2t} w(x,t), \tag{4.31}$$

$$\partial_{n_x} G_i = \frac{\alpha_i - \alpha_{i-1}}{(\lambda_{i-1} - \lambda_i)} \frac{|x|}{2t} w(x,t), \quad i = 2, \cdots, m, \tag{4.32}$$

$$\partial_{n_x} G_{m+1} = \frac{\alpha_{m+1} - \alpha_m}{\lambda_m} \frac{|x|}{2t} w(x,t). \tag{4.33}$$

The multi-phase version of the Theorem 9 will be given as follows.

Theorem 4.6 Let Ω_i $(i=1,\cdots,m)$ be domains in $\mathbb{R}^{n+1}$ satisfying the Conditions 4.1 and 4.2, with $\Omega_{i-1}\subset\Omega_i$, $0=\alpha_1<\alpha_2<\cdots<\alpha_{m+1}<\infty$, and suppose there exist non-constant u_i $(i=1,\cdots,m)$ solving the following problem (4.34) along with the boundary gradient condition (4.35) where c_i's are constants.

$$\begin{cases} Hu_1=-\delta_0 \quad \text{in} \quad \Omega_1, \\ Hu_i=0, \quad \text{in} \quad \Omega_i\backslash\Omega_{i-1}, \\ Hu_{m+1}=0, \quad \text{in} \quad \mathbb{R}^n\times\mathbb{R}^+\backslash\Omega_m, \\ u_i=u_{i-1}=-\alpha_i, \quad \text{on} \quad \partial\Omega_{i-1}, \\ u_{m+1}\to-\alpha_{m+1} \quad \text{as} \quad |x|\to\infty, \end{cases} \tag{4.34}$$

$$\begin{cases} \partial_{n_x}u_i=c_i\,\partial_{n_x}u_{i+1}, \quad \text{on} \quad \partial_{par}(\Omega_i\backslash\Omega_{i-1}), \quad i=1,\cdots,m-1, \\ \partial_{n_x}u_m=w(x,t)\partial_{n_x}u_{m+1}, \quad \text{on} \quad (\partial_{par}(\mathbb{R}^n\times\mathbb{R}^+\backslash\Omega_m))\backslash\{t=0\}. \end{cases} \tag{4.35}$$

Then, Ω_i, $(i=1,\cdots,m)$ are heat balls.

Proof. We will give the proof in three steps.

Step 1: (Largest heat ball from inside.) Let us first consider the largest heat ball $D_{\bar\lambda_i}$ in Ω_i $(i=1,2,\cdots,m)$ touching $\partial\Omega_i$ at some point $(\bar x_i,\bar t_i)$. By strong comparison principle, we get

$$G_1<u_1 \quad \text{in} \quad D_{\bar\lambda_1},$$
$$G_i<u_i \quad \text{in} \quad D_{\bar\lambda_i}\backslash\Omega_{i-1}.$$

Let $(\bar x_i,\bar t_i)\in\partial D_{\bar\lambda_i}\cap\partial\Omega_i$. Then, by Hopf's boundary lemma we get results (4.36 - 4.39).

$$\partial_{n_x}G_1<\partial_{n_x}u_1 \quad \text{at} \quad (\bar x_1,\bar t_1), \tag{4.36}$$

$$\partial_{n_x}G_i>\partial_{n_x}u_i \quad \text{at} \quad (\bar x_{i-1},\bar t_{i-1}), \quad i=2,3,\cdots,m, \tag{4.37}$$

$$\partial_{n_x}G_i<\partial_{n_x}u_i \quad \text{at} \quad (\bar x_i,\bar t_i), \quad i=2,3,\cdots,m, \tag{4.38}$$

$$\partial_{n_x}G_{m+1}>\partial_{n_x}u_{m+1} \quad \text{at} \quad (\bar x_m,\bar t_m). \tag{4.39}$$

Using (4.35), (4.38) and (4.39) at $(\bar x_m,\bar t_m)$, we obtain

$$w\partial_{n_x}G_{m+1}>w\partial_{n_x}u_{m+1}=\partial_{n_x}u_m>\partial_{n_x}G_m,$$

which by (4.32) and (4.33) results in inequality (4.40).

$$\alpha_{m+1}-\alpha_m>\frac{\alpha_m-\alpha_{m-1}}{\bar\lambda_{m-1}-\bar\lambda_m} \tag{4.40}$$

Using (4.35), (4.37) and (4.38), at $(\bar{x}_{i-1}, \bar{t}_{i-1})$ we obtain

$$\partial_{n_x} G_i > c_{i-1}\partial_{n_x} u_i = \partial_{n_x} u_{i-1} > \partial_{n_x} G_{i-1}, \quad i = 3, \cdots, m$$

which gives (4.41).

$$\frac{\alpha_i - \alpha_{i-1}}{\bar{\lambda}_{i-1} - \bar{\lambda}_i} > \frac{\alpha_{i-1} - \alpha_{i-2}}{c_{i-1}(\bar{\lambda}_{i-2} - \bar{\lambda}_{i-1})}, \quad i = 3, \cdots, m \tag{4.41}$$

It results in inequality (4.42).

$$\alpha_{m+1} - \alpha_m > \frac{\alpha_2 - \alpha_1}{c_2 \cdots c_{m-1}(\bar{\lambda}_1 - \bar{\lambda}_2)} \tag{4.42}$$

Finally, using (4.35), (4.36) and (4.37) at (x_1, t_1), we obtain

$$c_1 \partial_{n_x} G_2 > c_1 \partial_{n_x} u_2 = \partial_{n_x} u_1 > \partial_{n_x} G_1$$

and we get (4.43).

$$\alpha_2 - \alpha_1 > \frac{\bar{\lambda}_1 - \bar{\lambda}_2}{c_1}. \tag{4.43}$$

Equations (4.42) and (4.43) give us the inequality (4.44).

$$\alpha_{m+1} - \alpha_m > \frac{1}{c_1 \cdots c_{m-1}} \tag{4.44}$$

Step 2: (Smallest heat ball from outside.) Let us now take a reverse situation and consider the smallest heat balls $D_{\tilde{\lambda}_i}$ containing Ω_i $(i = 1, 2, \cdots, m)$. Then a similar argument as in the previous case shows that $G_i > u_i$ and at the point (x_1, t_1), we get

$$c_1 \partial_{n_x} G_2 < c_1 \partial_{n_x} u_2 = \partial_{n_x} u_1 < \partial_{n_x} G_1$$

which gives inequality (4.45).

$$\alpha_2 - \alpha_1 < \frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{c_1} \tag{4.45}$$

Similarly

$$c_{i-1}\partial_{n_x} G_i < c_{i-1}\partial_{n_x} u_i = \partial_{n_x} u_{i-1} < \partial_{n_x} G_{i-1}, \quad i = 3, \cdots, m$$

which gives inequality (4.46).

$$\frac{\alpha_i - \alpha_{i-1}}{\tilde{\lambda}_{i-1} - \tilde{\lambda}_i} < \frac{(\alpha_{i-1} - \alpha_{i-2})}{c_{i-1}(\tilde{\lambda}_{i-2} - \tilde{\lambda}_{i-1})} \quad i = 3, \cdots, m \tag{4.46}$$

Finally, at the point (x_m, t_m) we obtain

$$w\,\partial_{n_x} G_{m+1} < w\,\partial_{n_x} u_{m+1} = \partial_{n_x} u_m < \partial_{n_x} G_m$$

and get inequality (4.47).

$$\alpha_{m+1} - \alpha_m < \frac{\alpha_m - \alpha_{m-1}}{\tilde{\lambda}_{m-1} - \tilde{\lambda}_m} \tag{4.47}$$

Equations (4.45), (4.46) and (4.47) give us the inequality (4.48).

$$\alpha_{m+1} - \alpha_m < \frac{1}{c_1 \cdots c_{m-1}} \tag{4.48}$$

Step 3: (Putting things together.) Equations (4.44) and (4.48) give us a contradiction.

The following theorem is the multi-phase version of Theorem 10.

Theorem 4.7 Under the hypotheses of Theorem 13, with equations (4.35) replaced by equations (4.49) and (4.50) where c_i's are constants, Ω_i, $(i = 1, \cdots, m)$ are heat balls.

$$\partial_{n_x} u_i = c_i\,\partial_{n_x} u_{i+1}, \quad \text{on} \quad \partial_{par}(\Omega_i \backslash \Omega_{i-1}), \quad i = 1, \cdots, m-1, \tag{4.49}$$

$$\begin{aligned} &\partial_{n_x} u_m = \partial_{n_x} u_{m+1} - (w(x,t) - \alpha_{m+1})\frac{|x|}{2t}, \\ &\qquad \text{on } \ (\partial_{par}(\mathbb{R}^n \times \mathbb{R}^+ \backslash \Omega_m)) \backslash \{t = 0\}, \end{aligned} \tag{4.50}$$

Proof. If we consider the largest heat ball from inside, (4.38), (4.39) and (4.50) at $(\bar{x}_m, \bar{t}_m)$, we obtain

$$\partial_{n_x} G_{m+1} > \partial_{n_x} u_{m+1} = \partial_{n_x} u_m + (w - \alpha_{m+1})\frac{|\bar{x}_m|}{2\bar{t}_m} > \partial_{n_x} G_m + (w - \alpha_{m+1})\frac{|\bar{x}_m|}{2\bar{t}_m}$$

which gives inequality (4.51).

$$\frac{2\alpha_{m+1} - \alpha_m}{\bar{\lambda}_m} > 1 + \frac{\alpha_m - \alpha_{m-1}}{\bar{\lambda}_{m-1} - \bar{\lambda}_m} \tag{4.51}$$

Using (4.37), (4.38) and (4.49) at $(\bar{x}_{i-1}, \bar{t}_{i-1})$ we obtain (4.41). Using (4.36) and (4.49) at $(\bar{x}_1, \bar{t}_1)$ we obtain (4.43). Equations (4.51), (4.41) and (4.43) give

$$2\alpha_{m+1} - \alpha_m > \bar{\lambda}_m (1 + \frac{1}{c_1 \cdots c_{m-1}}).$$

By using the same argument, while working with the smallest heat ball from outside, we get

$$2\alpha_{m+1} - \alpha_m < \tilde{\lambda}_m (1 + \frac{1}{c_1 \cdots c_{m-1}}).$$

Since $D_{\bar{\lambda}_m} \subset D_{\tilde{\lambda}_m}$, we have $\bar{\lambda}_m > \tilde{\lambda}_m$ and obtain a contradiction.

The multi-phase case of Theorem 11 is stated in the following theorem.

Theorem 4.8 Let Ω_i $(i = 1, \cdots, m)$ be domains satisfying Conditions 4.1 and 4.2, with $\Omega_{i-1} \subset \Omega_i$, $0 = \alpha_1 < \alpha_2 < \cdots < \alpha_{m+1}$, and suppose there exist non-constant u_i $(i = 1, \cdots, m)$ satisfying (4.34) along with the boundary gradient condition (4.52) and $F_i, (i = 1, \cdots, m)$ denoting increasing functions with the property (4.53) with A denoting any constant.

$$\partial_\nu u_i = F_i(\partial_\nu u_{i+1}), \quad i = 1, \cdots, m, \tag{4.52}$$

$$r^{n+1} F_i(A) > F_i(r^{n+1} A), \quad \text{where} \quad r > 1. \tag{4.53}$$

Then, $\Omega_i \cap (t = \tau)$ $(i = 1, \cdots, m)$ are spheres for all τ.

Proof. Let Ω_i be domains fulfilling the Conditions 4.1 and 4.2, also let u_i be solutions satisfying (4.34). Let us suppose that $\Omega_i \cap (t = \tau)$ is not a ball. Then, we consider rotations in space $\tilde{\Omega}_i \neq \Omega_i$ of Ω_i, and let $\tilde{u}_i$ be the same rotation of u_i. Then we define the scaled function

$$\tilde{u}_{r_i}(x,t) := r_i^n \tilde{u}_i(r_i x, r_i^2 t), \quad \text{where} \quad r_i > 1, \quad i = 1, \cdots, m,$$

where $r_i > 1$ are the smallest possible so that $\tilde{\Omega}_i^r := \{x/r : x \in \tilde{\Omega}_i\}$, $\tilde{\Omega}_i^r \subset \Omega_i$ and $\partial \tilde{\Omega}_i^r \cap \partial \Omega_i \neq \emptyset$.

By using the strong comparison principle, $\tilde{u}_{r_i} < u_i$ in $\tilde{\Omega}_i^r \setminus \tilde{\Omega}_{i-1}^r$ and we have $\partial_{n_x} \tilde{u}_{r_i} < \partial_{n_x} u_i$. For touching points $(x_i, t_i) \in \partial \tilde{\Omega}_i^r \cap \partial \Omega_i$, we get (4.54).

$$r_i^{n+1} \partial_{n_x} \tilde{u}_i(r_i x_i, r_i^2 t_i) = \partial_{n_x} \tilde{u}_{r_i} < \partial_{n_x} u_i = F_i(\partial_{n_x} u_{i+1}) \tag{4.54}$$

On the other hand, $\tilde{u}_{r_{i+1}} < u_{i+1}$ in $\tilde{\Omega}_{i+1}^r \backslash \tilde{\Omega}_i^r$ and we have $\partial_{n_x} u_{i+1} < \partial_{n_x} \tilde{u}_{r_{i+1}}$. By using the monotonicity of F, we get (4.55).

$$F_i(\partial_{n_x} u_{i+1}) < F_i(\partial_{n_x} \tilde{u}_{r_{i+1}}) = F_i(r_i^{n+1} \partial_{n_x} \tilde{u}_{i+1}(r_i x_i, r_i^2 t_i)) \tag{4.55}$$

for the same touching point (x_i, t_i) on the boundary of $\tilde{\Omega}_i^r$. Equations (4.54) and (4.55) give us inequality (4.56).

$$r_i^{n+1} \partial_{n_x} \tilde{u}_i(r_i x_i, r_i^2 t_i) < F_i(r_i^{n+1} \partial_{n_x} \tilde{u}_{i+1}(r_i x_i, r_i^2 t_i)) \tag{4.56}$$

From (4.52), we can write the last inequality (4.56) resulting in (4.57).

$$r_i^{n+1} F(\partial_{n_x} \tilde{u}_{i+1}(r_i x_i, r_i^2 t_i)) < F_i(r_i^{n+1} \partial_{n_x} \tilde{u}_{i+1}(r_i x_i, r_i^2 t_i)) \tag{4.57}$$

The last inequality (4.57) contradicts with (4.53). Hence, we can conclude that $\Omega_i \cap (t = \tau)$ $(i = 1, \cdots, m)$ are spheres for all τ.

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